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PERFORMANCE OF THE PARAMETRIC BOOTSTRAP METHOD IN SMALL SAMPLE INTERVAL ESTIMATES

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Abstract

In this article, we investigate the performance of the parametric bootstrap (PB) method in interval estimation of parameters in various statistical problems and we present a numerical method to evaluate the merit of PB inferential procedures. We consider interval estimation of the Poisson mean, the correlation coefficient of a bivariate normal distribution, the largest eigenvalue of a normal covariance matrix, the ratio of normal means, and the common mean of several normal populations. PB interval estimates are compared with the ones based on exact methods and, in situations where exact methods are unavailable, their performance is evaluated by the numerical method. This evaluation method is useful to verify the validity of the PB method in complex problems.

1. Introduction

The bootstrap method in statistical inference was popularized by Efron [7] and can be applied in both parametric and nonparametric modes. Several papers have been written on the nonparametric application of the bootstrap to statistical problems. However, the results

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available on the performance of the parametric bootstrap (PB) method are limited. In their book, Efron and Tibshirani [9] devoted a few sections on parametric bootstrapping and illustrated the PB method to compute the standard error of the sample correlation coefficient. Lee [17] concluded that parametric bootstrap results may be more accurate than their nonparametric versions provided the model assumptions are at least approximately correct. Indurkhya [15] used a PB procedure to estimate the mean of a selected population and concluded that the PB estimate is in general superior to other estimators under the scalar loss function. For a good exposition of bootstrapping, its applications, and a review of literature, refer to the books by Hall [13], Efron and Tibshirani [9] and Davison and Hinkley [6].

We shall now explain the PB method for constructing interval estimates. Let $X_1, \ldots, X_n$ be a sample of independent observations from a population with distribution function $F(x;\,\theta)$, where θ is an unknown parameter. Let $\hat{\theta}$ be an estimator of θ based on the sample. $F(x;\,\hat{\theta})$ can be considered as an estimate of the population distribution function $F(x;\,\theta)$. The samples $x_{i1}, \ldots, x_{in}$, $i = 1, \ldots, m$ generated from $F(x;\,\hat{\theta})$ are called *parametric bootstrap samples*. The sampling distribution of $\hat{\theta}$ can be approximated by the frequency distribution of the $\hat{\theta}_i$'s, where $\hat{\theta}_i$ is the estimate of θ based on the ith PB sample from $F(x;\,\hat{\theta})$. In many situations, $\hat{\theta}_i$'s can be generated directly from the distribution of $\hat{\theta}$. The frequency distribution of the $\hat{\theta}_i$'s can be used to make inferences about θ. For example, when m is sufficiently large, $(\hat{\theta}(\alpha m/2),\ \hat{\theta}((1-\alpha/2)m))$, where $\hat{\theta}(k)$ denotes the kth smallest of the $\hat{\theta}_i$'s, serves as an approximate $100(1-\alpha)\%$ confidence interval for θ. Efron [8] refers to this method as the percentile method. Hall [12] suggested a modification to the Efron's percentile method, and this is to bootstrap $P(\hat{\theta},\,\theta) = (\hat{\theta}-\theta)/\hat{\sigma}(\hat{\theta})$, where $\hat{\sigma}(\hat{\theta})$ is an estimate of the standard deviation of $\hat{\theta}$, instead of bootstrapping $\hat{\theta}$. This method is referred to by many authors as the percentile-t method. A drawback of the percentile-t method is that the results based on it may not be transformation invariant. Nevertheless, as pointed out by many authors, it performs better than the percentile method in many situations. Inferences on a real valued function $g(\theta)$ can be obtained from the bootstrap distribution of $g(\hat{\theta})$ or $P(g(\hat{\theta}), g(\theta))$.

By the nature of the PB method, it is intuitive that the method will provide satisfactory results or results as good as those based on the Monte Carlo method, provided $\hat{\theta}$ is a consistent estimator of θ. Therefore, good performance of the PB method in a large sample study may not be surprising. It is of interest to see the small sample performance of the PB method in statistical inference about a parametric model.

In Section 2, we outline a numerical method to evaluate the performance of the PB method. In Section 3, we consider five examples: Interval estimation of (1) the Poisson mean, (2) the correlation coefficient of a bivariate normal distribution, (3) the largest eigenvalue of a normal covariance matrix, (4) the ratio of two normal means, and (5) the common mean of several normal populations. For examples (1) and (2), PB interval estimates are compared with the ones based on exact methods. For example (3), we note that an exact method for constructing confidence intervals is not available. Therefore, we used the numerical method given in Section 2 to understand the performance of the confidence intervals based on the percentile method and the percentile-t method. For example (4), a 95% PB interval estimate is computed for the example considered in Cox [5] and compared with the other interval estimates. For (5), a PB interval estimate is computed for the example considered by Jordan and Krishnamoorthy [16] and compared with other estimates given in their paper.

The overall performance of the percentile method for examples (1) and (2) is remarkably good even for small samples. For example (3), where we are interested in estimating the largest eigenvalue of a normal covariance matrix, the results based on the percentile-t method are more stable than the ones based on the percentile method. In examples (4) and (5), the PB intervals turned out to be narrower than those intervals to which they are compared. This calls for further evaluation of the PB intervals, and we used the numerical evaluation method of Section 2 to study their coverage probabilities. Finally, we make some concluding remarks in Section 4.

2. An Evaluation Method

In a statistical problem, if a parametric model is assumed, typically one attempts to make inferences about the parameter of interest using well-known exact methods. However, if the problem is more complex one may try numerical procedures such as Monte Carlo simulation and bootstrapping. When a numerical method is employed its validity needs to be evaluated in the context of the problem. To evaluate the performance of the PB confidence intervals, we present the following method:

Consider a parametric distribution function $F(x; \theta)$, where θ is an unknown parameter. Let $\hat{\theta}$ be an estimate of θ based on a sample $x_1, \ldots, x_n$ from $F(x; \theta)$. To initiate the evaluation method, let θ_0 be a value of θ chosen arbitrarily from the parameter space of θ. Let $\hat{\theta}_1, \ldots, \hat{\theta}_m$ be estimates based on m samples, each of size n, generated from $F(x; \theta_0)$. Let $\hat{\theta}_{1i}, \ldots, \hat{\theta}_{ki}$ be estimates generated from $F(x; \hat{\theta}_i)$, and I_i denotes the PB confidence interval for θ_0 based on $\hat{\theta}_{1i}, \ldots, \hat{\theta}_{ki}$, $i = 1, \ldots, m$. Let p represent the proportion of I_i's containing θ_0. If p is approximately equal to the specified confidence level, then the PB method will yield satisfactory results for values of θ near θ_0. The evaluation method should be repeated using other values of θ_0 to ensure the validity of the PB method over the entire parameter space. This evaluation procedure is illustrated in Sections 3.3, 3.4 and 3.5.

3. Examples

3.1. Interval estimation of the Poisson mean

An exact confidence interval for the Poisson mean θ is not difficult to obtain; we consider this example to show that PB confidence intervals are indeed in good agreement with the exact results.

Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean θ. Let k be an observed value of $\sum_{i=1}^{n} X_i$. An exact $100(1-\alpha)\%$ confidence interval for θ based on Clopper-Pearsons [4] method is given by

$$(\theta_l, \theta_u) = \left(\frac{1}{2n}\chi^2_{2k}(\alpha/2), \frac{1}{2n}\chi^2_{2k+2}(1-\alpha/2)\right) \tag{3.1}$$

(see, for example, Casella and Berger [3, p. 421]).

PB interval estimates for θ can be computed as follows: For given k and the sample size n, let $\hat{\theta} = k/n$. Generate 100,000 X's from Poisson$(n\hat{\theta})$. Then, $(X_{\alpha/2}/n, X_{1-\alpha/2}/n)$ is a $100(1-\alpha)\%$ PB confidence interval for θ.

In Table 1, 95% confidence intervals based on the exact method and the PB method are given. It is clear from the table values that the PB interval estimates are practically as good as the exact confidence intervals, even for small samples.

3.2. Interval estimation of the correlation coefficient

Let $\hat{\rho}$ denote the correlation coefficient of a random sample of size n from a bivariate normal population with mean vector μ and covariance matrix Σ, $N_2(\mu, \Sigma)$. Let $\hat{\rho}_0$ be an observed value of $\hat{\rho}$. An exact $100(1-\alpha)\%$ confidence interval for ρ is given by (ρ_l, ρ_u), where ρ_l is the solution of the equation

$$P(\hat{\rho} \geq \hat{\rho}_0 \mid \rho, n) = \alpha/2 \tag{3.2}$$

and ρ_u is the solution of

$$P(\hat{\rho} \leq \hat{\rho}_0 \mid \rho, n) = \alpha/2. \tag{3.3}$$

(See Anderson [1, Section 4.2.2].) For given α, $\hat{\rho}_0$ and n, equations (3.2) and (3.3) can be solved numerically using the probability density function (pdf) given by Hotelling [14]. This method is computationally very intensive because the pdf involves an infinite series which converges slowly when $|\rho| > 0.8$ and/or when n is large. Nevertheless, we computed the exact confidence intervals for ρ using (3.2) and (3.3) and presented them in Table 2.

Let $W_2(m, \Sigma)$ denote the Wishart distribution with $df = m$, and the parameter matrix Σ. For a given $\hat{\rho}_0$ and n, we computed the PB interval estimates for ρ as follows: For $j = 1$ to 100,000, generate $S = (s_{ij})$ from $W_2(n-1, \hat{\Sigma})$, where $\hat{\Sigma} = \begin{pmatrix} 1 & \hat{\rho}_0 \\ \hat{\rho}_0 & 1 \end{pmatrix}$; set $\hat{\rho}_j = s_{12}/\sqrt{s_{11}s_{22}}$. Then, the $100(\alpha/2)$th and $100(1-\alpha/2)$th percentiles of the $\hat{\rho}_j$'s form a $100(1-\alpha)\%$ confidence interval for ρ. We used Algorithm AS 53 of Smith and Hocking [19] to generate Wishart matrices.

In Table 2, we give exact confidence intervals and PB intervals (based on 100,000 PB samples) of ρ for various values of n. We see that the PB method provides satisfactory results even for small samples.

3.3. Interval estimation of the largest eigenvalue of a normal covariance matrix

Let S denote the sample covariance matrix based on a sample of n vector observations from a p-variate normal population with mean vector μ and covariance matrix Σ, $N_p(\mu, \Sigma)$. Assume that both μ and Σ are unknown. Let λ denote the large eigenvalue of Σ. Estimation of λ is important since it is the variance of the first principal component. A natural estimator of λ is the largest eigenvalue $\hat{\lambda}$ of S. A PB interval estimate of λ, using the percentile method, can be computed as follows: For given n and S, generate Wishart matrices W_i's from $W_p(n-1, S)$; set $V_i = W_i/(n-1)$. Compute the largest eigenvalue $\hat{\lambda}_i$ of the V_i for all i. Then, $100(\alpha/2)$th and $100(1-\alpha/2)$th percentiles of the $\hat{\lambda}_i$'s form a $100(1-\alpha)$ confidence interval for λ. One can also compute PB confidence interval using the percentile-t method. The pivot statistic $\sqrt{n-1}(l-\lambda)/(l\sqrt{2})$ given in Anderson [1, p. 469] can be used. Let $t_{\alpha/2}$ and $t_{1-\alpha/2}$ denote respectively the lower $100\alpha/2$th and the upper $100(1-\alpha/2)$th percentiles of $\sqrt{n-1}(l_i - \hat{\lambda})/(l_i\sqrt{2})$, $i = 1, \ldots, m$. The PB confidence interval for λ can be obtained by solving the inequality

$$t_{\alpha/2} \leq \frac{\sqrt{n-1}\,(\hat{\lambda} - \lambda)}{\hat{\lambda}\sqrt{2}} \leq t_{1-\alpha/2}$$

for λ. It should be noted that at present there is no exact method available for constructing confidence intervals for λ. Therefore, to understand the validity of the PB methods, we estimated the coverage probabilities as shown below:

Let Σ_0 be a known positive definite matrix and λ_0 be the largest eigenvalue of Σ_0.

For $k = 1, 1000$

Generate $W_k \sim W_p(n-1, \Sigma_0)$, and set $S_k = W_k/(n-1)$.

For $j = 1, 1000$

Generate $V_j \sim W_p(n-1, S_k)$; compute the largest eigenvalue λ_j of $V_j/(n-1)$.

(end j loop)

Compute a $100(1-\alpha)\%$ PB confidence interval for λ_0 based on the λ_j's; call it I_k.

If I_k contains λ_0, set $Z_k = 1$; else set $Z_k = 0$.

(end k loop)

$\sum_{k=1}^{1000} Z_k/1000$ is the estimated coverage probability of the PB confidence interval.

For an acceptable PB method, the above estimated coverage probability should be approximately equal to $1-\alpha$. We estimate the coverage probabilities of the PB confidence intervals based on the percentile method and the percentile-t method and present them in Table 3. We observe from Table 3 that both methods yield satisfactory results provided the sample size is large and Σ has a unique largest eigenvalue. The percentile method performs poorly, even for large samples, when the population eigenvalues are close to each other. In general, the percentile-t method performs satisfactorily provided the sample size is moderately large.

3.4. Interval estimates for the ratio of two normal means

Let us now consider the problem of estimating the ratio of the means of two normal populations $N(\mu_i, \sigma_i^2)$, $i = 1, 2$. This problem has been considered by many authors; however, for comparison purpose we consider the article by Cox [5] which gives a procedure that uses a variance-mean relationship to estimate μ_1/μ_2 and $(1 + \mu_1)/(1 + \mu_2)$. Cox illustrated his method along with other procedures using a data set. The summary statistics based on the data set are: $n_1 = n_2 = 7$, $\bar{x}_1 = 10.91$, $s_1^2 = 40.13$, $\bar{x}_2 = 3.94$, $s_2^2 = 6.95$. Cox's procedure uses the relationship between $(1 + \bar{x}_1)/(1 + \bar{x}_2)$ and s_1/s_2 to construct confidence intervals for μ_1/μ_2 and $(1 + \mu_1)/(1 + \mu_2)$ whereas the other procedures (see Table 4) do not.

We also computed PB confidence intervals for μ_1/μ_2 and $(1 + \mu_1)/(1 + \mu_2)$ using the percentile method. The PB samples are generated from $N(10.91, 40.13/7)$ and $N(3.94, 6.95/7)$. We did not assume any variance-mean relationship. The PB confidence intervals based on the percentile method, along with others given in Cox [5] are presented in Table 4. We notice from this table that the PB intervals are the shortest among those considered. This calls for an evaluation of the percentile method in the context of this problem. As in Section 3.3, we estimate the coverage probabilities of the PB interval estimates and present them in Table 5. We observe from this table that the percentile PB method performs very well even for moderately small samples.

3.5. Confidence interval for the common mean of several normal populations

Let $x_{j1}, \ldots, x_{jn_j}$ be a sample from $N(\mu, \sigma_j^2)$, $j = 1, \ldots, k$. Let $\bar{x}_j$ and s_j^2 denote respectively the mean and variance based on the jth sample. Fairweather [11] obtained an interval estimate for the common mean μ by inverting the sum of independent t-test statistics. Jordan and Krishnamoorthy [16] proposed a confidence interval for μ that can be obtained by inverting a linear combination of independent F statistics. The confidence intervals given in these papers are exact for the case $k = 2$. For the case $k \geq 3$, both papers provide approximate methods to compute the critical points needed for the confidence intervals.

We now illustrate the PB method for obtaining a confidence interval for Example 2 considered in Jordan and Krishnamoorthy [16]. This example is concerned with the estimation of selenium in non-fat milk powder by combining the results of four different analytical methods. The summary statistics are given in Table 6a.

A PB interval estimate for the common mean can be constructed as follows:

For $j = 1, 100000$:

1. Generate $y_{ij} \sim N(\bar{x}_i, s_i^2/n_i)$, $i = 1, \ldots, k$.

2. Generate $v_{ij} \sim \frac{n_i - 1}{s_i^2} \chi^2_{n_i - 1}$, $i = 1, \ldots, k$.

3. Set $\hat{\mu}_j = \frac{\sum_{i=1}^k n_i y_{ij}/v_{ij}}{\sum_{i=1}^k n_i/v_{ij}}$.

(end j loop)

4. The $100(\alpha/2)$th and $100(1 - \alpha/2)$th percentiles of $\hat{\mu}_j$'s form a $100(1 - \alpha)\%$ confidence interval for the common mean μ.

The PB interval estimate for the mean selenium in non-fat milk powder is given in Table 6b, along with other interval estimates. Because the PB estimates are narrower than the intervals of Fairweather [11] and Jordan and Krishnamoorthy [16], a study of the coverage probability of the PB intervals is necessary. The numerical evaluation method of Section 2 was used to estimate coverage probabilities for different parameter values, and the results are presented in Table 6c. The results reveal that the PB method works very well in the common mean problem.

4. Concluding Remarks

The examples considered in Sections 3.1, 3.2, 3.4, and 3.5 show that the performance of the percentile PB method is very good, and the resulting confidence intervals are very comparable to other intervals obtained either by exact or approximate methods. Even for the largest eigenvalue estimation problem (Section 3.3), the performance of the percentile method is not completely unsatisfactory. Given the results of these examples, we believe that the PB methods will be useful in many statistical problems, especially those for which exact methods are difficult to use or are not available. For other inferential methods such as hypothesis testing, the PB method can be applied similarly.

In general, if there is a pivot statistic for the parameter of interest, then the pivot can be used to obtain inferential procedures. If the pivot involves unknown parameters, then these parameters can be replaced by their sample estimates. For example, in multivariate calibration the pivot statistic used to find interval estimates of the unknown explanatory variables involves other unknown parameters (see Mathew and Zha [18] and Benton, Krishnamoorthy and Mathew [2]). By replacing these unknown parameters with their estimates and using the PB method, one can obtain interval estimates of the explanatory variables. Indeed, Benton et al. [2] demonstrated that such approach yields remarkable results in a univariate-multivariate calibration problem.

When the PB method is under consideration for a particular problem, its validity can be verified as presented in Section 2 and as shown in Sections 3.3-3.5. For the problem of interest, one can expect results similar to those obtained in the verification process using an arbitrary value of the parameter. For those cases to which it is applied, the PB method is a relatively easy way to obtain good approximate statistical procedures.

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Table 1. 95% Confidence intervals for Poisson mean θ

Sample size	Observed k	$\hat{\theta}$	Exact C. I.	PB C. I.
5	10	2	(0.96, 3.7)	(0.8, 3.4)
5	30	6	(4, 8.6)	(4, 8.2)
10	56	5.6	(4.2, 7.3)	(4.2, 7.1)
10	30	3	(2, 4.3)	(2, 4.1)
15	48	3.2	(2.4, 4.2)	(2.3, 4.1)
15	150	10	(8.5, 11.7)	(8.4, 11.6)
20	140	7	(5.9, 8.2)	(5.9, 8.3)
20	60	3	(2.3, 3.9)	(2.3, 3.8)

Table 2. 95% Confidence intervals for the correlation coefficient ρ

Sample size	$\hat{\rho}$	Exact C. I.	PB C. I.
6	0.8	(– 0.0290, 0.9643)	(0.1261, 0.9815)
6	0.4	(– 0.5470, 0.8775)	(– 0.5684, 0.9238)
10	0.4	(– 0.2916, 0.7998)	(– 0.2829, 0.8308)
10	0.6	(– 0.0489, 0.8759)	(– 0.0031, 0.8999)
15	0.8	(0.4730, 0.9244)	(0.5158, 0.9346)
15	0.3	(– 0.2437, 0.6883)	(– 0.2407, 0.7104)
20	0.7	(0.3635, 0.8652)	(0.3917, 0.8767)
20	0.5	(0.0704, 0.7617)	(0.0879, 0.7773)
30	0.4	(0.0448, 0.6574)	(0.0537, 0.6685)
30	0.8	(0.6111, 0.8971)	(0.6280, 0.9031)
50	0.4	(0.1352, 0.6063)	(0.1411, 0.6128)
50	0.8	(0.6673, 0.8762)	(0.6763, 0.8836)
75	0.6	(0.4294, 0.7254)	(0.4352, 0.7297)
100	0.6	(0.4566, 0.7106)	(0.4566, 0.7138)

Table 3. Estimated coverage probabilities and expected lengths of PB confidence intervals

$\Sigma_{3\times3} = (\sigma_{11}, \sigma_{12}, \sigma_{22}, \sigma_{13}, \sigma_{23}, \sigma_{33}) = (4, -2, 17, 6, 1, 35)$, eigenvalues = (36.14, 17.30, 2.56)

	n	10	20	30	40	50	100	500
Pivot	Coverage Prob	0.93	0.93	0.93	0.93	0.94	0.95	0.95
	Aver Lengths	84.3	49.9	38.9	32.7	29.3	20.5	9.0
Percentile	Coverage Prob	0.97	0.96	0.96	0.95	0.95	0.95	0.95
	Aver Lengths	71.4	46.1	37.3	32.0	28.5	20.1	9.0
$\Sigma_{3\times3} = (9, -3, 10, -3, -2, 11)$, eigenvalues = (13.18, 12.34, 4.48)								
Pivot	Coverage Prob	0.93	0.93	0.93	0.93	0.94	0.94	0.95
	Aver Lengths	33.1	18.5	14.3	11.8	10.4	7.1	3.0
Percentile	Coverage Prob	0.96	0.91	0.90	0.88	0.88	0.88	0.92
	Aver Lengths	32.7	20.0	15.6	13.0	11.3	7.6	3.1
$\Sigma_{3\times3} = (16, -48, 169, -60, 260, 1106)$, eigenvalues = (1177.6, 111.6, 1.7)								
Pivot	Coverage Prob	0.94	0.95	0.95	0.95	0.94	0.95	0.95
	Aver Lengths	3237	1818	1365	1171	1003	683	293
Percentile	Coverage Prob	0.90	0.92	0.94	0.94	0.93	0.95	0.95
	Aver Lengths	2135	1486	1196	1061	927	656	291
$\Sigma_{3\times3} = (4, -6, 10, -4, 7, 21)$, eigenvalues = (26.2, 8.51, 0.29)								
Pivot	Coverage Prob	0.91	0.92	0.94	0.93	0.95	0.95	0.95
	Aver Lengths	66.6	38.1	29.4	25.2	22.1	15.0	6.5
Percentile	Coverage Prob	0.94	0.95	0.95	0.95	0.95	0.95	0.95
	Aver Lengths	50.4	33.4	26.9	23.5	20.8	14.6	6.5
$\Sigma_{3\times3} = (10/3, 0, 4, 0.942809, 0, 8/3)$, eigenvalues = (4, 4, 2)								
Pivot	Coverage Prob	0.90	0.90	0.90	0.90	0.90	0.91	0.91
	Aver Lengths	10.3	5.6	4.3	3.7	3.2	2.2	0.93
Percentile	Coverage Prob	0.93	0.86	0.85	0.82	0.80	0.81	0.82
	Aver Lengths	10.6	6.3	4.8	4.1	3.5	2.4	0.97

$\Sigma_{3\times 3}$ = (8/3, − 0.816497, 3, 0.471405, − 0.577350, 7/3), eigenvalues = (4, 4, 2)								
Pivot	Coverage Prob	0.93	0.93	0.91	0.91	0.94	0.94	0.95
	Aver Lengths	8.7	5.1	4.1	3.4	3.1	2.2	1
Percentile	Coverage Prob	0.98	0.97	0.96	0.98	0.97	0.96	0.95
	Aver Lengths	8.5	5.4	4.2	3.6	3.2	2.2	1

Table 4. 95% Confidence intervals for $(c + \mu_1)/(c + \mu_2)$

Procedure	c	Confidence interval	Interval width
Cox [5]	0	(1.27, 6.68)	5.41
Elston [10]	0	(1.13, 7.85)	6.72
Log Transformation	0	(1.16, 8.90)	7.74
PB	0	(1.39, 5.96)	4.57
Cox [5]	1	(1.23, 4.75)	3.52
Log Transformation	1	(1.21, 5.10)	3.89
PB	1	(1.32, 4.39)	3.07

Table 5. Estimated coverage probabilities and expected lengths of 95% confidence intervals of the ratio of two normal means

$\mu_1 = 1.0,\ \sigma_1^2 = 1.0$

$n_1 = 10,\ n_2 = 10$	$\sigma_2^2 = 1.0$		$\sigma_2^2 = 5.0$		$\sigma_2^2 = 10.0$	
μ_2	Cov.	Av. Len.	Cov.	Av. Len.	Cov.	Av. Len.
1.0	.94	6.86	.96	19.75	.96	18.15
2.0	.93	0.78	.96	4.66	.96	8.38
3.0	.93	0.45	.94	0.93	.95	2.66
5.0	.92	0.25	.94	0.29	.93	0.40
10.0	.92	0.21	.92	0.13	.93	0.13

$n_1 = 10$, $n_2 = 25$						
1.0	.93	1.94	.96	15.15	.97	19.33
2.0	.92	0.66	.94	1.21	.96	3.44
3.0	.92	0.42	.93	0.50	.94	0.70
5.0	.92	0.24	.93	0.26	.93	0.28
10.0	.92	0.12	.93	0.12	.91	0.13
$n_1 = 25$, $n_2 = 10$						
1.0	.95	6.00	.95	18.63	.95	17.27
2.0	.94	0.58	.95	4.54	.96	8.37
3.0	.94	0.31	.94	0.78	.96	2.45
5.0	.93	0.17	.94	0.21	.94	0.31
10.0	.92	0.12	.94	0.08	.95	0.09
$n_1 = 25$, $n_2 = 25$						
1.0	.95	1.52	.95	1.50	.97	18.70
2.0	.94	0.45	.95	1.04	.96	3.05
3.0	.95	0.28	.95	0.28	.95	0.58
5.0	.94	0.16	.94	0.16	.94	0.20
10.0	.93	0.08	.94	0.08	.94	0.08
$n_1 = 35$, $n_2 = 35$						
1.0	.96	1.13	.96	10.29	.97	17.69
2.0	.94	0.38	.95	0.67	.96	1.80
3.0	.95	0.24	.95	0.30	.95	0.41
5.0	.94	0.13	.95	0.15	.95	0.17
10.0	.94	0.07	.94	0.07	.95	0.07

Table 6a. Estimation of selenium in non-fat milk powder

Methods	n_i	$\bar{x}_i$	s_i^2
Atomic absorption spectrometry	8	105.0	85.711
Neutron activation instrumental	12	109.75	20.748
Radiochemical	14	109.5	2.729
Isotope dilution mass spectrometry	8	113.25	33.640

Table 6b. 95% Confidence intervals for the common mean

Procedure	Confidence interval	Interval width
Fairweather [11]	(108.59, 110.81)	2.22
Jordan and Krishnamoorthy [16]	(108.52, 110.68)	2.16
PB	(108.75, 110.53)	1.78

Table 6c. Estimated coverage probabilities and expected lengths of 95% confidence intervals of the common mean problem

$n_1 = 10$	$n_2 = 10$	$\mu = 1.0$		$\mu = 5.0$		$\mu = 10.0$		$\mu = 20.0$	
σ_1^2	σ_2^2	Cov	Av. Len.	Cov	Av. Len.	Cov	Av. Len.	Cov	Av. Len.
5.0	5.0	.94	2.00	.93	2.01	.93	2.00	.93	2.00
	20.0	.94	2.56	.94	2.55	.92	2.52	.93	2.55
	40.0	.92	2.64	.93	2.69	.93	2.68	.92	2.66
20.0	20.0	.93	4.04	.94	4.01	.94	4.00	.92	4.01
	40.0	.93	4.65	.95	4.65	.93	4.65	.93	4.66
40.0	40.0	.94	5.70	.93	5.63	.94	5.70	.93	5.66
$n_1 = 25$	$n_2 = 25$								
5.0	5.0	.95	1.25	.95	1.24	.95	1.25	.94	1.26
	20.0	.95	1.58	.94	1.58	.94	1.58	.95	1.58
	40.0	.94	1.66	.95	1.65	.95	1.67	.94	1.66

20.0	20.0	.96	2.50	.94	2.51	.95	2.51	.94	2.52
	40.0	.94	2.87	.94	2.89	.94	2.89	.95	2.90
40.0	40.0	.95	3.53	.94	3.53	.94	3.55	.93	3.55
$n_1 = 35$	$n_2 = 35$								
5.0	5.0	.95	1.06	.93	1.05	.95	1.05	.94	1.06
	20.0	.94	1.32	.95	1.34	.94	1.33	.95	1.33
	40.0	.96	1.41	.95	1.40	.96	1.40	.94	1.40
20.0	20.0	.95	2.11	.94	2.11	.95	2.12	.95	2.12
	40.0	.94	2.43	.94	2.45	.95	2.44	.94	2.43
40.0	40.0	.94	2.43	.94	2.45	.95	2.44	.94	2.43
$n_1 = 10$	$n_2 = 25$								
5.0	5.0	.94	1.53	.95	1.53	.93	1.51	.94	1.53
	20.0	.93	2.19	.94	2.20	.93	2.22	.94	2.20
	40.0	.94	2.41	.94	2.43	.91	2.39	.93	2.42
20.0	20.0	.94	3.04	.94	3.06	.95	3.03	.93	3.03
	40.0	.94	3.75	.94	3.74	.94	3.75	.94	3.76
40.0	40.0	.94	4.32	.93	4.33	.94	4.30	.94	4.33
$n_1 = 25$	$n_2 = 10$								
5.0	20.0	.94	1.72	.94	1.73	.94	1.71	.93	1.71
	40.0	.93	1.74	.94	1.74	.94	1.74	.94	1.73
20.0	40.0	.95	3.32	.94	3.32	.94	3.31	.95	3.29

■