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IMPROVED ESTIMATORS OF EIGENVALUES OF $\Sigma_1\Sigma_2^{-1}$

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Abstract. *Let S_1 and S_2 be independent Wishart matrices, with $S_i \sim W_p(n_i, \Sigma_i)$, $i = 1,2$. In this paper the problem of estimating the eigenvalues $\delta_1, \delta_2, \dots, \delta_p$ ($\delta_1 \geq \delta_2 \geq \dots \geq \delta_p > 0$) of $\Sigma_1\Sigma_2^{-1}$ is considered. A random matrix F which is a function of S_1 and S_2 is defined such that the eigenvalues of F have the same distribution as that of $S_1S_2^{-1}$ and the probability density function of F depends on a positive definite matrix Δ whose eigenvalues are $\delta_1, \delta_2, \dots, \delta_p$. Then orthogonally invariant estimators $\hat{\Delta}(F)$ based on F of Δ are obtained. The eigenvalues of $\hat{\Delta}(F)$ are taken as the estimators of $\delta_1, \dots, \delta_p$. Monte Carlo study shows that these eigenvalues estimators perform very well under the entropy loss.*

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1. INTRODUCTION

Suppose that S_1 and S_2 are independent $p \times p$ Wishart random matrices with $S_i \sim W_p(n_i, \Sigma_i)$ where Σ_i is a positive definite matrix and n_i ($\geq p$) is degrees of freedom, $i = 1,2$. Let $\ell_1, \ell_2, \ldots, \ell_p$ ($\ell_1 > \ell_2 > \ldots > \ell_p > 0$) denote the eigenvalues of $S_1 S_2^{-1}$. We consider here the problem of estimating eigenvalues $\delta_1, \delta_2, \ldots, \delta_p$ ($\delta_1 \geq \delta_2 \geq \ldots \geq \delta_p > 0$) of $\Sigma_1 \Sigma_2^{-1}$. We need to estimate these eigenvalues, for example, in the problem of testing $\Sigma_1 = \Sigma_2$ against $\Sigma_1 \neq \Sigma_2$. The power function of any test statistic which is a function of $\ell_1, \ldots, \ell_p$, depends on Σ_1 and Σ_2 only through $\delta_1, \delta_2, \ldots, \delta_p$.

As pointed out by Muirhead and Verathaworn (1985), to estimate $\delta_1, \delta_2, \ldots, \delta_p$ through a decision theoretic approach, one should specify a loss function in terms of δ_i's and $\hat{\delta}_i$'s, where $\hat{\delta}_i$ is an estimator of δ_i ($i = 1,2,\ldots,p$). However, for this loss, computing the risk of an estimator under the joint distribution of $\ell_1, \ell_2, \ldots, \ell_p$ seems to be infeasible due to the complexity of the distribution of the ordered eigenvalues. Instead, we follow the approach given in Muirhead and Verathaworn (1985).

Define a random matrix

$$F = V_1^{1/2} V_2^{-1} V_1^{1/2}$$

where $V_1^{1/2} = \Sigma_2^{-1/2} S_1 \Sigma_2^{-1/2} \sim W_p(n_1, \Delta)$, $V_2 = \Sigma_2^{-1/2} S_2 \Sigma_2^{-1/2} \sim W_p(n_2, I)$ and $\Delta = \Sigma_2^{-1/2} \Sigma_1 \Sigma_2^{-1/2}$.

The density function of F is given by

$$c|\Delta|^{-n_1/2} |F|^{(n_1-p-1)/2} |I + \Delta^{-1} F|^{-n/2}, \quad F > 0 \tag{1.1}$$

where

$$c = \Gamma_p(n/2)/\left(\Gamma_p(n_1/2)\,\Gamma_p(n_2/2)\right),\quad \Gamma_p(a) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(\frac{2a-i+1}{2}\right)$$

$n_1 > p+1$, $n_2 > p+1$, $n = n_1 + n_2$ and Δ is a positive definite parameter matrix. The eigenvalues of F are the same as those of $S_1S_2^{-1}$ and their distribution depends only on the eigenvalues $\delta_1,\ldots,\delta_p$ of $\Sigma_1\Sigma_2^{-1}$ or equivalently of Δ. We then estimate Δ by $\hat{\Delta}(F)$ and take the eigenvalues of $\hat{\Delta}(F)$ as esimators of $\delta_1,\ldots,\delta_p$. Note that F is not observable. However, if we estimate Δ by only orthogonally invariant estimators of the form $\hat{\Delta}(F) = R\varphi(L)R'$ (where $F = RLR'$ is a spectral decomosition of F, $L = \text{diag}(\ell_1,\ldots,\ell_p)$ with $\ell_1 > \ell_2 > \ldots > \ell_p > 0$, and $\varphi(L) = \text{diag}(\varphi_1(L),\ldots,\varphi_p(L)))$ then the eigenvalues $\varphi_1(L), \varphi_2(L),\ldots, \varphi_p(L)$ of $\hat{\Delta}(F)$ are observable and they may be considered as estimators of $\delta_1,\ldots,\delta_p$.

Therefore we, hereafterwards, consider the problem of estimating Δ by orthogonally invariant estimators under the loss

$$L_1(\Delta,\hat{\Delta}) = \text{tr}(\hat{\Delta}\Delta^{-1}) - \log|\hat{\Delta}\Delta^{-1}| - p. \tag{1.2}$$

The usual unbiased (under the density (1.1)) and also orthogonally invariant estimator of Δ is given by

$$\hat{\Delta}_u(F) = \frac{n_2 - p - 1}{n_1} F. \tag{1.3}$$

Muirhead and Verathaworn (1985) have shown that $\hat{\Delta}_u(F)$ is the best multiple of F for the loss (1.2). They also derived an estimator $\hat{\Delta}_R = R\varphi(L)R'$ with

$$\varphi_i(L) = \ell_i/[(n_1 - p - 1)/2 + \frac{2(n - p - 1)}{n_2(n_2 - p - 1)} \ell_i \sum_{j \neq i} (\ell_i - \ell_j)^{-1}], \tag{1.4}$$

$i = 1,2,\ldots,p$. As an estimator of Δ, their Monte Carlo comparison study shows that $\hat{\Delta}_R(F)$ performs better than $\hat{\Delta}_u(F)$ when $\Delta \approx cI$ or when Δ has groups of equal eigenvalues. However, Monte Carlo study under the entropy loss function in terms of eigenvalues,

$$L_2(\Delta_e, \hat{\Delta}_e) = tr(\hat{\Delta}_e \Delta_e^{-1}) - \log|\hat{\Delta}_e \Delta_e^{-1}| - p \tag{1.5}$$

(where, for a matrix A, $A_e = diag(\lambda_1,\ldots,\lambda_p)$, λ_i's are the eigenvalues of A) indicates that the eigenvalues of $\hat{\Delta}_u(F)$ dominate the eigenvalues of $\hat{\Delta}_R(F)$ when the δ_i's are much dispersed.

In Section 2, we give some preliminary results which will be used to prove the dominance of the estimators given in this paper over the usual unbiased estimator $\hat{\Delta}_u(F)$.

It can be easily seen that the largest root of $\hat{\Delta}_u(F)$ overestimates δ_1 and the smallest root underestimates δ_p. This means that the roots of $\hat{\Delta}_u(F)$ tend to be more spread out than those of Δ. In Section 3, we give four simple orthogonally invariant estimators $\hat{\Delta}_1(F)$, $\hat{\Delta}_2(F)$, $\hat{\Delta}_3(F)$ and $\hat{\Delta}_4(F)$ (say). Each of them shrinks the larger eigenvalues of F and expands the smaller eigenvalues toward some central value. We show that $\hat{\Delta}_1(F)$ is uniformly better than $\hat{\Delta}_u(F)$ using the approximate unbiased estimate of the risk function of any orthogonally invariant estimator given by Muirhead and Verathaworn (1985). Numerical study indicates that $\hat{\Delta}_2(F)$ and $\hat{\Delta}_3(F)$ are better than $\hat{\Delta}_u(F)$ under certain conditions on n_1 and n_2.

Muirhead and Verathaworn (1985) derived the best lower triangular invariant minimax estimator $\hat{\Delta}_m(F)$ of Δ under the loss (1.2). Since this estimator is not orthogonally invariant, it is not of much interest in the present problem. Following the approach given in Krishnamoorthy and Gupta

(1987), we develop the orthogonally invariant estimator $\hat{\Delta}_4(F)$ from $\hat{\Delta}_m(F)$. This estimator $\hat{\Delta}_4(F)$ is of simple form and Monte Carlo study indicates that it is minimax. Also, the comparison study under the loss (1.5) indicates that, as an estimator of $\delta_1, \delta_2, \ldots, \delta_p$, the roots of $\hat{\Delta}_4(F)$ perform better than those of $\hat{\Delta}_u(F)$ significantly, and perform better than the eigenvalues of $\hat{\Delta}_R(F)$ when δ_i's are much dispersed. When δ_i's are approximately equal, the eigenvalues of $\hat{\Delta}_4(F)$ and those of $\hat{\Delta}_R(F)$ are not comparable.

2. SOME PRELIMINARY RESULTS

For the loss (1.2), omitting the constant term in it, Muirhead and Verathaworn(1985) have derived an unbiased estimate of the approximate risk of $\hat{\Delta}(F) = R\Phi(L)R'$ following the approach given in Haff (1982). The unbiased estimator is given by

$$\hat{R}(\Delta, R\Phi(L)R') = \frac{2(n -)p - 1}{n_2(n_2 - p - 1)} \sum_{i=1}^{p} \sum_{t>i} \frac{\varphi_i(L) - \varphi_t(L)}{\ell_i - \ell_t} + \frac{2(n - p - 1)}{n_2(n_2 - p - 1)} \sum_{i=1}^{p} \frac{\partial \varphi_i(L)}{\partial \ell_i}$$
$$+ \frac{(n_1 - p - 1)}{n_2} \sum_{i=1}^{p} \frac{\varphi_i(L)}{\ell_i} - \sum_{i=1}^{p} \log \varphi_i(L). \qquad (2.1)$$

We now consider the estimators of the form

$$\hat{\Delta}_d(F) = R\Phi(L)R' \qquad (2.2)$$

where $\Phi(L) = \mathrm{diag}(d_1 \ell_1, \ldots, d_p \ell_p)$, d_j's are constants such that $0 < d_1 < d_2 < \ldots < d_p$.

In the following lemma we give an upper bound for the risk of the estimators of the form (2.2).

LEMMA 2.1. *For any orthogonally invariant estimator* $\hat{\Delta}_d(F)$, *an upper bound for the* *unbiased estimate of the risk is given by*

$$\hat{R}(\Delta,\hat{\Delta}_d(F)) < \frac{2(n-p-1)}{n_2(n_2-p-1)}\sum_{i=1}^{p}(p-i+1)d_i + \frac{n_1-p-1}{n_2}\sum_{i=1}^{p} d_i - \sum_{i=1}^{p}\log(d_i\ell_i). \quad (2.3)$$

Proof. From (2.1), we get

$$\hat{R}(\Delta,\hat{\Delta}_d(F)) = \frac{2(n-p-1)}{n_2(n_2-p-1)}\sum_{i=1}^{p}\sum_{t>i}\frac{d_i\ell_i - d_t\ell_t}{\ell_i - \ell_t} + \frac{2(n-p-1)}{n_2(n_2-p-1)}\sum_{i=1}^{p} d_i$$

$$+ \frac{(n_1-p-1)}{n_2}\sum_{i=1}^{p} d_i - \sum_{i=1}^{p}\log(d_i\ell_i). \quad (2.4)$$

For $t > i$, $d_t > d_i$ and so

$$\sum_{i=1}^{n}\sum_{t>i}\frac{d_i\ell_i - d_t\ell_t}{\ell_i - \ell_t} < \sum_{i=1}^{n}\sum_{t>i}\frac{d_i\ell_i - d_i\ell_t}{\ell_i - \ell_t} = \sum_{i=1}^{n}(p-i)d_i \quad (2.5)$$

Using the inequality (2.5) in (2.4), we prove (2.3).

We next give an upper bound for the risk difference $R(\Delta,\hat{\Delta}_d(F)) - R(\Delta,\hat{\Delta}_u(F))$ in the following lemma.

LEMMA 2.2. *For any orthogonally invariant estimator* $\hat{\Delta}_d$, *we have*

$$R(\Delta,\hat{\Delta}_d(F)) - R(\Delta,\hat{\Delta}_u(F)) < \frac{2(n-p-1)}{n_2(n_2-p-1)}\sum_{i=1}^{p}(p-i+1)d_i + \frac{n_1-p-1}{n_2}\sum_{i=1}^{p}d_i$$

$$-\sum_{i=1}^{p}\log d_i + p\log(\frac{n_2-p-1}{n_1}) - p. \tag{2.6}$$

Proof. The risk of $\hat{\Delta}_u(F)$ under (1.2) is given by

$$R(\Delta,\hat{\Delta}_u(F)) = \frac{n_2-p-1}{n_1}\,\mathrm{tr}\,\mathcal{E}(F\Delta^{-1}) - \mathcal{E}\log|\Delta^{-1}\Delta_u(F)| - p$$

$$= -\mathcal{E}\log|\Delta^{-1}F| - p\log\frac{n_2-p-1}{n_1}. \tag{2.7}$$

Adding the constant terms in (1.2) to (2.3), we get

$$R(\Delta,\hat{\Delta}_d(F)) < \frac{2(n-p-1)}{n_2(n_2-p-1)}\sum_{i=1}^{p}(p-i+1)d_i + \frac{n_1-p-1}{n_2}\sum_{i=1}^{p}d_i$$

$$-\sum_{i=1}^{p}\log d_i - \mathcal{E}\log|\Delta^{-1}F| - p. \tag{2.8}$$

From (2.7) and (2.8), we prove (2.6).

3. ORTHOGONALLY INVARIANT ESTIMATORS OF Δ.

Here we introduce three orthogonally invariant estimators of Δ, namely,

$$\hat{\Delta}_i(F) = R\varphi_i^*(L)R' \tag{3.1}$$

where $\varphi_i^*(L) = \text{diag}(d_{i1}, d_{i2}, \ldots, d_{ip})$ and

$$d_{1j} = (n_2 - p - 1)/(n_1 + p + 1 - 2j)$$
$$d_{2j} = (n_2 - p - 1 + j)(n_2 - p - 2 + j)/(n_1(n_2 - 1))$$
$$d_{3j} = n_1 d_{1j} d_{2j}/(n_2 - p - 1)$$

$i = 1,2,3$ and $j = 1,2,\ldots,p$. Note that $0 < d_{i1} < d_{i2} < \ldots < d_{ip}$ for $i = 1,2,3$. These constants d_{1j}'s and d_{2j}'s are chosen, respectively, from the best invariant minimax estimators of the normal covariance matrix Σ (James and Stein (1961)) and Σ^{-1} (Krishnamoorthy and Gupta (1987)).

We need the following lemma to prove that $\hat{\Delta}_1(F)$ dominates $\hat{\Delta}_u(F)$.

<u>LEMMA 3.1</u>. *For any positive integers* k *and* p, $k > p$,

$$\sum_{j=1}^{p} \log(k + p + 1 - 2j) < p \log k.$$

<u>Proof</u>. Using the fact that

$$\log(a + b) + \log(a - b) = \log(a^2 - b^2) \leq \log(a^2) = 2\log(a),$$

when p is even, we have

$$\sum_{j=1}^{p} \log(k + p + 1 - 2j) = \sum_{j=1}^{p/2} [\log(k + p + 1 - 2j) + \log(k - p - 1 + 2j)] < p \log(k).$$

When p is odd, writing

$$\sum_{j=1}^{p} \log(k + p + 1 - 2j) = \log(k) + \sum_{\substack{j=1 \\ j\neq(p+1)/2}}^{p} \log(k + p + 1 - 2j)$$

the inequality can be similarly proved.

THEOREM 3.1. *For the loss (1.2), we have*

$$R(\Delta,\hat{\Delta}_1(F)) < R(\Delta,\hat{\Delta}_u(F)) \tag{3.2}$$

for all $n_1, n_2 > p + 1$.

Proof. Since the estimator $\hat{\Delta}_1(F)$ is of the form (2.2), from (2.6) we have,

$$R(\Delta,\hat{\Delta}_1(F)) - R(\Delta,\hat{\Delta}_u(F)) < \frac{2(n-p-1)}{n_2}\sum_{j=1}^{p}\frac{(p-j+1)}{n_1+p+1-2j}$$
$$+\frac{(n_1-p-1)(n_2-p-1)}{n_2}\sum_{j=1}^{p}(n_1+p+1-2j)^{-1}+\sum_{j=1}^{p}\log(n_1+p+1-2j)-p\log n_1-p. \tag{3.3}$$

It follows from Lemma 3.1 that the r.h.s. of (3.3) is less than zero if

$$\frac{2(n-p-1)}{n_2}\sum_{j=1}^{p}\frac{(p-j+1)}{(n_1+p+1-2j)}+\frac{(n_1-p-1)(n_2-p-1)}{n_2}\sum_{j=1}^{p}(n_1+p+1-2j)^{-1}<p. \tag{3.4}$$

Writing the l.h.s. of (3.4) as

$$\frac{2(n-p-1)}{n_2}\sum_{j=1}^{p}\frac{(p-j+1)}{(n_1+p+1-2j)}+\frac{(n_2-p-1)}{n_2}\sum_{j=1}^{p}\frac{[(n_1+p+1-2j)-2(p+1-j)]}{(n_1+p+1-2j)} \tag{3.5}$$

and after some simplification, it can be shown that (3.4) holds if and only if

$$\sum_{j=1}^{p} \frac{(p-j+1)}{(n_1+p+1-2j)} < \frac{p(p+1)}{2n_1} \Leftrightarrow \sum_{j=1}^{p} \left(\frac{p-j+1}{n_1+p+1-2j} - \frac{p-j+1}{n_1} \right) < 0. \tag{3.6}$$

Now to prove (3.6), let $a_j = \frac{p-j+1}{n_1+p+1-2j} - \frac{p-j+1}{n_1}$ and $x_j = a_j + a_{p+j+1}$. For each j, $x_j < 0$.

If p is even, $\sum_{j=1}^{p} a_j = \sum_{j=1}^{p/2} x_j < 0$. If p is odd $a_{(p+1)/2} = 0$ and $\sum_{j=1}^{p} a_j = \sum_{j=1}^{(p-1)/2} x_j < 0$. This then proves the inequality (3.2).

For the estimators $\hat{\Delta}_i(F)$, $i = 2,3$, proving that the r.h.s. of the following inequality

$$R(\Delta,\hat{\Delta}_i(F)) - R(\Delta,\hat{\Delta}_u(F)) \leq \frac{2(n-p-1)}{n_2(n_2-p-1)} \sum_{j=1}^{p} (p-j+1)d_{ij} + \frac{(n_1-p-1)}{n_2} \sum_{j=1}^{p} d_{ij} - \sum_{j=1}^{p} \log(d_{ij})$$
$$+ p \log\left(\frac{n_2-p-1}{n_1}\right) - p \tag{3.7}$$

is less than or equal to zero seems to be difficult. Numerical computations show that it is negative for $\hat{\Delta}_2(F)$ if $n_1, n_2 \geq p+1$ and for $\hat{\Delta}_3(F)$ if $n_1, n_2 \geq 2p+1$.

We next develop an orthogonally invariant estimator following an approach given by Krishnamoorthy and Gupta (1987). Muirhead and Verathaworn (1985) have derived the best lower triangular invariant estimator $\hat{\Delta}_m(F)$ of Δ, for the loss (1.2), which is given by

$$\hat{\Delta}_m(F) = TDT'$$

where T is a lower triangular matrix with positive diagonal elements such that $TT' = F$, $D = \mathrm{diag}(d_{41},\ldots,d_{4p})$ and

$$d_{4j} = \frac{(n_2 - p - 1 + j)(n_2 - p - 2 + j)}{(n_1 + 1 - j)(n_2 - p - 1 + j) + (p - j)(n - p - 1)}, \quad j = 1,2,\ldots,p. \tag{3.8}$$

Since the group of lower triangular matrices is solvable $\hat{\Delta}_m(F)$ is minimax. But this estimator is not very useful in the present problem, because its eigenvalues are not observable. However, we can develop an orthogonally invariant estimator from $\hat{\Delta}_m(F)$ as follows:

Notice that the loss function (1.2) is nonsingular invariant and $\hat{\Delta}_m(F)$ is a constant risk minimax estimator. Therefore, for any orthogonal matrix Γ,

$$\hat{\Delta}_{m\Gamma}(F) = \Gamma\hat{\Delta}_m(\Gamma'F\Gamma)\Gamma' \tag{3.9}$$

is also a constant risk minimax estimator (the proof is similar to the one given in the problem of estimating normal covariance matrix. See, for example, Sharma and Krishnamoorthy (1983)). In (3.9), if we let $\Gamma = R$, where R is an orthogonal matrix such that $RLR' = F$, $L = \text{diag}(\ell_1,\ldots,\ell_p)$ with $\ell_1 > \ell_2 > \ldots \ell_p > 0$, then $\hat{\Delta}_{m\Gamma}(F)$ becomes an orthogonally invariant estimator

$$\hat{\Delta}_4(F) = R\varphi_4^*(L)R' \tag{3.10}$$

where $\varphi_4^*(L) = \text{diag}(d_{41}\ell_1,\ldots,d_{4p}\ell_p)$. We also observe that $\hat{\Delta}_4(F)$ shrinks the larger eigenvalues and expands the smaller eigenvalues of F towards some central value.

<u>REMARK 3.1</u>. Note that the eigenvalues of the estimators $\hat{\Delta}_i(F)$ $(i = 1,\ldots,4)$ and $\hat{\Delta}_R(F)$ are unordered. We also observe, from (1.4), that some eigenvalues of $\hat{\Delta}_R(F)$ may not be positive. Therefore, it is desirable to use isotonizing algorithms to insure that

$$\varphi_1(L) \geq \varphi_2(L) \geq \ldots \geq \varphi_p(L) > 0, \tag{3.11}$$

where $\varphi_i(L)$'s are eigenvalues of an orthogonally inviarant estimator. To construct the eigenvalue estimates from $\hat{\Delta}_R(F)$ we use the algorithm given in Lin and Perlman (1985) and from $\hat{\Delta}_i(F)$ (i = 1,...,4) we follow the algorithm given in Barlow et al. (1972). These algorithms essentially group the adjacent eigenvalues to the extent necessary to insure the order (3.11).

4. MONTE CARLO STUDY

In this section, we carry out a Monte-Carlo study to compare the eigenvalues of the new estimators with those of $\hat{\Delta}_u(F)$ and $\hat{\Delta}_R(F)$ under the loss (1.5). For $p = 2,3,4$ and different values of (n_1,n_2), we compute the risk based on 1000 independent S_1's and 1000 independent S_2's generated, respectively, from $W_p(n_1,\Delta)$ and $W_p(n_2,I)$. As all these estimators considered in this paper are scale and orthogonally invariant, we take Δ to be $\mathrm{diag}(1,c_2,c_3,\ldots,c_p)$. For $p = 4$, we take the same Δ's considered in Muirhead and Verathaworn (1985).

In each table, the column values represent the risks of isotonized eigenvalues of the estimator of Δ. Since we observed that the eigenvalues of $\hat{\Delta}_1(F)$ and $\hat{\Delta}_2(F)$ are uniformly dominated by those of $\hat{\Delta}_3(F)$, we do not present these values in the tables.

Table 1 indicates that for $p = 2$, the eigenvalues of $\hat{\Delta}_u(F)$ perform better than those of $\hat{\Delta}_R(F)$ when one of the eigenvalues of Δ is smaller than half times of the other value. Otherwise the eigenvalues of $\hat{\Delta}_R(F)$ dominate those of $\hat{\Delta}_u(F)$. The eigenvalues of $\hat{\Delta}_3(F)$ and $\hat{\Delta}_4(F)$ dominate those of $\hat{\Delta}_u(F)$ uniformly. We also note that when $\Delta \approx cI$, the eigenvalues of $\hat{\Delta}_3(F)$, $\hat{\Delta}_4(F)$ and $\hat{\Delta}_R(F)$ are performing equally, while if the eigenvalues of Δ are very spreadout the eigenvalues of $\hat{\Delta}_3(F)$ and $\hat{\Delta}_4(F)$ dominate those of $\hat{\Delta}_R(F)$.

Tables 2 and 3 represent the risks for $p = 3$ and 4 respectively. From these tables, we see that, when $\Sigma \approx cI$, the eigenvalues of $\hat{\Delta}_3(F)$, $\hat{\Delta}_4(F)$ and $\hat{\Delta}_R(F)$ are not comparable. That is, their performance depends on the values of (n_1,n_2). However, the eigenvalues of $\hat{\Delta}_4(F)$ out perform those of $\hat{\Delta}_R(F)$ when the eigenvalues of Δ are very spreadout. For all p's, n_1's and n_2's considered in these three tables, the performance of the eigenvalues of $\hat{\Delta}_3(F)$ is as good as those of $\hat{\Delta}_4(F)$.

TABLE 1

Risk of the eigenvalues of $\hat{\Delta}$ under loss (1.5) for $p = 2$

$n_1 = 10,\ n_2 = 15$

c_2	$\hat{\Delta}_u$	$\hat{\Delta}_3$	$\hat{\Delta}_4$	$\hat{\Delta}_R$
1.0	0.55497	0.40477	0.40857	0.41379
.95	0.53026	0.38830	0.39170	0.39962
.90	0.50688	0.37315	0.37616	0.38529
.85	0.48494	0.35940	0.36204	0.37393
.80	0.46452	0.34713	0.34939	0.36809
.70	0.42863	0.32743	0.32902	0.36904
.50	0.37972	0.31024	0.31060	0.39494
.40	0.36765	0.31428	0.31407	0.42405
.30	0.36506	0.32704	0.32628	0.46877
.20	0.37101	0.34715	0.34598	0.51264
.10	0.38145	0.37080	0.36919	0.54484
.05	0.38622	0.38077	0.37900	0.52850

$n_1 = 20,\ n_2 = 20$

c_2	$\hat{\Delta}_u$	$\hat{\Delta}_3$	$\hat{\Delta}_4$	$\hat{\Delta}_R$
1.0	0.32206	0.24837	0.24975	0.23155
.95	0.30279	0.23417	0.23540	0.22056
.90	0.28510	0.22144	0.22257	0.21519
.85	0.26908	0.21040	0.21134	0.20883
.80	0.25479	0.20094	0.20175	0.20648
.70	0.23162	0.18720	0.18776	0.20510
.50	0.20749	0.18052	0.18065	0.23208
.40	0.20583	0.18686	0.18680	0.24870
.30	0.20978	0.19731	0.19714	0.26750
.20	0.21503	0.20845	0.20813	0.28029
.10	0.21952	0.21674	0.21635	0.27264
.05	0.22073	0.21936	0.21894	0.25989

$n_1 = 20,\ n_2 = 10$

c_2	$\hat{\Delta}_u$	$\hat{\Delta}_3$	$\hat{\Delta}_4$	$\hat{\Delta}_R$
1.0	0.53648	0.38816	0.39129	0.40820
.95	0.51161	0.37147	0.37427	0.38743
.90	0.48809	0.35609	0.35858	0.37718
.85	0.46601	0.34218	0.34430	0.36517
.80	0.44545	0.32966	0.33154	0.35711
.70	0.40924	0.30960	0.31091	0.34497
.50	0.35959	0.29196	0.29222	0.36287
.40	0.34795	0.29650	0.29631	0.38870
.30	0.34646	0.31023	0.30962	0.42730
.20	0.35448	0.33310	0.33214	0.46637
.10	0.36749	0.35989	0.35857	0.49751
.05	0.37270	0.37014	0.36868	0.48845

TABLE 2

Risk of the eigenvalues of $\hat{\Delta}$ under loss (1.5) for $p = 3$

(c_2,c_3)	$\hat{\Delta}_u$	$\hat{\Delta}_3$	$\hat{\Delta}_4$	$\hat{\Delta}_R$
		$n_1 = 10,\ n_2 = 15$		
(1,1)	1.1964	0.72289	0.73489	0.65751
(.95,.95)	1.1507	0.69738	0.70794	0.63008
(.95,.8)	1.0523	0.63947	0.64714	0.58060
(.8,.7)	0.95929	0.59311	0.49765	0.53849
(.7,.5)	0.82864	0.53229	0.53240	0.49220
(.6,.3)	0.72557	0.50457	0.50039	0.52681
(.5,.5)	0.84555	0.56215	0.56175	0.52908
(.5,.4)	0.77364	0.52325	0.52066	0.50148
(.5,.3)	0.71760	0.50270	0.49812	0.51539
(.4,.2)	0.67993	0.50798	0.50123	0.57436
(.3,.1)	0.65764	0.53757	0.52850	0.67627
(.9,.1)	0.81438	0.62837	0.62435	0.83070
		$n_1 = 20,\ n_2 = 20$		
(1,1)	0.67931	0.44410	0.44936	0.38259
(.95,.95)	0.64450	0.42138	0.42607	0.36275
(.95,.8)	0.56786	0.37065	0.37418	0.32154
(.8,.7)	0.50344	0.33314	0.33547	0.30022
(.7,.5)	0.41827	0.28874	0.28944	0.28539
(.6,.3)	0.36596	0.27823	0.27748	0.32878
(.5,.5)	0.44989	0.32221	0.32281	0.31705
(.5,.4)	0.39631	0.28912	0.28889	0.30278
(.5,.3)	0.36291	0.27680	0.27589	0.32251
(.4,.2)	0.35068	0.28595	0.28435	0.35650
(.3,.1)	0.35063	0.30974	0.30745	0.40278
(.9,.1)	0.43938	0.35876	0.35844	0.40420
		$n_1 = 20,\ n_2 = 10$		
(1,1)	1.1993	0.70271	0.71332	0.70419
(.95,.95)	1.1515	0.67646	0.68583	0.67130
(.95,.8)	1.0536	0.62093	0.62784	0.63002
(.8,.7)	0.95435	0.57225	0.57640	0.57617
(.7,.5)	0.82203	0.51347	0.51382	0.53350
(.6,.3)	0.71963	0.49007	0.48687	0.53178
(.5,.5)	0.82881	0.53589	0.53576	0.54255
(.5,.4)	0.76016	0.50076	0.49875	0.51755
(.5,.3)	0.70758	0.48516	0.48147	0.52134
(.4,.2)	0.66972	0.49265	0.48719	0.54939
(.3,.1)	0.65068	0.52674	0.51938	0.62491
(.9,.1)	0.81206	0.61147	0.60846	0.75699

TABLE 3
Risk of the eigenvalues of $\hat{\Delta}$ under loss (1.5) for p = 4

$n_1 = 10,\ n_2 = 10$

(c_1,c_2,c_3,c_4)	$\hat{\Delta}_u$	$\hat{\Delta}_3$	$\hat{\Delta}_4$	$\hat{\Delta}_R$
(1,1,1,1)	2.8265	1.3727	1.3966	1.4404
(25,1,1,1)	2.0453	1.3707	1.3399	1.2317
(10,10,1,1)	1.9402	1.3219	1.2794	1.3273
(8,4,2,1)	1.6350	1.0312	0.9772	0.9794
(100,50,10,1)	1.5759	1.2479	1.1797	1.4240
(25,25,1,1)	1.9658	1.4145	1.3702	1.3993
(10,10,10,1)	2.2548	1.3360	1.3156	1.6068
(1.75,1.25,.75,.25)	1.7619	1.0629	1.0158	1.0511
		$n_1 = 25,\ n_2 = 25$		
(1,1,1,1)	0.90066	0.54023	0.54845	0.40818
(25,1,1,1)	0.64431	0.48371	0.48354	0.38438
(10,10,1,1)	0.57996	0.46238	0.46105	0.43932
(8,4,2,1)	0.38321	0.30277	0.29919	0.40065
(100,50,10,1)	0.40875	0.38180	0.37701	0.43657
(25,25,1,1)	0.58236	0.47169	0.47017	0.40771
(10,10,1,1)	0.66833	0.47906	0.48131	0.47526
(1.75,1.25,.75,.25)	0.41799	0.31847	0.31594	0.41331
		$n_1 = 10,\ n_2 = 10$		
(1,.9,.2,.1)	1.7673	1.1427	1.0929	1.1902
(1,.8,.1,.01)	1.7026	1.3073	1.2454	1.5429
(1,.8,.3,.1)	1.7257	1.0999	1.0483	1.1581
(1,.2,.1,.1)	1.7036	1.1392	1.0887	0.99184
(1,.5,.3,.1)	1.6400	1.0510	0.9958	1.0572
(1,.7,.4,.1)	1.7306	1.0925	1.0424	1.1928
(1,.7,.6,.2)	1.8919	1.0749	1.0375	1.0360
(1,.8,.3,.1)	1.7257	1.0999	1.0483	1.1581
(1,.1,.05,.05)	1.7157	1.1891	1.1374	1.0268
		$n_1 = 10,\ n_2 = 50$		
(1,1,1,1)	1.4019	0.81425	0.82204	0.58245
(25,1,1,1)	1.0466	0.75170	0.75052	0.61184
(10,10,1,1)	0.93451	0.72153	0.71825	0.78692
(8,4,2,1)	0.68241	0.50149	0.49586	0.63147
(100,50,10,1)	0.69343	0.62597	0.61844	0.82181
(25,25,1,1)	0.94562	0.74783	0.74436	0.70914
(10,10,10,1)	1.0503	0.73735	0.73770	0.89174
(1.75,1.25,.75,.25)	0.72901	0.52120	0.51665	0.66563
(1,.4,.1,.05)	0.68545	0.62122	0.61347	0.82800
(1,.5,.1,.05)	0.68736	0.56479	0.55806	0.69431
(1,.6,.2,.10)	0.68975	0.52898	0.52305	0.68771
(1,.8,.2,.15)	0.77770	0.57399	0.56946	0.68123
(1,.9,.1,.1)	0.89556	0.69637	0.69247	0.76804
(1,.7,.2,.1)	0.71062	0.54531	0.53969	0.69238
(1,.99,.1,.01)	0.82035	0.69978	0.69439	0.86253
(1,.8,.2,.1)	0.73857	0.56519	0.55994	0.70912

REFERENCES

] Barlow, R.E., Bartholomew, D. J., Bremner, J. M., and Brunk, H.D. (1972). *Statistical inference under order restrictions*. John Wiley and Sons, New York.

] Haff, L. R. (1982). Solutions of Euler-Lagrange equations for certain multivariate normal estimation problems. Unpublished manuscript.

] James, W. and Stein, C. (1961). Estimation with quadratic loss. *Fourth Berkeley Symp. on Math. Statist. & Prob.*, University of California Press, Berkeley.

] Krishnamoorthy, K. and Gupta, A. K. (1989). Improved minimax estimators of normal precision matrix. (to appear in the *Canadian J. Statist.)*

·] Lin, S. P. and Perlman, M. D. (1985). A Monte Carlo comparison of four estimators for a covariance matrix. *Multivariate Analysis - VI* (P.R. Krishnaiah, ed.), 411-429. North-Holland, Amsterdam.

;] Muirhead, R. J. and Verathaworn, T. (1985). On estimating the latent roots of $\Sigma_1\Sigma_2^{-1}$. In *Multivariate Analysis VI* (P. R. Krishnaiah, ed.), 431-477. North Holland, Amsterdam.

'] Sharma, D. and Krishnamoorthy, K. (1983). Orthogonal equivariant minimax estimators of bivariate normal covariance matrix and precision matrix. *Calcutta Statist. Assoc. Bull. 32*, 23-45.

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