$= \{\beta_1, \ldots, \beta_k\}$, when we have

$$\left| \bigcap_{j=1}^{k} A_{\beta_j} \right| = 2^{n-k} + 1.$$

Hence for all $1 \leq i \leq n$ and $\{s_1, \ldots, s_i\} \subseteq \{1, \ldots, n\}$, Relation (7) holds except when $i = k$ and $\{s_1, \ldots, s_k\} = \{\beta_1, \ldots, \beta_k\}$. In this case

$$P\left(\bigcap_{j=1}^{k} A_{\beta_j}\right) = 2^{-k} + 2^{-n} \neq \prod_{j=1}^{k} \mathrm{P}(A_{\beta_j}) = 2^{-k}.$$

As mentioned in Section 1, by using the result for random variables in Wang (1990), we can construct a set of n random events which are dependent at the nth level and independent at all other levels. Thus we have a case in which all the equalities in (1) fail except one. [Other similar examples can be found in Stoyanov (1987).] Example 4, however, covers the most general situation, since *that only* exception can occur at *any* level k, $k = 2, \ldots, n$, *and* we can choose which one of the $\binom{n}{k}$ equalities in (1) is to be the exception.

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Testing Means Using Hypothesis-Dependent Variance Estimates

ARVIND K. SHAH and K. KRISHNAMOORTHY*

Some of the commonly used univariate and multivariate tests on means are studied where the usual hypothesis-independent variance estimates are replaced with the corresponding hypothesis-dependent variance estimates. This approach leads to tests that are equivalent to the corresponding traditional tests in the cases considered here except in one case. Understanding this can be beneficial to practitioners and students of statistics.

KEY WORDS: Analysis of variance; Hotelling's test; MANOVA; *t* test.

1. INTRODUCTION

While introducing the concepts of hypothesis testing to beginning statistics students, the analogy between the process of jury trial of an accused and the process of hypothesis testing is often drawn. The hypothesis testing approach is also compared with the approach of "proof by contraction." The students are repeatedly reminded about the logic of proceeding under the assumption of the null hypothesis being true unless contradicted through the sample evidence (at some specified level of significance).

As various hypothesis testing procedures are introduced, some students become puzzled by the fact that the information specified in the null hypothesis is not fully utilized in some of the tests, especially in computing the variance estimates. For example, under the one sample t test, the sample mean, $\bar{x}$, is utilized (instead of the specified mean, μ_0) in computing the variance estimate, while in testing for the binomial proportion, the specified proportion value, p_0, is utilized in computing the variance estimate.

This article examines some of the common univariate and multivariate tests from this viewpoint. It turns out that in all but one test considered, the use of hypothesis-dependent variance estimates leads to tests that are equivalent to the corresponding traditional tests utilizing the hypothesis-independent variance estimates. The realization of this fact may be interesting and educational to practitioners and students of statistics.

2. UNIVARIATE TESTS

The one sample t test is commonly used to test the null hypothesis

$$H_0 : \mu = \mu_0,$$

based on a random sample from a normal distribution with unknown variance σ^2. This test is the likelihood ratio test as well as the uniformly most powerful unbiased test. The test statistic t is given by

$$t = (\bar{x} - \mu_0)/(s^2/n)^{1/2},$$

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where $\bar{x}$ is the sample average based on a random sample of size n. The sample variance s^2 is an unbiased estimate of σ^2 computed as

$$\hat{\sigma}^2 = s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/(n - 1).$$

The question of interest is why don't we take advantage of the specified value of μ under the null hypothesis in computing an estimate of σ^2, as

$$\hat{\sigma}^2 = s_0^2 = \sum_{i=1}^{n} (x_i - \mu_0)^2/n?$$

In turn, why don't we use t_0 as our test statistic to test the hypothesis in (2·1) where

$$t_0 = (\bar{x} - \mu_0)^2/(s_0^2/n)^{1/2}?$$

The answers to these questions are rather simple. It can be easily shown that the t_0 statistic can be written in terms of the t statistic as follows:

$$t_0 = t[n/(n - 1 + t^2)]^{1/2}. \qquad (2.1)$$

The one-to-one correspondence between the statistics t and t_0 and their corresponding distributions (percentile points) are now apparent. Note that even though $\bar{x}$ and s_0^2 are dependent, the distribution of t_0 can still be easily obtained from the distribution of t through (2.1). In summary, the t and t_0 tests lead to the same inference and hence are identical tests. It is interesting to note that a gain of one degree of freedom really does not improve the test. So why bother with the t_0 test? For this and related discussions, see Lefante and Shah (1986) and Good (1986).

In the same spirit, one can raise a similar question about the F statistic for testing the equality of the k treatment means, that is,

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k,$$

in a completely randomized design. The same logic leads to a new statistic, F_0, which can be expressed in terms of the usual F statistic as shown below:

$$F_0 = \frac{\text{treatment mean squares}}{\text{total mean squares}} = \frac{(N - 1)F}{(k - 1)F + (N - k)}.$$

Once again, the one-to-one correspondence between the statistics F and F_0 for the completely randomized design and their corresponding distributions lead to the same inference and hence are identical tests. So there is no need to bother with the F_0 test.

3. MULTIVARIATE TESTS

Now we continue with this same idea in the one-sample and k-sample multivariate testing procedures. Suppose that we have independent vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from a p-variate normal population with unknown mean vector $\boldsymbol{\xi}$ and unknown covariance matrix Σ. The usual test statistic for testing

$$H_0 : \boldsymbol{\xi} = \boldsymbol{\xi}_0$$

is the Hotelling's T^2, which is given by

$$T^2 = n(\bar{\mathbf{X}} - \boldsymbol{\xi}_0)'S^{-1}(\bar{\mathbf{X}} - \boldsymbol{\xi}_0), \qquad (3.1)$$

where $\mathbf{X}$ is the sample mean vector and

$$S = \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'/(n - 1)$$

is the sample covariance matrix. If we replace S in (3.1) by the hypothesis-dependent covariance estimate,

$$S_0 = \sum_{i=1}^{n} (\mathbf{X}_i - \boldsymbol{\xi}_0)(\mathbf{X}_i - \boldsymbol{\xi}_0)'/n,$$

do we get a different test? That is, does

$$T_0^2 = n(\bar{\mathbf{X}} - \boldsymbol{\xi}_0)'S_0^{-1}(\bar{\mathbf{X}} - \boldsymbol{\xi}_0)$$

lead to a different test? It is easy to show that T_0^2 can be expressed as

$$T_0^2 = T^2[n/(n - 1 + T^2],$$

which is a one-to-one function of T^2.

Thus, for the same reasons given in the t test case, we conclude that the test procedures based on T^2 and T_0^2 are identical. Interestingly, Kshirsager (1972, problem 40, p. 490) has noted a statistic T'^2 which equals $(n - 1)T_0^2/n$.

Finally, we investigate the effect of hypothesis-dependent covariance estimates in the MANOVA setup for testing the equality of several normal mean vectors. Let $\boldsymbol{\xi}_i$ and Σ_i denote the unknown mean vector and the unknown covariance matrix of ith population, $i = 1, \cdots, k$. Further let the jth observation from the ith population, $i = 1, \cdots, k$ and $j = 1, 2, \cdots, n_i$ be denoted by $\mathbf{X}_{ij}$ and the ith sample mean vector by $\bar{\mathbf{X}}_i$. Define the hypothesis matrix

$$H = \sum_{i=1}^{k} n_i(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})(\bar{\mathbf{X}}_i - \bar{\mathbf{X}})',$$

where $\bar{\mathbf{X}} = \Sigma_{i=1}^{k} n_i\bar{\mathbf{X}}_i/(\Sigma_{i=1}^{k} n_i)$, and the error matrix

$$E = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\mathbf{X}_{ij} - \bar{\mathbf{X}}_i)(\mathbf{X}_{ij} - \bar{\mathbf{X}}_i)'.$$

Then, the usual test statistics to test

$$H_0 : \boldsymbol{\xi}_1 = \cdots = \boldsymbol{\xi}_k$$

are some real valued functions of the eigenvalues of HE^{-1} (Morrison 1990). For example, Roy's root test statistic is $\theta_s = c_s/(1 + c_s)$, where c_s is the largest eigenvalue of HE^{-1} among the s nonzero eigenvalues of HE^{-1}. The Lawley–Hotelling test statistic is $\mathrm{tr}(HE^{-1})$. Again, the question of interest is whether we get different test procedures if we use the hypothesis-dependent error matrix

$$E_0 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\mathbf{X}_{ij} - \overline{\mathbf{X}})(\mathbf{X}_{ij} - \overline{\mathbf{X}})'$$

instead of E. We show that two commonly used real valued functions of the eigenvalues of HE_0^{-1}, namely the largest eigenvalue and the trace of HE_0^{-1}, lead to the well-known test procedures available in the literature.

First let us consider the largest eigenvalue of HE_0^{-1} as a test statistic to test H_0. Noting the relation that $E_0 = H + E$, it can be easily shown that the largest eigenvalue of HE_0^{-1} is indeed Roy's root test statistic θ_s. If we propose $\text{tr}(HE_0^{-1})$ as a test statistic, then this leads to a different test than the Lawley–Hotelling test. However, this is the well-known Pillai's trace test statistic, as $\text{tr}(HE_0^{-1}) = \text{tr}[H(H + E)^{-1}]$.

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More on Shortest Confidence Intervals

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Confidence intervals for a single unknown parameter are often derived by using a pivotal quantity. We present an elementary method for deriving the shortest such interval. The method yields intervals in several cases which have previously required separate analyses using more advanced techniques. In addition, it is suitable for introduction in a senior undergraduate level course in mathematical statistics.

KEY WORDS: Parametric family of densities; Pivotal quantity method.

1. INTRODUCTION

Using a confidence interval to estimate an unknown parameter is a classical statistical technique that is introduced in almost every statistics course. To clarify the problem, we will follow Guenther (1969) and Ferentinos (1990) and consider the case where we are given a random sample $X_1, X_2, \ldots, X_n$ from a density $f(x, \theta)$ and a pivotal quantity $Q(X_1, X_2, \ldots, X_n, \theta)$ whose distribution does not depend on θ. Finding a pivotal quantity will not be discussed, but the choice of a "good" pivotal quantity is essential for the resulting confidence interval to be useful. Usually the pivotal quantities are developed from either maximum likelihood estimates or sufficient statistics.

A confidence interval will be "constructed" in the following manner. Numbers a and b are chosen to satisfy the probability statement:

$$P[a < Q < b] = 1 - \alpha. \tag{1.1}$$

Often this expression can be solved for θ, in the form of an interval

$$P[W_1(X_1, X_2, \ldots, X_n) < \theta < W_2(X_1, X_2, \ldots, X_n)] = 1 - \alpha. \tag{1.2}$$

After observing data $x_1, x_2, \ldots, x_n$, the numbers $w_i = W_i(x_1, x_2, \ldots, x_n)$ are calculated and form the lower and upper endpoints of a $1 - \alpha$ confidence interval based on Q.

Ferentinos (1990) has a discussion of finding pivotal quantities based on sufficient statistics in families with truncation parameters, and presents a different method for finding the shortest interval.

If a and b can be found so that $w_2 - w_1$ is as small as possible, then the resulting confidence interval is called shortest confidence interval based on Q (Ferentinos 1990).

To illustrate the problem, consider two cases based on the normal distribution with expectation μ and variance σ^2, where a set of sufficient statistics are

$$\overline{X} = \sum_{k=1}^{n} X_k/n \quad \text{and} \quad S^2 = \sum_{k=1}^{n} (X_k - \overline{X})^2/(n - 1).$$

To find a confidence interval for μ, choose Q to be either $n^{1/2}(\overline{X} - \mu)/\sigma$ or $n^{1/2}(\overline{X} - \mu)/S$ depending on whether σ is assumed to be known or unknown. If σ is unknown then (1.2) becomes:

$$P[\overline{X} - Sb/n^{1/2} < \mu < \overline{X} - Sa/n^{1/2}] = 1 - \alpha.$$

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