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Combining Independent Information in a Multivariate Calibration Problem

K. Krishnamoorthy

University of Southwestern Louisiana

and

Darren J. Johnson

United States National Biological Service

The problem of combining independent information from different sources in a multivariate calibration setup is considered. The dimensions of the response vectors from various sources may be unequal. A linear combination of the classical estimators based on the individual sources is proposed as an estimator for the unknown explanatory variable. It is shown that the combined estimator has finite mean provided the sum of the dimensions of the response vectors exceeds one and has finite mean squared error if it exceeds two. Expressions for asymptotic bias and mean squared error are given. © 1997 Academic Press

1. INTRODUCTION

Multivariate calibration involves two sets of variables, namely, response set and explanatory set. In practical situations, the explanatory variables represent the true characteristics of interest determined by a difficult and expensive method and the response variables represent the measurements related to the same characteristics obtained by easy and inexpensive methods. Usually, the response set is regarded as random and the explanatory variables are considered as fixed. The problem of interest is to establish a relationship between the two sets of variables based on the experimental data in order to make inferences about the unknown explanatory variables of a future observation based on its known values of the response variables. Applications and practical examples of multivariate calibration can be

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found, for example, in Brown [1, 2], Heise and Marbach [5], Lieftinck-Koeijers [9], and Oman and Wax [12].

This article is concerned with the situation where the experimental data are available from k different sources for the same objective. As pointed out by Johnson and Krishnamoorthy [7], such situations arise when a calibration experiment is conducted by different laboratories, using different methods or different measuring instruments. They also give a practical example in the univariate case. Stolaski [13] points out that measuring ozone with a variety of instruments promises to reduce calibration errors. In short, we have calibration data (X_j, Y_{ij}), $i=1, ..., k$ and $j=1, ..., n$, where X_j represents true values of the jth unit and Y_{ij} represents the measurements related to the X_j obtained by the ith measuring instrument or method. It is assumed that Y_{ij}'s are linearly related to the X_j's.

Thus, the calibration models and the prediction models assumed in this paper are, respectively,

$$Y_{ij} = \alpha_i + \beta_i X_j + \varepsilon_{ij}, \qquad i=1, ..., k, \quad j=1, ..., n, \tag{1.1}$$

and

$$Y_{oi} = \alpha_i + \beta_i X_o + \varepsilon_{oi}, \qquad i=1, 2, ..., k, \tag{1.2}$$

where

Y_{ij}: $p_i \times 1$ vector of responses,

X_j: 1×1 fixed explanatory variable,

α_i: $p_i \times 1$ intercept parameter,

β_i: $p_i \times 1$ vector of regression parameters,

ε_{ij}: $p_i \times 1$ error vector, and

Y_{oi}: $p_i \times 1$ is observed response for the unknown X_o.

It is assumed that all the errors are independent with

$$\varepsilon_{ij} \sim N_{p_i}(0, \Sigma_i) \qquad \text{and} \qquad \varepsilon_{oi} \sim N_{p_i}(0, \Sigma_i). \tag{1.3}$$

The problem is to estimate X_o based on the calibration models (1.1) and the Y_{oi}'s from the prediction models (1.2).

Define

$$\begin{pmatrix} \bar{X} \\ \bar{Y}_i \end{pmatrix} = \frac{1}{n} \sum_{j=1}^{n} \begin{pmatrix} X_j \\ Y_{ij} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_{XX_i} & S_{XY_i} \\ S_{Y_iX} & S_{Y_iY_i} \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} X_j - \bar{X} \\ Y_{ij} - \bar{Y}_i \end{pmatrix} \begin{pmatrix} X_j - \bar{X} \\ Y_{ij} - \bar{Y}_i \end{pmatrix}'. \tag{1.4}$$

The least-squares estimators of α_i and β_i based on the calibration samples are given by

$$\hat{\alpha}_i = \bar{Y}_i - \hat{\beta}_i \bar{X}_i \quad \text{and} \quad \hat{\beta}_i = S_{Y_i X} S_{XX}^{-1}. \tag{1.5}$$

Let

$$S_i = (n-2)^{-1} A_i = (n-2)^{-1} \sum_{j=1}^{n} (Y_{ij} - \hat{\alpha}_i - \hat{\beta}_i X_{ij})(Y_{ij} - \hat{\alpha}_i - \hat{\beta}_i X_{ij})'. \tag{1.6}$$

We note that $Y_{o1}, ..., Y_{ok}, \bar{Y}_1, ..., \bar{Y}_k, \hat{\beta}_1, ..., \hat{\beta}_k, A_1, ..., A_k$ are all statistically independent with

$$\begin{aligned} Y_{oi} &\sim N_{p_i}(\alpha_i + \beta_i' X_o, \Sigma_i), & \bar{Y}_i &\sim N_{p_i}(\alpha_i + \beta_i' \bar{X}, \Sigma_i / n), \\ \Sigma_i^{-1/2} \hat{\beta}_i S_{xx}^{1/2} &\sim N_{p_i}(\Sigma_i^{-1/2} \beta_i S_{xx}^{1/2}, I_{p_i}) & \text{and} \quad A_i &\sim W_{p_i}(n-2, \Sigma_i), \end{aligned} \tag{1.7}$$

where $W_p(m, \Sigma)$ denotes the Wishart distribution with m degrees of freedom and parameter matrix Σ.

When $k = p = 1$, it is well known that the classical estimator

$$\hat{X}_{o1} = \bar{X} + (Y_{o1} - \bar{Y}_1)/\hat{\beta}_1 \tag{1.8}$$

has infinite absolute moments. However, for $k = 1$ and assuming that the error covariance matrix is known, Lieftinck-Koeijers [9] showed that a natural generalization of the classical estimator (1.8) has finite mean if $p_1 \geqslant 3$ and finite MSE if $p_1 \geqslant 5$. Her results were strengthened and extended to the unknown covariance matrix case by Nishii and Krishnaiah [11]. For estimating a vector X_o of dimension q, they showed that the classical estimate has finite mean provided $p_1 \geqslant q + 1$ and finite MSE provided $p_1 \geqslant q + 2$, and also give exact expressions for the bias and MSE when $q = 1$. Brown and Spiegelman [3] considered the case of known diagonal error covariance matrix and gave insights as to why Nishii and Krishnaiah's conditions are sharper than those of Lieftinck-Koeijers. Fujikoshi and Nishii [4] give asymptotic expressions for the first two moments of the classical estimator when $p_1 \geqslant q \geqslant 1$. For confidence estimation of X_o, we refer the readers to Brown [1] and Mathew and Subramanyam [10].

In this article, we consider the models (1.1) and (1.2). The classical estimator of X_o based on the ith model alone is given by

$$\hat{X}_{oi} = \bar{X} + \frac{\hat{\beta}_i' S_i^{-1} (Y_{oi} - \bar{Y}_i)}{\hat{\beta}_i' S_i^{-1} \hat{\beta}_i}. \tag{1.9}$$

An estimator based on all calibration data and the prediction data is given by

$$\hat{X}_c = \bar{X} + \frac{\Sigma_{i=1}^{k} \hat{\beta}_i' S_i^{-1} (Y_{oi} - \bar{Y}_i)}{\sum_{j=1}^{k} \hat{\beta}_j' S_j^{-1} \hat{\beta}_j}. \tag{1.10}$$

This is a natural generalization of the univariate estimator proposed by Johnson and Krishnamoorthy [7] and can be regarded least-squares estimator for it minimizes $\sum_{i=1}^{k} (Y_{oi}-\hat{\alpha}_i-\hat{\beta}_i X_o)' S_i^{-1}(Y_{oi}-\hat{\alpha}_i-\hat{\beta}_i X_o)$ with respect to X_o.

In the following section, we give some preliminary lemmas which are needed to show that the $\hat{X}_c$ has finite mean and finite MSE. In Section 3, we show that the $\hat{X}_c$ has finite mean if $p=\sum_{i=1}^{k} p_i \geqslant 2$ and finite MSE if $p \geqslant 3$. We also give asymptotic expressions for the bias and MSE. In passing, we clarify some discrepancies between the results of Lieftinck-Koeijers [9] and those of Nishii and Krishnaiah [11]. Finally, in Section 4 we point out some practical applications of the results of the paper and some future research in this area.

2. SOME PRELIMINARY RESULTS

The following lemmas are needed to find expressions for the bias and MSE of the combined estimate $\hat{X}_c$ in (1.10). Lemma 2.1 can be easily verified.

LEMMA 2.1. *Let v be a chi-squared random variable with m degrees of freedom, $v \sim \chi^2_m$. Then, for any real k,*

$$E[v^k f(v)] = 2^k \Gamma(m/2+k)\, E[f(v_o)]/\Gamma(m/2),$$

where $v_o \sim \chi^2_{m+2k}$, provided the indicated expectations exist.

LEMMA 2.2 (Hwang [6]). *Let Z be a Poisson random variable with mean λ and $|g(-1)| < \infty$. Then,*

$$E[Zg(Z-1)] = \lambda E[g(Z)],$$

provided the indicated expectations exist.

LEMMA 2.3. *Let $\eta_1, ..., \eta_k$ be nonnegative numbers such that $\sum_{i=1}^{k} \eta_i = 1$ and $v_1, ..., v_k$ are independent random variables with $v_i \sim \chi^2_{m_i}$, $i=1, ..., k$. Let $m_{(i)}$ denote the ith smallest of the m_j's. Further, let $r \geqslant s \geqslant 0$, $r+s \geqslant 1$, $m_{(1)} \geqslant 2(r+1)$, and $t_{12} = 2^{-r-s}\Gamma(m_1/2-r)\,\Gamma(m_2/2-s)/[\Gamma(m_1/2)\,\Gamma(m_2/2)]^2$. Then,*

$$t_{12}(m_{(1)}-2r-2)^{r+s} < E\left[\frac{v_1^{-r} v_2^{-s}}{(\sum_{i=1}^{k} \eta_i v_i^{-1})^{r+s}}\right] < t_{12}\, 2^{r+s}\Gamma((m_{(k)}/2+r+s))/\Gamma(m_{(k)}/2).$$

Proof. Using Lemma 2.1, we get

$$E\left[\frac{v_1^{-r}v_2^{-s}}{(\sum_{i=1}^k \eta_i v_i^{-1})^{r+s}}\right] = t_{12} E\left(\sum_{i\neq 1,2} \eta_i v_i^{-1} + \eta_1 v_{o1}^{-1} + \eta_2 v_{o2}^{-1}\right)^{-r-s} \tag{2.1}$$

where $v_{o1} \sim \chi^2_{m_1-2r}$ and $v_{o2} \sim \chi^2_{m_2-2s}$. Since $r+s \geqslant 1$, it follows from Jensen's inequality that

$$\left(\sum_{i\neq 1,2} \eta_i v_i^{-1} + \eta_1 v_{o1}^{-1} + \eta_2 v_{o2}^{-1}\right)^{-r-s} < \left(\sum_{i\neq 1,2} \eta_i v_i + \eta_1 v_{o1} + \eta_2 v_{o2}\right)^{r+s}$$
$$\leqslant \sum_{i\neq 1,2} \eta_i v_i^{r+s} + \eta_1 v_{o1}^{r+s} + \eta_2 v_{o2}^{r+s}. \tag{2.2}$$

Substituting (2.2) in (2.1) and then taking expectation we get the desired upper bound. To get the lower bound, write $V = (\sum_{i\neq 1,2} \eta_i v_i^{-1} + \eta_1 v_{o1}^{-1} + \eta_2 v_{o2}^{-1})$. Since V^{-r-s} is a convex function of V, using Jensen's inequality, we get

$$E(V^{-r-s}) > [E(V)]^{-r-s}$$
$$= \left(\sum_{i\neq 1,2} \eta_i/(m_i-2) + \eta_1/(m_1-2r-2) + \eta_2/(m_2-2s-2)\right)^{-r-s}$$
$$\geqslant (m_{(1)}-2r-2)^{r+s}. \tag{2.3}$$

Substituting (2.3) in (2.1), we get the desired lower bound.

LEMMA 2.4. *Let $U_1, ..., U_k$ be independent normal random vectors with $U_i \sim N_{p_i}(\mu_i, I_{p_i})$, where I_{p_i} denotes the identity matrix of order p_i, $i=1, ..., k$. Let $p = \sum_{i=1}^k p_i$. Then,*

(i) $$E\left(\sum_{i=1}^k \|U_i\|^2\right)^{-1} = E(p+2Z-2)^{-1} \quad \textit{if} \quad p \geqslant 3,$$

(ii) $$E\left[\frac{\sum_{i=1}^k U_i'\mu_i}{\sum_{j=1}^k \|U_j\|^2}\right] = 2\lambda E(p+2Z)^{-1} \quad \textit{if} \quad p \geqslant 2,$$

(iii) $$E\left[\frac{\sum_{i=1}^k U_i'\mu_i}{\sum_{j=1}^k \|U_j\|^2}\right]^2 = 2\lambda E\left[\frac{2Z+1}{(p+2Z)(p+2Z-2)}\right] \quad \textit{if} \quad p \geqslant 3,$$

(iv) $$E\left[\frac{\sum_{i=1}^k \|U_i\|^2 \lambda_i}{(\sum_{i=1}^k \|U_j\|^2)^2}\right] = E\left[\frac{\sum_{i=1}^k p_i\lambda_i}{(p+2Z)(p+2Z-2)}\right]$$
$$+ 2E\left[\frac{\sum_{i=1}^k \lambda_i^2}{(p+2Z)(p+2Z+2)}\right] \quad \textit{if} \quad p \geqslant 3,$$

where Z is a Poisson random variable with mean $\lambda = \sum_{i=1}^{k} \lambda_i$ *and* $\lambda_i = \|\mu_i\|^2/2$, $i = 1, ..., k$.

Proof. (i), (ii), and (iii) are trivial generalization of Lemma 3 of Nishii and Krishnaiah [11].

(iv) Note that $\|U_i\|^2 \sim \chi^2_{p_i+2Z_i}$, where Z_i's are independent random variables with $Z_i \sim \text{Poisson}(\lambda_i)$, $i = 1, ..., k$. So, conditioning on Z_i's, Lemma 2.1 can be applied to get

$$E\left[\frac{\sum_{i=1}^{k} \|U_i\|^2 \lambda_i}{(\sum_{i=1}^{k} \|U_i\|^2)^2}\right] = E\left[\frac{\sum_{i=1}^{k} (p_i + 2Z_i) \lambda_i}{(p+2Z)(p+2Z-2)}\right],$$

where $Z = \sum_{i=1}^{k} Z_i \sim \text{Poisson}(\lambda)$. Now, applying Lemma 2.2, we prove (iv).

Remark 2.1. It is interesting to observe that Lemma 1 of Nishii and Krishnaiah [11] (which is crucial to show that the classical estimate in the multivariate case has finite absolute moments) can also be proved by writing an absolute standard normal random variable as a positive square root of a χ_1^2 random variable and then applyng Lemma 2.1 in a straightforward manner.

The following lemma is due to Nishii and Krishnaiah [11]. However, we prove it here using a slightly different approach. Further, the arguments used to prove the lemma facilitate the proof of Theorem 3.2.

LEMMA 2.5. *Let* $U_1 \sim N_{p_1}(\mu_1, I_{p_1})$ *independently of* $V_1 \sim W_{p_1}(m_1, I_{p_1})$ *where* I_{p_1} *denotes the identity matrix of order* p_1 *and* $W_{p_1}(m_1, I_{p_1})$ *denotes the Wishart distribution with degrees of freedom* m_1 *and* $E(V_1) = m_1 I_{p_1}$. *Then,*

(i) $$E\left[\frac{U_1' V_1^{-1} \mu_1}{U_1' V_1^{-1} U_1}\right] = 2\lambda_1 E(p_1 + 2Z_1)^{-1} \qquad \text{if} \quad p_1 \geq 2,$$

(ii) $$E\left[\frac{U_1' V_1^{-2} U_1}{(U_1' V_1^{-1} U_1)^2}\right] = (m_1 - 1)\, E(p_1 + 2Z_1 - 2)^{-1}/(m_1 - p_1) \qquad \text{if} \quad p_1 \geq 3,$$

(iii) $$E\left[\frac{U_1' V_1^{-1} \mu_1}{U_1' V_1^{-1} U_1}\right]^2 = 2\lambda_1 E\left[\frac{2Z_1 + 1}{(p_1 + 2Z_1)(p_1 + 2Z_1 - 2)} + \frac{1}{(p_1 + 2Z_1 - 2)(m_1 - p_1)}\right] \qquad \text{if} \quad p_1 \geq 3.$$

Proof. (i) Let Γ be an orthogonal matrix with first row $(u_{11}/\|U_1\|, ..., u_{1p_1}/\|U_1\|)$ so that $\Gamma U_1 = U_o = (\|U_1\|, 0, ..., 0)'$. Then,

$$E\left[\frac{U_1' V_1^{-1}\mu_1}{U_1' V_1^{-1} U_1}\right] = E\left[\frac{U_1'\Gamma'\Gamma V_1^{-1}\Gamma'\Gamma\mu_1}{U_1'\Gamma'\Gamma V_1^{-1}\Gamma'\Gamma U_1}\right]$$
$$= E\left[\frac{U_o' V_1^{-1}(U_1'\mu_1/\|U_1\|, \mu_1'\Gamma_2')'}{U_o' V_1^{-1} U_o}\right], \tag{2.4}$$

where Γ_2 denotes the matrix formed by the last $(p_1 - 1)$ rows of Γ. In (2.4), we replaced $\Gamma' V_1 \Gamma$ by V_1 since they are identically distributed. Let $V_1 = TT'$, where T is the upper triangular matrix with positive diagonal elements (Cholesky decomposition). It is well known that t_{ij}'s are independent with $t_{ii}^2 \sim \chi^2_{m_1 - p_1 + i}$ for $i = 1, ..., p_1$ and $t_{ij} \sim N(0, 1)$ for $i < j$. Partition T as

$$T = \begin{pmatrix} t_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{so that} \quad T^{-1} = \begin{pmatrix} t_{11}^{-1} & -t_{11}^{-1} T_{12} T_{22}^{-1} \\ 0 & T_{22}^{-1} \end{pmatrix}$$
$$\text{and} \quad V_1^{-1} = \begin{pmatrix} t_{11}^{-2} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \tag{2.5}$$

where $G_{12} = -t_{11}^{-2} T_{12} T_{22}^{-1}$. Thus, in terms of the elements of T^{-1}, (2.4) can be expressed as

$$E\left[\frac{U_1' V_1^{-1}\mu_1}{U_1' V_1^{-1} U_1}\right] = E\left[\frac{t_{11}^{-2} U_1'\mu_1 + \|U_1\|\, G_{12}\Gamma_2\mu_1}{t_{11}^{-2}\, \|U_1\|^2}\right]$$
$$= E\left[\frac{U_1'\mu_1}{\|U_1\|^2}\right]. \tag{2.6}$$

The rhs of (2.6) follows from the fact that U_1 and T are independent and $E(G_{12}) = -E(t_{11}^{-2} T_{12} T_{22}^{-1}) = 0$. Now, applying Lemma 2.4 (ii) to (2.6), we prove (i).

(ii) Using orthogonal transformaton and Cholesky decomposition as in the proof of (i), it can be shown that

$$E\left[\frac{U_1' V_1^{-2} U_1}{(U_1' V_1^{-1} U_1)^2}\right] = E\left[\frac{1 + T_{12} T_{22}^{-1} T_{22}'^{-1} T_{12}'}{\|U_1\|^2}\right]. \tag{2.7}$$

Note that $T_{22} T_{22}' \sim W_{p_1 - 1}(m_1, I_{p_1 - 1})$ and hence $E[(T_{22} T_{22}')^{-1}] = (m_1 - p_1)^{-1} I_{p_1 - 1}$. Using this result and noting that $E(T_{12}' T_{12}) = I_{p_1 - 1}$, we get $E[T_{12} T_{22}^{-1} T_{22}'^{-1} T_{12}'] = E[\text{tr}(T_{12} T_{22}^{-1} T_{22}'^{-1} T_{12}')] = (m_1 - p_1)^{-1} I_{p_1 - 1}$. Substituting this expression in (2.7) and then using Lemma 2.4(i), we prove (ii).

(iii) It follows from (2.6) that

$$E\left[\frac{U_1' V_1^{-1}\mu_1}{U_1' V_1^{-1} U_1}\right]^2 = E\left[\frac{t_{11}^{-2} U_1'\mu_1 + \|U_1\| G_{12}\Gamma_2\mu_1}{t_{11}^{-2}\|U_1\|^2}\right]^2$$

$$= E\left[\frac{U_1'\mu_1}{\|U_1\|^2}\right]^2 + E\left[\frac{T_{12}T_{22}^{-1}\Gamma_2\mu_1\mu_1'\Gamma_2' T_{22}'^{-1}T_{12}'}{\|U_1\|^2}\right]$$

$$= E\left[\frac{U_1'\mu_1}{\|U_1\|^2}\right]^2 + E\left[\frac{\mu_1'\mu_1}{(m_1-p_1)\|U_1\|^2}\right]. \tag{2.8}$$

Now, using Lemma 2.4 (i) and (iii) in (2.8), we prove (iii).

LEMMA 2.6. *Let $U_1, ..., U_k$ be independent random vectors with $U_i \sim N_{p_i}(\mu_i, I_{p_i})$, and $v_1, ..., v_k$ be independent random variables with $v_i \sim \chi^2_{m_i}$, $i = 1, ..., k$. Let $p = \sum_{i=1}^k p_i$. For large m_i's,*

(i) $$E\left[\frac{\sum_{i=1}^k U_i'\mu_i v_i^{-1}}{\sum_{j=1}^k \|U_j\|^2 v_j^{-1}}\right] \doteq 2\lambda E(p+2Z)^{-1} \qquad \textit{for} \quad p \geqslant 2,$$

(ii) $$E\left[\frac{\sum_{i=1}^k \sum_{j\neq i}(U_i'\mu_i)(U_j'\mu_j)\, v_i^{-1} v_j^{-1}}{(\sum_{j=1}^k \|U_j\|^2 v_i^{-1})^2}\right]$$

$$\doteq E\left[\frac{\sum_{i=1}^k \sum_{j\neq i}(U_i'\mu_i)(U_j'\mu_j)}{(\sum_{j=1}^k \|U_j\|^2)^2}\right],$$

(iii) $$E\left[\frac{\sum_{i=1}^k \lambda_i \|U_i\|^2 v_i^{-2}/m_i}{(\sum_{j=1}^k \|U_j\|^2 v_j^{-2})^2}\right] \doteq 0 \qquad \textit{for} \quad p \geqslant 3,$$

where Z is a Poisson random variable with mean $\lambda = \sum_{i=1}^k \lambda_i$ and $\lambda_i = \|\mu_i\|^2/2$, $i = 1, ..., k$.

Proof. (i) Let $U_i = (u_{i1}, ..., u_{ip_i})'$ and $\mu_i = (\mu_{i1}, ..., \mu_{ip_i})'$. The expectation in the lhs of (i) is absolutely convergent provided

$$E\left[\frac{|\mu_{ij}u_{ij}|\, v_i^{-1}}{\sum_{j=1}^k \|U_j\|^2 v_j^{-1}}\right] = E\left[\frac{|\mu_{ij}u_{ij}|\, v_i^{-1}}{\sum_{j=1}^k \|U_j\|^2 \sum_{j=1}^k \eta_j v_j^{-1}}\right] \tag{2.9}$$

is finite, where $\eta_i = \|U_i\|^2/\sum_{j=1}^k \|U_j\|^2$, $i = 1, ..., k$. For notational simplicity, let us consider (2.9) with $i = 1$ and $j = 1$. Conditioning on U_i's and applying Lemma (2.3), we see that (2.9) is less than a constant time $E[|u_{11}\mu_{11}|/\sum_{j=1}^k \|U_j\|^2]$, which is finite provided $p \geqslant 2$ (from Lemma 2.4(ii)). Next, to get the rhs of (i), we note that the density of u_{11} can be expressed as $\exp(-\mu_{11}^2/2 + \mu_{11}u_{o1})\,\phi(u_{o1})$, where $\phi(\cdot)$ denotes the standard normal density. Using this result and letting $u_{o1} \sim N(0, 1)$ and $\|U_{o1}\|^2 = u_{o1}^2 + \sum_{j=2} u_{1j}^2$, it can be shown that

$$E\left[\frac{\mu_{11}u_{11}v_1^{-1}}{\sum_{j=1}^k \|U_j\|^2 v_j^{-1}}\right]$$

$$= \mu_{11} e^{-\mu_{11}^2/2} \sum_{l=0}^{\infty} \frac{\mu_{11}^l u_{o1}^{l+1} v_1^{-1}}{l!\,(\|U_{o1}\|^2 + \sum_{j=2}^k \|U_j\|^2) \sum_{j=1}^k \eta_{oj} v_j^{-1}}$$

$$= \mu_{11}^2 e^{-\mu_{11}^2/2} E \sum_{k=0}^{\infty} \frac{\mu_{11}^{2k} (u_{o1}^2)^{k+1} v_1^{-1}}{(2k+1)!\,(\|U_{o1}\|^2 + \sum_{j=2}^k \|U_j\|^2) \sum_{j=1}^k \eta_{oj} v_j^{-1}} \quad (2.10)$$

where $\eta_{oj} = \|U_j\|^2/(\|U_{o1}\|^2 + \sum_{j=2}^k \|U_j\|^2)$ for $j \neq 1$ and $\eta_{o1} = 1 - \sum_{j=2}^k \eta_{oj}$. To get the last equation in (2.10), we note that $E[u_{o1}^{l+1} v_1^{-1}/((\|U_{o1}\|^2 + \sum_{j=2}^k \|U_j\|^2) \sum_{j=1}^k \eta_{oj} v_j^{-1}) = 0$ for $l = 0, 2, 4, \ldots$, since the distribution of u_{o1} is symmetric about zero. We can now apply Lemma 2.3 to get bounds for the expectation in (2.10). Indeed, conditioning on all the U's and applying the lemma to (2.10), we get

$$\frac{m_{(1)} - 4}{m_1 - 2} E\left(\frac{\mu_{11}u_{11}}{\sum_{j=1}^k \|U_j\|^2}\right) < E\left[\frac{\mu_{11}u_{11}v_1^{-1}}{\sum_{j=1}^k \|U_j\|^2 v_j^{-1}}\right] < \frac{m_{(k)}}{m_1 - 2} E\left(\frac{\mu_{11}u_{11}}{\sum_{j=1}^k \|U_j\|^2}\right), \quad (2.11)$$

where $p_{(j)}$ denotes the jth smallest of the p_i's. Thus, for large m_i's, $E[\mu_{11}u_{11}v_1^{-1}/(\sum_{j=1}^k \|U_j\|^2 v_j^{-1})] \doteq E(\mu_{11}u_{11}/\sum_{j=1}^k \|U_j\|^2)$. Using this result and Lemma 2.4(ii), we get

$$E\left[\frac{\sum_{i=1}^k U_i' \mu_i v_i^{-1}}{\sum_{j=1}^k \|U_j\|^2 v_j^{-1}}\right] = E\left[\frac{\sum_{i=1}^k U_i' \mu_i}{\sum_{j=1}^k \|U_j\|^2}\right] \doteq 2\lambda E(p + 2Z)^{-1}, \quad (2.12)$$

where Z is a Poisson random variable with mean $\lambda = \sum_{i=1}^k \|\mu_i\|^2/2 = S_{xx} \sum_{i=1}^k \beta_i' \Sigma_i^{-1} \beta_i/2$. Thus, we complete the proof of (i).

(ii) Similar to the proof (i) and hence is omitted.

(iii) As in the proof of (i), by applying Lemma 2.3, we see that for lage m_i's

$$E\left[\frac{\sum_{i=1}^k \lambda_i \|U_i\|^2 v_i^{-2}/m_i}{(\sum_{j=1}^k \|U_j\|^2 v_j^{-2})^2}\right]$$

$$\doteq E\left[\frac{\sum_{i=1}^k \lambda_i \|U_i\|^2/m_i}{(\sum_{j=1}^k \|U_j\|^2)^2}\right]$$

$$= E\left[\frac{\sum_{i=1}^k p_i \lambda_i/m_i}{(p + 2Z)(p + 2Z - 2)}\right] + 2E\left[\frac{\sum_{i=1}^k \lambda_i^2/m_i}{(p + 2Z)(p + 2Z + 2)}\right], \quad (2.13)$$

where Z is a Poisson random variable with mean $\lambda = \sum_{i=1}^{k} \lambda_i$, and $\lambda_i = \|\mu_i\|^2/2$, $i = 1, ..., k$. The last equation in (2.13) follows from Lemma 2.4(iv). It is easy to show that, for $p \geqslant 3$, $E((p+2Z)(p+2Z-2))^{-1} < E((Z+1)(Z+2))^{-1} = \lambda^{-2}(1-(1+\lambda)\exp(-\lambda)) < \lambda^{-2}$. Using this relation along with the inequality that $\sum_{i=1}^{k} \lambda_i^2/\lambda^2 < 1$, it can be easily shown that the second expectation in (2.13) is approximately zero for large m_i's. Similarly, it can be shown that for large m_i's, the first expectation in (2.13) is also approximately zero. Thus, we complete the proof of (iii).

3. BIAS AND MSE OF $\hat{X}_c$

We first give expressions for the bias and MSE of the estimate in (1.10) for the case $k = 1$ in the following theorem.

THEOREM 3.1. *For $k = 1$ and $p_1 \geqslant 2$,*

$$\text{BIAS}(\hat{X}_c) = (\bar{X} - X_o)\, E[1 - 2\lambda_1 E(p_1 + 2Z_1)^{-1}] \tag{3.1}$$

and for $n \geqslant p_1 + 2$ and $p_1 \geqslant 3$,

$$\begin{aligned} \text{MSE}(\hat{X}_c) = {} & c_n S_{xx} \frac{n-3}{n-p_1-2} E\left[\frac{1}{p_1+2Z_1-2}\right] \\ & + (\bar{X} - X_o)^2 E\left[1 - \frac{4\lambda_1}{p_1+2Z_1}\right. \\ & \left. + \frac{2\lambda_1}{p_1+2Z_1-2}\left(\frac{2Z_1+1}{p_1+2Z_1} + \frac{1}{n-p_1-2}\right)\right], \end{aligned} \tag{3.2}$$

where $c_n = 1 + 1/n$ and Z_1 is a Poisson random variable with mean $\lambda_1 = \sqrt{S_{xx}}\beta_1' \Sigma_1^{-1}\beta_1/2$.

Proof. Letting $a_1' = \hat{\beta}_1' S_1^{-1}/(\hat{\beta}_1' S_1^{-1}\hat{\beta}_1)$, we note that

$$(\hat{X}_o - X_o) \mid \hat{\beta}_1, S_1 \sim N(\bar{X} - X_o + a_1'\beta_1(X_o - \bar{X}), c_n a_1' \Sigma^{-1} a_1). \tag{3.3}$$

Therefore,

$$\begin{aligned} E[E(\hat{X}_c - X_o) \mid \hat{\beta}_1', S_1] &= \bar{X} - X_o + E\left[\frac{\hat{\beta}_1' S_1^{-1}(X_o - \bar{X})}{\hat{\beta}_1' S_1^{-1}\hat{\beta}_1}\right] \\ &= (\bar{X} - X_o)\, E\left[1 - \frac{U_1' V_1^{-1}\mu_1}{U_1' V_1^{-1} U_1}\right], \end{aligned} \tag{3.4}$$

where $U_1 = \sqrt{S_{xx}}\Sigma_1^{-1/2}\hat{\beta}_1 \sim N_{p_1}(\mu_1, I_{p_1})$, $\mu_1 = \sqrt{S_{xx}}\Sigma_1^{-1/2}\beta_1$, and $V_1 = (n-2)\,\Sigma_1^{-1/2}S_1\Sigma_1^{-1/2} \sim W_{p_1}(n-2, I_{p_1})$. Now, applying Lemma 2.5(i) to (3.4), we get the desired expression for the bias. To find the MSE of $\hat{X}_c$, note that

$$E[E(\hat{X}_c - X_o)^2 \mid \hat{\beta}_1', S_1]$$

$$= c_n E(a_1'\Sigma_1 a_1) + (\bar{X} - X_o)^2 E\left[\frac{\hat{\beta}_1 S_1^{-1}(\hat{\beta}_1 - \beta_1)}{\hat{\beta}_1' S_1^{-1}\hat{\beta}_1}\right]^2$$

$$= c_n S_{xx} E\left[\frac{U_1' V_1^{-2} U_1}{(U_1' V_1^{-1} U_1)^2}\right] + (\bar{X} - X_o)^2 E\left[1 - \frac{U_1' V_1^{-1}\mu_1}{U_1' V_1^{-1} U_1}\right]^2, \qquad (3.5)$$

where U_1 and V_1 are defined as above. Now, using Lemma 2.5 in (3.5), we get the expresson for the MSE.

Remark 3.1. Nishii and Krishnaiah [11] pointed out that Lieftinck-Koeijers' expression for the bias (with known covariance matrix) is finite even when $p_1 = 2$. On the contrary, we note that her expression for the bias $(\bar{x} - x_o)(p_1 - 2)\,E(p_1 + 2Z - 2)^{-1}$ is finite only when $p_1 \geqslant 3$, whereas the expression (3.1) is finite for $p_1 \geqslant 2$. Applying Lemma 2.2 to (3.1), we see that for $p_1 \geqslant 3$ (otherwise the lemma is not applicable) the expression (3.1) is equal to Lieftinck-Koeijers' expression given above. It is interesting to note that whether the error covariance matrix is known or unknown, the bias of the classical estimate remains the same. Further, Lieftinck-Koeijers's expression (7.2) for the MSE is finite only when $p_1 \geqslant 5$. Again, applying Lemma 2.2 to (3.2), it can be verified that for $p_1 \geqslant 5$ and large n, (3.2) is in agreement with (7.2) of her paper. We also checked that the bias (3.1) is in agreement with Nishii and Krishnaiah's ([11, expression 4.2]) result. However, their (4.3) is indeed $E(\hat{x} - \bar{x})^2$ not the MSE. Further, the expression for $E(b'S^{-1}\gamma_*/b'S^{-1}b)^2$ given in their paper is incorrect. After correcting these errors, we checked that the MSE (3.2) is in agreement with their result.

THEOREM 3.2. *Let* $p = \sum_{i=1}^k p_i$. *For* $k \geqslant 1$ *and large* n,

$$\text{BIAS}(\hat{X}_c) \doteq (\bar{X} - X_o)(1 - 2\lambda E(p + 2Z)^{-1}) \qquad \textit{if} \quad p \geqslant 2, \qquad (3.6)$$

and

$$\text{MSE}(\hat{X}_c) \doteq S_{xx} E(p + 2Z - 2)^{-1} + (\bar{X} - X_o)^2\left[1 + 2\lambda E\left(\frac{2Z+1}{(p+2Z)(p+2Z-2)} - \frac{2}{p+2Z}\right)\right] \qquad \textit{for} \quad p \geqslant 3, \qquad (3.7)$$

where Z *is a Poisson random variable with mean* $\lambda = S_{xx}\sum_{i=1}^k \beta_i' \Sigma_i^{-1}\beta_i/2$.

Proof. It follows from (3.3) that

$$E[E(\hat{X}_c - X_o) \mid \hat{\beta}_i, S_i] = (\bar{X} - X_o) + (X_o - \bar{X})\, E\left(\frac{\sum_{i=1}^{k} \hat{\beta}_i' S_i^{-1} \beta_i}{\sum_{j=1}^{k} \hat{\beta}_j' S_j^{-1} \hat{\beta}_j}\right)$$

$$= (\bar{X} - X_o)\left(1 - E\left[\frac{\sum_{i=1}^{k} U_i' V_i^{-1} \mu_i}{\sum_{j=1}^{k} U_j' V_j^{-1} U_j}\right]\right), \qquad (3.8)$$

where $U_i = \sqrt{S_{xx}}\,\Sigma_i^{-1/2}\hat{\beta}_i \sim N_{p_i}(\mu_i, I_{p_i})$, $\mu_i = \sqrt{S_{xx}}\,\Sigma_i^{-1/2}\beta_i$, and $V_i = (n-2)\,\Sigma_i^{-1/2} V_i \Sigma_i^{-1/2} \sim W_{p_i}(n-2, I_{p_i})$. Note that $U_1, ..., U_k$, $V_1, ..., V_k$ are all statistically independent. As in Lemma 2.4, using orthogonal transformation and Cholesky decomposition, we get

$$E\left[\frac{U_i V_i^{-1} \mu_i}{\sum_{j=1}^{k} U_j' V_j^{-1} U_j}\right] = E\left[\frac{U_i' \mu_i t_{i11}^{-2}}{\sum_{j=1}^{k} \|U_j\|^2 t_{j11}^{-2}}\right], \qquad (3.9)$$

where t_{i11} is the $(1, 1)$ element of T_i, $V_i = T_i T_i'$, and T_i is the upper triangular matrix with positive diagonal elements. Recall that t_{i11}^2's are independent chi-squared random variables with $m_i = n - p_i - 1$ degrees of freedom, $i = 1, ..., k$. So, we can apply Lemma 2.6(i) to get the expression for the bias given in (3.6). To find the MSE of $\hat{X}_c$, we note first that

$$E(\hat{X}_c - X_o)^2 = E[E(\hat{X}_c - X_o)^2 \mid \hat{\beta}_i, S_i]$$

$$= c_n E\left[\frac{\sum_{i=1}^{k} \hat{\beta}_i' S_i^{-1} \Sigma_i S_i^{-1} \hat{\beta}_i}{(\sum_{j=1}^{k} \hat{\beta}_j' S_j^{-1} \hat{\beta}_j)^2}\right]$$

$$+ (\bar{X} - X_o)^2\, E\left[\frac{\sum_{i=1}^{k} \hat{\beta}_i' S_i^{-1} (\beta_i - \hat{\beta}_i)}{\sum_{j=1}^{k} \hat{\beta}_j' S_j^{-1} \hat{\beta}_j}\right]$$

$$= c_n E\left[\frac{\sum_{i=1}^{k} U_i' V_i^{-2} U_i}{(\sum_{j=1}^{k} U_j' V_j^{-1} U_j)^2}\right]$$

$$+ (\bar{X} - X_o)^2\, E\left[1 - \frac{\sum_{i=1}^{k} U_i' V_i^{-1} \mu_i}{\sum_{j=1}^{k} U_j' V_j^{-1} U_j}\right]^2. \qquad (3.10)$$

As in Lemma 2.5, using orthogonal transformations and Cholesky decompositions, it can be shown that

$$E\left[\frac{\sum_{i=1}^{k} U_i' V_i^{-2} U_i}{(\sum_{j=1}^{k} U_j' V_j^{-1} U_j)^2}\right] = E\left[\frac{\sum_{i=1}^{k} \|U_i\|^2 t_{i11}^{-4}(1 + T_{i12} T_{i22}^{-1} T_{i22}'^{-1} T_{i12}')}{(\sum_{j=1}^{k} \|U_j\|^2 t_{j11}^{-2})^2}\right]$$

$$= (n-3)\, E\left[\frac{\sum_{i=1}^{k} \|U_i\|^2 t_{i11}^{-4}/(n - p_i - 2)}{(\sum_{j=1}^{k} \|U_j\|^2 t_{j11}^{-2})^2}\right]$$

$$= (n-3)\, E\left[\frac{\sum_{i=1}^{k} \|U_i\|^2 t_{i11}^{-4}/(n - p_i - 2)}{(\sum_{j=1}^{k} \|U_j\|^2)^2 (\sum_{j=1}^{k} \eta_j t_{j11}^{-2})^2}\right], \qquad (3.11)$$

where $\eta_i = \|U_i\|^2/\sum_{j=1}^k \|U_j\|^2$, $i=1, ..., k$. Now, conditioning on U_i's, applying Lemma 2.3 and then using Lemma 2.4(i), we see that for large n,

$$E\left[\frac{\sum_{i=1}^k U_i' V_i^{-2} U_i}{(\sum_{j=1}^k U_j' V_j^{-1} U_j)^2}\right] \doteq E\left(\sum_{i=1}^k \|U_i\|^2\right)^{-1} = E(p+2Z-2)^{-1}. \quad (3.12)$$

Next,

$$E\left[\frac{\sum_{i=1}^k (U_i' V_i^{-1} U_i)^2}{(\sum_{j=1}^k U_j' V_j^{-1} U_j)^2}\right] = E\left[\frac{\sum_{i=1}^k (U_i'\mu_i)^2 t_{i11}^{-4}}{(\sum_{j=1}^k \|U_j\|^2 t_{j11}^{-2})^2}\right]$$
$$+ 2E\left[\frac{\sum_{i=1}^k \lambda_i \|U_i\|^2 t_{i11}^{-4}/(m_i-1)}{(\sum_{j=1}^k \|U_j\|^2 t_{j11}^{-2})}\right]. \quad (3.13)$$

Using Lemma (2.3), we can show that, for large n, the first expectation in (3.13) is approximately equal to $E[\sum_{i=1}^k (U_i'\mu_i)^2/(\sum_{j=1}^k \|U_j\|^2)^2]$ and, using Lemma 2.6(iii), we see that the second expectation is nearly zero. Hence,

$$E\left[\frac{\sum_{i=1}^k (U_i' V_i^{-1}\mu_i)^2}{(\sum_{j=1}^k U_j' V_j^{-1} U_j)^2}\right] \doteq E\left[\frac{\sum_{i=1}^k (U_i'\mu_i)^2}{(\sum_{j=1}^k \|U_j\|^2)^2}\right]. \quad (3.14)$$

Again, as in the proof of Lemma 2.5(i), using orthogonal transformations and Cholesky decompositions, it can be shown that

$$E\left[\frac{\sum_{i=1}^k \sum_{j\neq i} U_i' V_i^{-1}\mu_i U_j' V_j^{-1}\mu_j}{(\sum_{j=1}^k U_j' V_j^{-1} U_j)^2}\right] = E\left[\frac{\sum_{i=1}^k \sum_{j\neq i} (U_i'\mu_i)(U_j'\mu_j)\, t_{i11}^{-2} t_{j11}^{-2}}{(\sum_{j=1}^k \|U_j\|^2 t_{j11}^{-2})^2}\right]$$
$$\doteq E\left[\frac{\sum_{i=1}^k \sum_{j\neq i} U_i'\mu_i U_j'\mu_j}{(\sum_{j=1}^k \|U_j\|^2)^2}\right], \quad (3.15)$$

for large n (Lemma 2.6(ii)). Combining (3.14) and (3.15), and then using Lemma 2.4(iii), we get

$$E\left[\frac{\sum_{i=1}^k U_i' V_i^{-1}\mu_i}{\sum_{j=1}^k U_j' V_j^{-1} U_j}\right]^2 \doteq E\left[\frac{\sum_{i=1}^k U_i'\mu_i}{\sum_{j=1}^k \|U_j\|^2}\right]^2$$
$$= 2\lambda E\left[\frac{2Z+1}{(p+2Z)(p+2Z-2)}\right]. \quad (3.16)$$

Further, from the expression for the bias, for large n, we have

$$E\left[\frac{\sum_{i=1}^k U_i' V_i^{-1}\mu_i}{\sum_{j=1}^k U_i^{-1} V_j^{-1} U_j}\right] = 2\lambda E(p+2Z)^{-1}. \quad (3.17)$$

Substituting (3.12), (3.16), and (3.17) in (3.10), we get the desired expression (3.7) for the MSE.

Remark 3.2. As argued in the proof of Lemma 2.6(iii), it can be shown that both the bias and MSE approach zero when λ tends to ∞. This means, from the definition of λ, that both the bias and MSE will be small if the slopes are large and/or the error covariance matrices are "small." Further, for large n, S_{xx} is expected to be large and so will be λ, and as a result bias will be small. However, the $\text{MSE} \to \sum_{i=1}^{k} \hat{\beta}_i' \Sigma_i^{-1} \hat{\beta}_i$ as $n \to \infty$ (see Johnson and Krishnamoorthy [7]).

Remark 3.3. Simpler approximations to the bias and MSE can be obtained using the Taylor series expansion. It can be shown, along the lines of Lieftinck-Koeijers [9], that

$$\text{BIAS}(\hat{X}_c) \doteq (\bar{X} - X_o)(p/(p+2\lambda) - 8\lambda^2/(p+2\lambda)^3) \tag{3.18}$$

and

$$\text{MSE}(\hat{X}_c) \doteq \frac{S_{xx}}{2\lambda}\left[1 + \frac{(\bar{X}-X_o)^2}{S_{xx}} + \frac{p-1}{\lambda}\right]. \tag{3.19}$$

Remark 3.4. To understand the validity of the approximations of the bias and MSE of $\hat{X}_c$, we estimated (3.8) for bias and (3.10) for MSE using simulation (100,000 runs), and computed (3.18) for bias and (3.19) for MSE numerically for $n = 25$ ($k = 2$, $p_1 = p_2 = 2$; $k = 3$, $p_1 = 3$, $p_2 = 1$, $p_3 = 1$)

TABLE I

Simulated and Approximated Values of Bias and MSE

	$k=3$				$k=5$			
	Simulation		Approximation		Simulation		Approximation	
λ	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
2	0.63	1.77	0.67	1.50	1.02	1.51	1.02	1.75
3	0.41	1.10	0.47	0.94	0.79	1.02	0.80	1.06
4	0.30	0.77	0.35	0.69	0.64	0.75	0.65	0.75
5	0.23	0.59	0.28	0.54	0.53	0.59	0.55	0.58
6	0.18	0.47	0.23	0.44	0.45	0.47	0.47	0.47
7	0.16	0.41	0.19	0.38	0.39	0.39	0.40	0.40
8	0.14	0.36	0.17	0.33	0.35	0.35	0.36	0.34
10	0.11	0.28	0.13	0.26	0.29	0.27	0.30	0.27
15	0.07	0.18	0.08	0.17	0.19	0.18	0.20	0.18
20	0.05	0.14	0.06	0.13	0.14	0.14	0.15	0.13

and various values of λ. They are given in Table I for $\bar{x}=0$, $S_{xx}=1$, and $x_o=-2$. These values are in good agreement even for small values of λ when $k=5$; but when $k=3$ they are in agreement for moderately large values of λ.

4. CONCLUSION AND DISCUSSION

The results of this paper are applicable to practical problems where response variables are measured using alternative measuring devices or methods. For instance, when we have two machines and one is capable of measuring two characteristics of each unit and another can read at least one of these two characteristics, then the combined estimator $\hat{X}_c$ in (1.10) has finite bias an MSE. Thus, it is advantangeous to use a variety of instruments to measure the response variables in order to make better inferences about the explanatory variable.

Regarding confidence estimation, Johnson and Krishnamoorthy [7] provide two confidence sets for X_o when p_i's are equal to 1. One of them is obtained by inverting the absolute sum of independent t-statistics which is always nonempty and a condition, similar to the one in Brown [1], is needed for it to be a finite interval; another is obtained by inverting the sum of independent F-statistics which can be empty and two conditons are needed for it to be a finite interval. The later confidence set can be easily extended to the present setup along the lines of Johnson and Krishnamoorthy [7]. In general, developing a satisfactory confidence set based on all independent multvariate models seems to be difficult even if the confidence sets based on individual models are bonafined nonempty ellipsoids (Jordan and Krishnamoorthy [8]).

This paper further opens up a few problems. For instance, it is plausible that the results can be extended to the case where the dimension of the explanatory variable exceeds one. In this case, the combined estimator will involve "matrix weights" which make the problem more complex. We further note that this article is concerned only with the classical approach. However, in the area of calibration, the so-called inverse approach is also commonly used (in particular, when the explanatory variables also involve measurement errors) to estimate the explanatory variables. Therefore, one may want to investigate a method of combining inferences about X_o using the inverse approach.

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