Combining Independent Studies in a Calibration Problem

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The problem of calibration in which the response variable is measured by k different methods or using different instruments is considered. It is well known that the usual classical estimator for the unknown explanatory variable has infinite mean and mean squared error when $k = 1$. In this article a linear combination of the classical estimators is proposed. It is shown that the combined estimator has finite mean provided that $k \geq 2$ and finite mean squared error provided that $k \geq 3$. Expressions for asymptotic bias and mean squared error are given. Also, two confidence sets for the unknown exploratory variable are developed, and sufficient conditions under which they will be finite intervals are given. The results are illustrated by a practical example.

KEY WORDS: Bias; Classical estimator; Confidence set; Controlled experiment; Mean squared error; Prediction experiment.

1. INTRODUCTION

The well-known calibration problem can be briefly described as follows. There are two related responses, x and y, where x represents the true value of the characteristic of interest determined by a difficult and expensive method and y represents the measurements related to the same characteristic of interest obtained by an easy and cheap method. The objective here is to establish a relationship between x and y using calibration data $(x_1, y_1), \ldots, (x_n, y_n)$ available from n experimental units. The established relationship based on the calibration data can be used to make inferences about the unknown x value of a future individual based on its known y value. The responses x and y may be vectors. Some specific problems where the calibration techniques are used include: predicting oil, protein, and starch x in maize based on near-infrared spectroscopy measurements y (Orman 1991); determining glucose x in whole blood by attenuated total reflective infrared spectroscopy y (Heise and Marbach 1989); and predicting output from ground water flow models using hydrogeologic information (Cooley 1993). Other examples for multivariate calibration and general discussion on the practical uses of calibration have been given by Brown and Sundberg (1989), Oman and Wax (1984), and Rosenblatt and Spiegelman (1981). For a good exposition of this area, we refer the readers to the book by Brown (1993).

In this article we consider the situation in which calibration data are available from k different sources for the same objective. Such situations arise, for instance, when a calibration experiment is conducted by different laboratories, using different methods or using different measuring instruments. (See Sec. 4 for a specific example.) We then have data $(x_j, y_{ij}), i = 1, \ldots, k$ and $j = 1, \ldots, n$, where x_j represents the true x value of the jth unit and y_{ij} represents the measurement related to the x_j obtained by the ith measuring instrument or method. In these situations the variances of y_{ij}'s may possibly be different. In other words, we have "controlled calibration data" from k different sources with unknown and arbitrary variances for the same objective. We assume that the y_i's are linearly related to x.

Thus the calibration model that we assume in this article is

$$y_{ij} = \alpha_i + \beta_i x_j + \varepsilon_{ij}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n. \tag{1}$$

We assume that the ε_{ij}'s are independent with $\varepsilon_{ij} \sim N(0, \sigma_i^2)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$. Further, we assume that the responses of the prediction experiment follow the same model assumptions as those of the calibration experiment (1). That is,

$$y_{0i} = \alpha_i + \beta_i x_0 + \varepsilon_i, \qquad i = 1, \ldots, k, \tag{2}$$

where the ε_i's are independent random variables that follow $N(0, \sigma_i^2)$ and are independent of the ε_{ij}'s in the calibration model (1). The variances σ_i^2's are unknown and arbitrary, and there is no restriction on the α_i's and β_i's. The problem is to make inferences about the unknown x_0 based on the calibration model (1) and the y_{0i}'s from the prediction model (2).

Define $\bar{y}_i = \sum_{j=1}^n y_{ij}/n, \bar{x} = \sum_{j=1}^n x_j/n, S_{xx} = \sum_{j=1}^n (x_j - \bar{x})^2$, and $S_i^2 = \sum_{j=1}^n (\hat{y}_{ij} - y_{ij})^2/(n-2)$. Let $\hat{\alpha}_i$ and $\hat{\beta}_i$ denote the least squares estimators of α and β based on the ith model; that is,

$$\hat{\beta}_i = \sum_{j=1}^n (y_{ij} - \bar{y})(x_j - \bar{x})/S_{xx}$$

and

$$\hat{\alpha}_i = \bar{y}_i - \hat{\beta}_i \bar{x}, \qquad i = 1, \ldots, k. \tag{3}$$

We note that $y_{01}, \ldots, y_{0k}, \bar{y}_1, \ldots, \bar{y}_k, \hat{\beta}_1, \ldots, \hat{\beta}_k$, and $S_1^2, \ldots, S_k^2$ are all statistically independent with

$$y_{0i} \sim N(\alpha_i + \beta_i x_0, \sigma_i^2), \qquad \bar{y}_i \sim N(\alpha_i + \beta_i \bar{x}, \sigma_i^2/n),$$

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$$\hat{\beta}_i \sim N(\beta_i, \sigma_i^2/S_{xx}), \quad \text{and} \quad (n-2)S_i^2/\sigma_i^2 \sim \chi^2_{n-2} \tag{4}$$

for $i = 1, \ldots, k$. The classical estimator based on the ith model alone is given by

$$\hat{x}_{0i} = \bar{x} + (y_{0i} - \bar{y}_i)/\hat{\beta}_i. \tag{5}$$

This estimator is obtained by solving the equation $y_{0i} = \hat{\alpha}_i + \hat{\beta}_i x_0$ for x_0. It has infinite mean and mean squared error (MSE) on account of the denominator $\hat{\beta}_i$ (see, e.g., Piegorsch and Casella 1985). Krutchkoff (1967) introduced the concept of inverse estimation and compared the inverse estimator and the classical estimator (5) with respect to the MSE criterion by a Monte Carlo study. His results led to a long controversy. The MSE as a criterion for comparing the classical estimator and the inverse estimator has been criticized by Williams (1969a,b). Also, Williams (1969b) showed that no unbiased estimator has finite variance and recommended $\hat{x}_{0i}$ on the grounds that it is based on a set of sufficient statistics. Halperin (1970) used Pitman nearness criterion for comparison and supported using $\hat{x}_{0i}$ rather than the inverse estimator. However, in the multivariate case (i.e., **y** is a $k \times 1$ vector with covariance matrix $\boldsymbol{\Sigma}$) Lieftinck-Koeijers (1988) showed that when $\boldsymbol{\Sigma}$ is known, a generalization of the classical estimator (5) has finite mean provided that $k \geq 3$ and finite MSE if $k \geq 5$. Her results were improved and extended to the unknown covariance matrix case by Nishii and Krishnaiah (1988). They indeed showed that the classical estimate has finite mean if $k \geq 2$ and finite MSE if $k \geq 3$, and also gave expressions for them. Brown and Spiegelman (1991) also showed the existence of absolute moments of the classical estimate in a more general situation assuming that the error variances are known. (For confidence estimation in the multivariate case, we refer the readers to Brown 1982 and Mathew and Subramaniam 1994.) The multivariate setup is appropriate when several responses are available from each unit, whereas the present setup is suitable when only a single response is observable and various methods or different measuring devices are used to measure it. Of course, there should not be any confusion between the present problem and that of comparative calibration (Williams 1969a), as in the latter no standard measurement on x is available and the objective is to compare different measuring instruments in a symmetric way.

In the following section we propose first a linear combination of the k classical estimators in (5) and show that the combined estimator has finite mean when $k \geq 2$ and finite MSE when $k \geq 3$ under the model assumptions in (1) and (2). We also give asymptotic expressions for both bias and MSE of the combined estimator in terms of the expectations involving Poisson random variables. Further, we develop two exact confidence sets for x_0 based on all k calibration data sets and the prediction experiment data using a generalization of the joint sampling approach (Brown 1982). We give sufficient conditions for the confidence sets to be finite intervals in Section 3. We illustrate the results using an experimental data set in Section 4. Finally, in Section 5 we make some remarks regarding the practical implications of the results obtained in this article.

2. A COMBINED ESTIMATOR OF x_0

Suppose that the β_i's and σ_i^2's are known and let $\hat{x}_i = \bar{x} + (y_{0i} - \bar{y}_i)/\beta_i, i = 1, \ldots, k$. Consider $\hat{x}_l = \sum_{i=1}^k a_i \hat{x}_i$, where the a_i's are nonnegative constants such that $\sum_{i=1}^k a_i = 1$. It follows from (4) that $E(\hat{x}_l) = x_0$, and it can be easily verified that the var$(\hat{x}_l)$ attains its minimum when $a_i = (\beta_i^2/\sigma_i^2)/(\sum_{j=1}^k \beta_j^2/\sigma_j^2), i = 1, \ldots, k$. Hence $\hat{x}_l$ with these choices for the a_i's is the best linear unbiased estimator of x_0. Therefore, when the β_i's and σ_i^2's are unknown, replacing them by appropriate estimators, we propose

$$\hat{x}_c = \sum_{i=1}^k w_i \hat{x}_{0i} \quad \text{with} \quad w_i = (\hat{\beta}_i^2/S_i^2)\Bigg/\left(\sum_{j=1}^k \hat{\beta}_j^2/S_j^2\right), \tag{6}$$

where $\hat{x}_{0i}$ is given in (5) as an estimator of x_0. The estimator (6) can also be regarded as a generalized least squares estimator, for it minimizes $\sum_{i=1}^k (y_{0i} - \hat{\alpha}_i - \hat{\beta}_i x_0)^2/S_i^2$ with respect to x_0.

2.1 Bias and Mean Squared Error of $\hat{x}_c$

In the following theorem we give conditions under which the bias and the MSE of $\hat{x}_c$ are finite and give asymptotic expressions for them. The proof of the theorem is deferred to the Appendix.

Theorem 2.1. The estimator $\hat{x}_c$ has finite mean if $n \geq 5$ and $k \geq 2$ and finite MSE if $n \geq 7$ and $k \geq 3$. Further,

$$\text{bias}(\hat{x}_c) = (\bar{x} - x_0)(1 - 2\lambda E(k + 2Z)^{-1}) + O(n^{-1}) \tag{7}$$

and

$$\begin{aligned} \text{MSE}(\hat{x}_c) &= S_{xx}E(k + 2Z - 2)^{-1} + (\bar{x} - x_0)^2 \\ &\quad \times \left[1 + 2\lambda E\left(\frac{2Z+1}{(k+2Z)(k+2Z-2)} - \frac{2}{k+2Z}\right)\right] \\ &\quad + O(n^{-1}), \end{aligned} \tag{8}$$

where $Z \sim \text{Poisson}(\lambda)$ and $\lambda = (S_{xx}/2)\sum_{i=1}^k \beta_i^2/\sigma_i^2$.

Because $(k + 2Z)^{-1}$ is a convex function of $Z, E(k + 2Z)^{-1} \geq (k + 2\lambda)^{-1}$. Further, for $k \geq 2, E(k + 2Z)^{-1} \leq E(2 + 2Z)^{-1} = (1 - \exp(-\lambda))/(2\lambda)$. Therefore, $2\lambda/(k + 2\lambda) \leq 2\lambda E(k + 2Z)^{-1} \leq 1 - \exp(-\lambda)$. In view of this inequality, it is evident from (7) that $\hat{x}_c$ overestimates x_0 if $x_0 < \bar{x}$ and underestimates it if $x_0 > \bar{x}$. We also note that $2\lambda E(k + 2Z)^{-1} \to 1$ as $\lambda \to \infty$. Similarly, it can be shown that for fixed $S_{xx}, S_{xx}E(k + 2Z - 2)^{-1} \to 0$ as $\lambda \to \infty$, and the terms inside the square brackets in (8) approach zero as λ tends to ∞. This means that both the bias$(\hat{x}_c)$ and the MSE$(\hat{x}_c)$ will be small if the slopes are large and/or error variances are small.

Remark 2.1. It is interesting to note that Lieftinck-Koeijers's expression for the bias in the multivariate case (with known covariance matrix) is $(\bar{x} - x_0)(k - 2)E(k + 2Z - 2)^{-1}$, which is finite only when $k \geq 3$, whereas the expression (7) is finite for $k \geq 2$. Further, applying the Poisson identity (Hwang 1982)—that for $Z \sim \text{Poisson}(\lambda), EZ[g(Z-1)] = \lambda E[g(Z)]$, provided that $|g(-1)| < \infty$—to (7), we see that for $k \geq 3$ (otherwise, the Poisson identity is not applicable), the expression (7) is equivalent to Lieftinck-Koeijers's expression given earlier. Similarly, by applying the Poisson identity to (8), it can be shown for $k \geq 5$ that (8) is in agreement with (7.2) of her paper. We also checked that the bias (7) is in accordance with Nishii and Krishnaiah's (1988, ex. 4.2) result for the multivariate case with unknown covariance matrix. However, their (4.3) is indeed $E(\hat{x}-\bar{x})^2$, not the MSE. Further, the expression for $E(b'S^{-1}\gamma_*/b'S^{-1}b)^2$ given in their paper is incorrect. After correcting these errors, we verified that the MSE (8) is in agreement with their result.

Remark 2.2. Simpler approximations to the bias and MSE can be obtained using the Taylor series expansion. Letting $S_{xx} = n\hat{\sigma}_x^2$ and $\lambda_0 = \lambda/S_{xx}$, it can be shown along the lines of Lieftinck-Koeijers (1988) that

$$\text{bias}(\hat{x}_c) = (\bar{x} - x_0)(k/(k+2\lambda) - 8\lambda^2/(k+2\lambda)^3) + O(n^{-1}) \quad (9)$$

and

$$\text{MSE}(\hat{x}_c) = \frac{1}{2\lambda_0}\left[1 + \frac{(\bar{x}-x_0)^2}{n\hat{\sigma}_x^2} + \frac{k-1}{n\hat{\sigma}_x^2\lambda_0}\right] + O(n^{-1}). \quad (10)$$

For large n, S_{xx} is expected to be large, as is λ, and as a result bias$(\hat{x}_c)$ will be small. However, we observe from (10) that MSE$(\hat{x}_c) \to 1/(2\lambda_0)$ as $n \to \infty$. These results are in agreement with those of Lieftinck-Koeijers (1988).

Remark 2.3. It seems very difficult to derive exact expressions for bias and MSE of $\hat{x}_c$, even for some special cases. Therefore, to see for what values of n and λ the approximations are satisfactory, we estimated (A.5) for bias and (A.11) for MSE using simulation (100,000 runs) and computed (9) for bias and (10) for MSE numerically at various values of λ for $n = 20$ and $k = 3, 5$. These are given in Table 1 for $\bar{x} = 0, S_{xx} = 1$, and $x_0 = -3$.

It is clear that these numbers are in good agreement for moderately large λ. We also did computations at different values of n and λ; these are not reported here, because they all exhibited the same pattern as the numbers in Table 1. In general, we found that the approximations are satisfactory as long as $\lambda \geq 8$ and $n \geq 20$.

3. CONFIDENCE ESTIMATION OF x_0

To develop a confidence set for x_0, we note that the distribution of any combined estimator of x_0 will involve nuisance parameters, and hence the standard method has serious limitations for the purpose of finding an exact confidence set. Therefore, we follow an approach (essentially inverting a combined test for the parameter of interest) that was used successfully to develop exact confidence intervals for the common mean of several normal populations (see Jordan and Krishnamoorthy 1996 and the references therein). We first develop a confidence set by inverting the sum of independent F statistics.

Let $c_n = 1 + 1/n$ and $F_i = [y_{0i} - \bar{y}_i - \hat{\beta}_i(x_0 - \bar{x})]^2/[S_i^2(c_n + (x_0 - \bar{x})^2/S_{xx})], i = 1, \ldots, k$. Under the model assumptions, the F_i's are iid F random variables with 1 and $n-2$ degrees of freedom. Let ξ_α denote the upper $100(1-\alpha)$th percentile point of the sum of k iid F random variables with 1 and $n-2$ df. Then the set of x_0 values that satisfies

$$\sum_{i=1}^{k} \frac{[y_{0i} - \bar{y}_i - \hat{\beta}_i(x_0 - \bar{x})]^2}{S_i^2(c_n + (x_0 - \bar{x})^2/S_{xx})} \leq \xi_\alpha \quad (11)$$

is an exact $(1-\alpha)$ level confidence set for x_0. When $k = 1$, this confidence set is identical to the classical confidence set for x_0, which is always nonempty; it is a finite interval if $t_1^2 = S_{xx}\hat{\beta}_1^2/S_1^2 > \xi_\alpha$ or, equivalently, when the null hypothesis that $\beta_1 = 0$ is rejected at the level α based on the test statistic t_1^2 that follows F distribution with 1 and $n-2$ df when $\beta_1 = 0$. Otherwise, it could be the entire real line with a finite interval removed or the entire real line (see, e.g., Fisch and Strehlau 1993). However, for $k \geq 2$, we show that the confidence set (11) is not always nonempty, and that two conditions are needed for the confidence set to be a finite interval.

After some algebraic manipulation, we see that (11) is equivalent to

Table 1. Simulated and Approximated Values of Bias and MSE

	k = 3				k = 5			
	Simulation		Approximation		Simulation		Approximation	
λ	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
2	.97	3.27	1.01	2.91	1.52	3.11	1.53	3.17
3	.62	2.18	.70	1.85	1.19	2.10	1.20	1.97
4	.44	1.58	.53	1.35	.96	1.50	.98	1.38
5	.36	1.22	.42	1.07	.80	1.17	.82	1.18
6	.27	.92	.34	.88	.68	.95	.71	.91
7	.23	.80	.29	.73	.58	.79	.62	.77
8	.22	.71	.25	.65	.52	.68	.55	.67
10	.16	.55	.19	.52	.42	.54	.44	.53
15	.11	.36	.12	.34	.28	.36	.29	.35
20	.07	.27	.09	.26	.22	.27	.23	.26

$$(x_0-\bar{x})^2(1-\gamma_f)-2(x_0-\bar{x})(\hat{x}_c-\bar{x}) + \sum_{i=1}^{k} w_i(y_{0i}-\bar{y}_i)^2/\hat{\beta}_i^2-\gamma_f c_n S_{xx}\le 0, \quad (12)$$

where $\gamma_f=\xi_\alpha/\sum_{i=1}^k t_i^2$ and $t_i^2=S_{xx}\hat{\beta}_i^2/S_i^2, i=1,\ldots,k$. Clearly, the left side of (12) is a convex function of x_0 if $1-\gamma_f>0$ or, equivalently,

$$\sum_{i=1}^{k} t_i^2>\xi_\alpha, \quad (13)$$

and in this case the confidence set may be a finite interval with endpoints given by the roots of the quadratic equation in (12). Indeed, solving the inequality (12) for x_0, we get

$$\hat{x}_c+\frac{\gamma_f(\hat{x}_c-\bar{x})}{1-\gamma_f} \pm \frac{[(\hat{x}_c-\bar{x})^2-(1-\gamma_f) \times(\sum_{i=1}^k w_i(y_{0i}-\bar{y}_i)^2/\hat{\beta}_i^2-\gamma_f c_n S_{xx})]^{1/2}}{1-\gamma_f} \quad (14)$$

as an interval estimator of x_0. The interval (14) is finite if the expression under the square root is positive. Using the fact that variance is location invariant, it can be shown that the expression under the square root in (14) is positive if and only if

$$\sum_{i=1}^{k} w_i(\hat{x}_{0i}-\hat{x}_c)^2 < \gamma_f\left(\sum_{i=1}^{k} w_i(y_{0i}-\bar{y}_i)^2/\hat{\beta}_i^2+(1-\gamma_f)c_n S_{xx}\right). \quad (15)$$

We now argue that the conditions (13) and (15) are more likely to hold under the model assumptions (1) and (2). From the definition of t_i^2's and (4), we see that the t_i^2's are independent and follow F distribution with 1 and $n-2$ df when $\beta_1=\cdots=\beta_k=0$, and hence $\sum_{i=1}^k t_i^2$ is a test statistic for testing

$$H_0\colon \beta_1=\cdots=\beta_k=0 \quad \text{vs.} \quad H_a\colon \beta_i\ne 0 \quad \text{for some } i. \quad (16)$$

Therefore, from the definition of ξ_α, the condition (13) is more likely to hold when one or more slopes are significantly different from zero. Because the w_i's are nonnegative and $\sum_{i=1}^k w_i=1$, the left side of (15) is the variance of the individual classical estimates centered at $\hat{x}_c$, which is expected to be small. Thus under the model assumptions, the interval (14) is more likely to be nonempty and finite. When the condition (13) fails to hold, the left side of (12) is a concave function of x_0, and hence the confidence set will be either the entire real line with the finite interval (14) removed or the entire real line.

Remark 3.1. Note that if $1-\gamma_f>0$, then the left side of (12) attains its minimum at $x_0=\hat{x}_c+\gamma_f(\hat{x}_c-\bar{x})/(1-\gamma_f)$, and this minimum is less than or equal to zero if and only if the condition (15) holds. Therefore, the confidence set (11) is not always nonempty. As pointed out by Jordan and Krishnamoorthy (1996) in the context of common mean problem, if the classical estimates are especially different, then in fact a combined confidence set should not be constructed based on all samples. In these situations the confidence set developed by using any combined method may be dubious. The condition (15) helps in deciding whether to combine the independent estimates or not. However, if one decides to construct a confidence set based on all samples at any cost, then the following confidence set (17), which is always nonempty, can be used.

We now develop a confidence set by inverting the absolute sum of independent t statistics. This approach was used by Fairweather (1972) to develop a confidence set for the common mean of several normal populations. Let η_α denote the upper $100(1-\alpha)$th percentile point of $|\sum_{i=1}^k z_i|$, where z_i's are independent Student's-t random variables with $n-2$ df. Because $[y_{0i}-\bar{y}_i-\hat{\beta}_i(x_0-\bar{x})]/[S_i(c_n+(x_0-\bar{x})^2/S_{xx})^{1/2}]$'s are independent Student's-t variables with $n-2$ df, the set of x_0 values that satisfy

$$\left|\sum_{i=1}^{k}\frac{[y_{0i}-\bar{y}_i-\hat{\beta}_i(x_0-\bar{x})]}{S_i(c_n+(x_0-\bar{x})^2/S_{xx})^{1/2}}\right|\le\eta_\alpha \quad (17)$$

is an exact confidence set for x_0 with confidence coefficient $1-\alpha$. This confidence set is always nonempty, because the left side of (17) becomes zero at $x_0=\hat{x}_t=\sum_{i=1}^k u_i\hat{x}_{0i}$, where $u_i=(\hat{\beta}_i/S_i)/[\sum_{j=1}^k\hat{\beta}_j/S_j], i=1,\ldots,k$. This confidence set is a finite interval,

$$\hat{x}_t+\frac{\gamma_t(\hat{x}_t-\bar{x})}{1-\gamma_t}\pm\frac{[\gamma_t(\hat{x}_t-\bar{x})^2+\gamma_t c_n S_{xx}(1-\gamma_t)]^{1/2}}{1-\gamma_t}, \quad (18)$$

provided that $1-\gamma_t=1-\eta_\alpha^2/(\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i)^2>0$ or, equivalently, $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|>\eta_\alpha$. Note that under H_0 in (16), $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|$ is distributed as $|\sum_{i=1}^k t_i|$. In general, $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|$ is not a good test statistic for testing (16), as it may not provide evidence against H_0 when some of the β_i's are well below zero and others are well above zero. However, it is very unlikely to happen in the present problem, because the responses are measured on the same x values using different methods or instruments, and the β_i's should all be of the same sign. Therefore, we expect the $\hat{\beta}_i$'s to be of the same sign as well. Note that one can always change the sign of a $\hat{\beta}_i$ by appropriately transforming the corresponding y values (for example, multiplying by -1). Thus $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|$ is expected to be large if H_0 in (16) is not true. Further, because $(\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i)^2>S_{xx}\sum_{i=1}^k\hat{\beta}_i^2/S_i^2$ when the $\hat{\beta}_i$'s are all of the same sign and $\eta_\alpha^2<\xi_\alpha$ (see Jordan and Krishnamoorthy 1996, tables 5 and 6), $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|>\eta_\alpha$ whenever $S_{xx}\sum_{i=1}^k\hat{\beta}_i^2/S_i^2>\xi_\alpha$. Thus in the present setup the condition $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|>\eta_\alpha$ is expected to hold more often than (13) when H_0 is not true. If $|\sqrt{S_{xx}}\sum_{i=1}^k\hat{\beta}_i/S_i|<\eta_\alpha$, then the confidence set (17) will be either the entire real line with the finite interval (18) removed or the entire real line.

Remark 3.2. One may want to consider $\hat{x}_t = \sum_{i=1}^{k} u_i \hat{x}_{0i}$ in (18) as a point estimator of x_0. We note that $\hat{x}_t$ is not a natural estimator to begin with, and the weight u_i's used to combine the independent classical estimators can be negative. This can be overcome by choosing weights $u_{i*} = |\hat{\beta}_i/S_i|/(\sum_{j=1}^{k} |\hat{\beta}_j/S_j|)$, and defining $\hat{x}_{t*} = \sum_{i=1}^{k} u_{i*}\hat{x}_{i0}$. Recently, Moore and Krishnamoorthy (1995) compared similar estimators for the common mean of several normal populations and found that generally the one similar to $\hat{x}_c$ is superior to the one similar to $\hat{x}_{t*}$. In view of their result, $\hat{x}_{t*}$ may be inferior to $\hat{x}_c$.

The percentile points ξ_α of the sum of two independent F random variables and η_α of the sum of two independent Student's-t variables can be obtained numerically. Jordan and Krishnamoorthy (1996) computed exact values of ξ_α and η_α for some selected values of m. For $k \geq 3$, it is difficult to compute the exact values of ξ_α and η_α. However, they can be approximated conveniently. For example, ξ_α can be approximated by the distribution of $dF_{k,\nu}$, where $F_{k,\nu}$ denotes F random variable with k and ν df. The unknown positive parameters d and ν can be estimated by equating the first two moments of the sum of k independent $F_{1,m}$ random variables with those of $dF_{k,\nu}$. When $m \geq 5$, the estimated values are

$$\nu = [(k+2)(m-4)+12]/3$$

and

$$d = (\nu-2)km/[\nu(m-2)].$$

This approximation is not only simple to use, but also (as shown in Jordan and Krishnamoorthy 1995, 1996), gives very satisfactory results for $m \geq 8$. Similarly, the percentile points η_α can be approximated by the distribution of st_a, where s and a are positive parameters to be estimated and t_a denotes the Student's-t variable with a df (Fairweather 1972).

4. AN EXAMPLE

A controlled experiment was conducted at the National Biological Service, Louisiana, to predict the amount of sodium chloride solution in dionized water based on electric conductivity. Two machines, the conductivity controller (CC) and the Fisher conductivity meter (FCM), were used to measure the electric conductivity. The experiment was run 31 times for different known amounts of sodium chloride solution (x in ml), and conductivity measurements (y_1, y_2 in micromoles/cm^3) were recorded. These values are given in Table 2. Simple linear regression models were fit with y_1 regressed on x and y_2 regressed on x separately based on all 31 observations. The fitted model for the CC machine is $y_1 = 1.8904 + .3264x$, and that for the FCM machine is $y_2 = 1.8038 + .6045x$. The values of R^2 for both models are about .99. The normal probability plots indicate that

Table 2. Calibration Data for Predicting Sodium Chloride Solution (x in ml) Based on Electric Conductivity Measurements y_1 and y_2 (in micromoles/cm^3)

x	y_1	y_2	$\hat{x}_{01}$	$\hat{x}_{02}$	$\hat{x}_c$	$\hat{x}_t$	I_1	I_2	I_c	I_t
0	1.6	1.5	−.99	−.56	−.76	−.77	(−2.2, .2)	(−1.71, .57)	(−1.79, .24)	(−1.6, .05)
5	1.8	1.9	−.36	.13	−.1	−.12	(−1.58, .84)	(−1.03, 1.26)	(−1.13, .9)	(−.94, .71)
1.0	2.0	2.2	.27	.62	.46	.47	(−.95, 1.48)	(−.53, 1.76)	(−.58, 1.48)	(−.38, 1.28)
1.5	2.2	2.6	.9	1.3	1.11	1.1	(−.33, 2.11)	(.15, 2.44)	(.08, 2.13)	(.27, 1.93)
2.0	2.4	2.9	1.53	1.8	1.67	1.66	(.3, 2.74)	(.65, 2.93)	(.63, 2.7)	(.83, 2.49)
2.5	2.6	3.2	2.15	2.3	2.23	2.22	(.92, 3.36)	(1.16, 3.42)	(1.18, 3.26)	(1.39, 3.05)
3.0	2.8	3.6	2.77	2.97	2.88	2.87	(1.55, 3.98)	(1.83, 4.1)	(1.84, 3.91)	(2.04, 3.7)
3.5	3.0	3.9	3.39	3.47	3.43	3.43	(2.17, 4.6)	(2.33, 4.59)	(2.39, 4.47)	(2.6, 4.25)
4.0	3.2	4.2	4.01	3.96	3.99	3.98	(2.79, 5.22)	(2.83, 5.08)	(2.94, 5.02)	(3.16, 4.81)
4.5	3.4	4.5	4.63	4.46	4.54	4.54	(3.42, 5.84)	(3.33, 5.58)	(3.5, 5.57)	(3.72, 5 36)
5.0	3.6	4.8	5.25	4.95	5.09	5.09	(4.04, 6.45)	(3.83, 6.07)	(4.07, 6.11)	(4.28, 5.91)
5.5	3.8	5.2	5.86	5.62	5.74	5.74	(4.66, 7.06)	(4.5, 6.74)	(4.71, 6.75)	(4.92, 6.55)
6.0	3.9	5.5	6.16	6.12	6.14	6.14	(4.96, 7.36)	(5., 7.23)	(5.11, 7.17)	(5.32, 6.96)
6.5	4.1	5.8	6.78	6.62	6.69	6.69	(5.58, 7.98)	(5.5, 7.73)	(5.67, 7.71)	(5.88, 7.51)
7.0	4.3	6.1	7.4	7.11	7.24	7.25	(6.2, 8.59)	(5.99, 8.23)	(6.23, 8.26)	(6.44, 8.06)
7.5	4.5	6.4	8.01	7.61	7.8	7.8	(6.83, 9.2)	(6.49, 8.72)	(6.8, 8.8)	(6.99, 8.61)
8.0	4.6	6.7	8.31	8.1	8.2	8.2	(7.11, 9.51)	(6.99, 9.22)	(7.18, 9.22)	(7.39, 9.02)
8.5	4.8	7.0	8.93	8.6	8.75	8.76	(7.74, 10.12)	(7.48, 9.71)	(7.74, 9.76)	(7.95, 9.57)
9.0	5.0	7.3	9.54	9.1	9.31	9.31	(8.36, 10.73)	(7.98, 10.21)	(8.31, 10.3)	(8.5, 10.12)
9.5	5.1	7.6	9.84	9.59	9.71	9.71	(8.65, 11.04)	(8.48, 10.71)	(8.69, 10.73)	(8.9, 10.53)
10.0	5.3	7.9	10.46	10.09	10.26	10.27	(9.27, 11.66)	(8.91, 11.21)	(9.26, 11.27)	(9.46, 11.08)
11.0	5.6	8.5	11.38	11.08	11.22	11.23	(10.18, 12.58)	(9.97, 12.2)	(10.21, 12.24)	(10.41, 12.04)
12.0	6.0	9.1	12.62	12.07	12.33	12.34	(11.43, 13.81)	(10.96, 13.2)	(11.35, 13.32)	(11.53, 13.15)
13.0	6.3	9.7	13.54	13.07	13.29	13.3	(12.34, 14.74)	(11.95, 14.2)	(12.29, 14.3)	(12.48, 14.12)
14.0	6.6	11.0	14.46	15.3	14.95	14.92	(13.25, 15.67)	(14.28, 16.33)	(14.06, 15.84)	(14.13, 15.70)
15.0	6.9	11.4	15.37	15.94	15.69	15.68	(14.16, 16.6)	(14.86, 17.04)	(14.72, 16.68)	(14.87, 16.49)
16.0	7.2	11.6	16.29	16.22	16.26	16.26	(15.07, 17.53)	(15.09, 17.38)	(15.21, 17.32)	(15.42, 17.1)
17.0	7.5	12.0	17.21	16.85	17.02	17.03	(15.97, 18.46)	(15.72, 18.01)	(15.98, 18.07)	(16.19, 17.87)
18.0	7.7	13.0	17.77	18.59	18.22	18.21	(16.54, 19.03)	(17.45, 19.75)	(17.24, 19.21)	(17.36, 19.05)
20.0	8.2	14.0	19.22	20.21	19.73	19.73	(18.01, 20.46)	(19.03, 21.41)	(18.79, 20.7)	(18.88, 20.58)
24.0	9.1	15.0	21.57	21.25	21.34	21.38	(20.69, 22.46)	(20.7, 21.82)	(20.77, 21.93)	(20.89, 21.86)

the distribution of the error terms does not depart from a normal distribution.

Table 2 gives the classical estimates $\hat{x}_{01}$ and $\hat{x}_{02}$ based on individual models and the combined estimates $\hat{x}_c$ and $\hat{x}_t$. These statistics and the confidence sets are computed using the leave-one-out method; for example, to compute point and interval estimates for the first x-value, the observation (0, 1.6, 1.5) is not used to fit the models. Interestingly, for this example, $\hat{x}_c$ and $\hat{x}_t$ are almost the same for all x values. For all four estimates, we computed the average squared differences between the actual values and the estimated values to understand their inaccuracies. These are .404 for $\hat{x}_1$, .369 for $\hat{x}_2$, .356 for $\hat{x}_c$, and .350 for $\hat{x}_t$. Further, we computed the 95% interval estimates I_1 and I_2 using (12), I_c using (15) with $\xi_{.05} = 6.626$ (exact), and I_t using (19) with $\eta_{.05} = 2.888$ (exact). Comparing the interval estimates, we see that I_t is always narrower than the other three intervals and I_c is narrower than I_1 and I_2. All four intervals contain the corresponding true x values in all the cases except when $x = 14$ and 24. None of these four intervals contains 24, and only the interval I_1 contains 14. Overall, this example demonstrates that combining independent measurements is advantageous for better estimation.

5. CONCLUSIONS

This article studied the problem of calibration in which the response variable is measured by different instruments or determined by various methods. We demonstrated that it is advantageous to combine different measurements on the response variable to make better inferences about the explanatory variable. The results of this article are applicable to practical problems where only one response variable is observable and alternative measuring devices or methods are used to measure the response variable. The confidence sets (12) and (18) are more likely to be finite intervals in practical situations, as evidenced in the example of Section 4. Further, the techniques used to derive our results can be extended to k independent multivariate regression models. We are currently investigating this extension and plan to report it elsewhere.

APPENDIX

The following lemmas are needed to prove Theorem 2.1. (The proof of Lemma A.1 can be found in, e.g., Casella and Berger 1990, sec. 4.7.)

Lemma A.1. Let V be a chi-squared random variable with m df, $V \sim \chi^2_m$. For any real k,

$$E[V^k f(V)] = 2^k \Gamma[(m+2k)/2] E[f(V_0)]/\Gamma(m/2),$$

where $V_0 \sim \chi^2_{m+2k}$, provided that the indicated expectations exist.

The following Lemmas A.2 and A.3 are due to Nishii and Krishnaiah (1988). They proved these lemmas using an integral over the unit sphere given by Gradshteyn and Ryzhik (1980, 4.635). We observe that Lemma A.2, which essentially shows that the classical estimate in the multivariate case has finite moments, can also be proved using Lemma A.1 in a straightforward manner.

Lemma A.2. Let $u_1, \ldots, u_k$ be independent random variables with $u_i \sim N(\mu_i, 1), i = 1, \ldots, k$. Let $Z_1, \ldots, Z_k$ be independent random variables with $Z_i \sim \text{Poisson}(\lambda_i)$, where $\lambda_i = \mu_i^2/2, i = 1, \ldots, k$. Define $Z = \sum_{i=1}^k Z_i$ so that $Z \sim \text{Poisson}(\lambda)$, where $\lambda = \sum_{i=1}^k \lambda_i$. Then

(a) when $\mu_1 = \cdots = \mu_k = 0, k \geq 2, a > -1, b > -1$, and $c > 0$,

$$E\left(\frac{|u_1|^a |u_2|^b}{(\sum_{j=1}^k u_j^2)^c}\right) = 2^{(a+b-2c)/2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\frac{\Gamma\left(\frac{k+a+b-2c}{2}\right)}{\pi\Gamma\left(\frac{k+a+b}{2}\right)}$$

and is finite provided that $k + a + b > 2c$. When the μ_i's are not equal to zero for some i, then

(b) $E[|u_1|/(\sum_{i=1}^k u_i^2)] = \sqrt{2}E\{[\Gamma(1+Z_1)]/\Gamma((1+2Z_1)/2)(k+2Z-1)]\} < \infty$ if $k \geq 2$, and

(c) $E[|u_1||u_2|/(\sum_{i=1}^k u_i^2)^2] = 2E\{[\Gamma(1+Z_1)\Gamma(1+Z_2)]/[\Gamma((1+2Z_1)/2)\Gamma((1+2Z_2)/2)(k+2Z)(k+2Z-2)]\} < \infty$ if $k \geq 3$.

Proof. Note that when $\mu_1 = \cdots = \mu_k = 0$, the u_i^2's are independent with $u_i^2 \sim \chi^2_1$. Therefore, by writing $|u_i| = (u_i^2)^{1/2}$ and then applying Lemma A.1, we prove (a). When the μ_i's are not equal to zero, note that u_i^2 follows a noncentral chi-squared distribution with 1 df and noncentrality parameter μ_i^2, which can be expressed as a Poisson mixture of central chi-squared distributions. That is, $u_i^2 \sim \chi^2_{1+2Z_i}$, where $Z_i \sim \text{Poisson}(\lambda_i), i = 1, \ldots, k$. Using this fact and conditioning on Z_i, we can apply Lemma A.1 to get the expressions (b) and (c) given in the lemma.

Lemma A.3. Let $u_1, \ldots, u_k$ be independent random variables with $u_i \sim N(\mu_i, 1), i = 1, \ldots, k$. Then

(a) $E[1/(\sum_{i=1}^k u_i^2)] = E(k+2Z-2)^{-1}$ if $k \geq 3$,

(b) $E[u_i/(\sum_{i=1}^k u_i^2)] = \mu_i E(k+2Z)^{-1}$ if $k \geq 2$, and

(c) $E[(\sum_{i=1}^k u_i\mu_i)^2/(\sum_{i=1}^k u_i^2)^2] = 2\lambda E\{(2Z+1)/[(k+2Z)(k+2Z-2)]\}$ if $k \geq 3$,

where Z is a Poisson random variable with mean $\lambda = \sum_{i=1}^k \mu_i^2/2$.

Lemma A.4. Let $p_1, \ldots, p_k$ be nonnegative numbers such that $\sum_{i=1}^k p_i = 1$, and let $v_1, \ldots, v_k$ be independent chi-squared random variables with m df. Let $p \geq q \geq 0, p+q \geq 1, m \geq 2(p+1)$, and $t_1 = 2^{-p-q}\Gamma((m-2q)/2)\Gamma((m-2p)/2)/[\Gamma(m/2)]^2$. Then

$$t_1(m-2p-2)^{p+q} < E\left(\frac{v_i^{-p}v_j^{-q}}{(\sum_{l=1}^k p_l v_l^{-1})^{p+q}}\right) < t_1 2^{p+q}\Gamma((m+2p+2q)/2)/\Gamma(m/2).$$

Proof. Using Lemma A.1, we get

$$E\left(\frac{v_i^{-p}v_j^{-q}}{(\sum_{l=1}^k p_l v_l^{-1})^{p+q}}\right) = t_1 E\left(\sum_{l\neq i,j} p_l v_l^{-1} + p_i v_{0i}^{-1} + p_j v_{0j}^{-1}\right)^{-p-q}, \quad \text{(A.1)}$$

where $v_{0i} \sim \chi^2_{m-2p}$ and $v_{0j} \sim \chi^2_{m-2q}$. Because $p + q \geq 1$, it follows from Jensen's inequality that

$$\left(\sum_{l\neq i,j} p_l v_l^{-1} + p_i v_{0i}^{-1} + p_j v_{0j}^{-1}\right)^{-p-q}$$
$$< \left(\sum_{l\neq i,j} p_l v_l + p_i v_{0i} + p_j v_{0j}\right)^{p+q}$$
$$\leq \sum_{l\neq i,j} p_l v_l^{p+q} + p_i v_{0i}^{p+q} + p_j v_{0j}^{p+q}. \quad \text{(A.2)}$$

Substituting (A.2) in (A.1) and then using the result that $E(\sum_{l\neq i,j} p_l v_l^{p+q} + p_i v_{0i}^{p+q} + p_j v_{0j}^{p+q}) \leq E(v_l^{p+q})$, we get the desired upper bound. To get the lower bound, write $V = (\sum_{l\neq i,j} p_l v_l^{-1} + p_i v_{0i}^{-1} + p_j v_{0j}^{-1})$. Because V^{-p-q} is a convex function of V, using Jensen's inequality, we get

$$E(V^{-p-q}) > [E(V)]^{-p-q}$$
$$= \left(\sum_{l\neq i,j} p_l/(m-2) + p_i/(m-2p-2) + p_j/(m-2q-2)\right)^{-p-q}$$
$$\geq (m-2p-2)^{p+q}. \quad \text{(A.3)}$$

Thus we get the lower bound given in the lemma.

Proof of Theorem 2.1

Let $a_i = (\hat{\beta}_i/S_i^2)/(\sum_{i=1}^k \hat{\beta}_i^2/S_i^2), i = 1,\ldots,k$. It follows from (4) that

$$(\hat{x}_c - x_0)|\hat{\beta}_i's, S_i's \sim N\left((\bar{x}-x_0)\sum_{i=1}^k a_i(\hat{\beta}_i - \beta_i), c_n \sum_{i=1}^k a_i^2\sigma_i^2\right), \quad \text{(A.4)}$$

where $c_n = 1 + 1/n$. Let $u_i = \hat{\beta}_i\sqrt{S_{xx}}/\sigma_i, \mu_i = \beta_i\sqrt{S_{xx}}/\sigma_i$, and $v_i = (n-2)S_i^2/\sigma_i^2$ for $i = 1,\ldots,k$. Then $u_1,\ldots,u_k$ and $v_1,\ldots,v_k$ are all independent with $u_i \sim N(\mu_i, 1)$ and $v_i \sim \chi^2_{n-2}, i = 1,\ldots,k$. In terms of these new variables, it follows from (A.4) that

$$E(\hat{x}_c - x_0)$$
$$= (\bar{x}-x_0)E\left(\sum_{i=1}^k u_i(u_i-\mu_i)v_i^{-1} \Big/ \sum_{j=1}^k u_j^2 v_j^{-1}\right)$$
$$= (\bar{x}-x_0)\left(1 - E\left(\sum_{i=1}^k u_i\mu_i v_i^{-1} \Big/ \sum_{j=1}^k u_j^2 v_j^{-1}\right)\right). \quad \text{(A.5)}$$

It is clear from (A.5) that the absolute moment of $\hat{x}_c$ is finite if

$$E\left(\frac{|u_i|v_i^{-1}}{\sum_{j=1}^k u_j^2 v_j^{-1}}\right) = E\left(\frac{|u_i|v_i^{-1}}{(\sum_{j=1}^k u_j^2)(\sum_{j=1}^k p_j v_j^{-1})}\right) \quad \text{(A.6)}$$

is finite, where $p_i = u_i^2/(\sum_{j=1}^k u_j^2), i = 1,\ldots,k$. Now, conditioning on u_i's and applying Lemma A.4, we see that the left side of (A.5) is less than $(n-2)E(|u_i|/\sum_{j=1}^k u_j^2)/(n-4)$, which is finite provided that $k \geq 2$ (from Lemma A.2b) and $n \geq 5$. To find an expression for the bias of $\hat{x}_c$ we need to evaluate $E(\mu_i u_i v_i^{-1}/\sum_{j=1}^k u_j^2 v_j^{-1})$. For notational convenience, let us evaluate for $i = 1$. Note that the density function of u_1 can be written as $\exp(-\mu_1^2/2 + \mu_1 y)\phi(y)$, where $\phi(\cdot)$ denotes the standard normal density. Using this fact and letting $u_{01} \sim N(0,1)$, it can be checked that

$$E\left(\frac{\mu_1 u_1 v_1^{-1}}{\sum_{j=1}^k u_j^2 v_j^{-1}}\right)$$
$$= \mu_1\exp(-\mu_1^2/2)\sum_{l=0}^\infty E\,\frac{(\mu_1)^l u_{01}^{l+1}v_1^{-1}}{l!(u_{01}^2 + \sum_{j\neq 1} u_j^2)(\sum_{j=1}^k p_{0j}v_j^{-1})}$$
$$= \mu_1^2\exp(-\mu_1^2/2)$$
$$\times \sum_{k=0}^\infty E\,\frac{(\mu_1^2)^k(u_{01}^2)^{k+1}v_1^{-1}}{(2k+1)!(u_{01}^2 + \sum_{j\neq 1} u_j^2)(\sum_{j=1}^k p_{0j}v_j^{-1})}, \quad \text{(A.7)}$$

where $p_{0j} = u_j^2/(u_{01}^2 + \sum_{j\neq 1} u_j^2)$ for $j \neq 1$ and $p_{01} = 1 - \sum_{j\neq 1} p_{0j}$. To get the last equation, we used the fact that conditioning on v_i's and u_j's, $E[u_{01}^{l+1}v_i^{-1}/(u_{01}^2 + \sum_{j\neq 1} u_j^2)(\sum_{j=1}^k p_{0j}v_j^{-1})] = 0$ for $l = 0,2,4,\ldots$. Now we can apply Lemma A.4 to get bounds for the foregoing expectation. Indeed, conditioning on all the u's and applying Lemma A.4 to the right side of (A.7), we get

$$\frac{n-6}{n-4}E\left(\mu_1 u_1 \Big/ \sum_{j=1}^k u_j^2\right) < E\left(\mu_1 u_1 v_1^{-1} \Big/ \sum_{j=1}^k u_j^2 v_j^{-1}\right) < \frac{n-2}{n-4}E\left(\mu_1 u_1 \Big/ \sum_{j=1}^k u_j^2\right). \quad \text{(A.8)}$$

Thus it follows from (A.8) and Lemma A.3b that

$$\frac{n-6}{n-4}2\lambda E(k+2Z)^{-1} < E\left(\sum_{i=1}^k \mu_i u_i v_i^{-1} \Big/ \sum_{j=1}^k u_j^2 v_j^{-1}\right) < \frac{n-2}{n-4}2\lambda E(k+2Z)^{-1}, \quad \text{(A.9)}$$

where Z is a Poisson random variable with mean $\lambda = \sum_{i=1}^k \mu_i^2/2 = (S_{xx}/2)\sum_{i=1}^k \beta_i^2/\sigma_i^2$. Because $2\lambda E(k+2Z)^{-1} \leq 1$ for $k \geq 2$ (see Sec. 2), it follows from (A.9) that

$$E\left(\sum_{i=1}^k \mu_i u_i v_i^{-1} \Big/ \sum_{j=1}^k u_j^2 v_j^{-1}\right) = 2\lambda E(k+2Z)^{-1} + O(n^{-1}). \quad \text{(A.10)}$$

Using this expression in (A.5), we get the desired expression for the bias of $\hat{x}_c$ stated in Theorem 2.1.

To find the MSE of $\hat{x}_c$, we have from (A.4) and the definition of the a_i's,

$$E(\hat{x}_c - x_0)^2$$
$$= E[E(\hat{x}_c - x_0)^2|\hat{\beta}_i, S_i^2]$$
$$= E\left(c_n\frac{\sum_{i=1}^k \hat{\beta}_i^2\sigma_i^2/S_i^4}{(\sum_{j=1}^k \hat{\beta}_j^2/S_j^2)^2} + (\bar{x}-x_0)^2\frac{(\sum_{i=1}^k \hat{\beta}_i(\hat{\beta}_i - \beta_i)/S_i^2)^2}{(\sum_{j=1}^k \hat{\beta}_j^2/S_j^2)^2}\right)$$

$$= E\left(c_n S_{xx}\frac{\sum_{i=1}^k u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2} + (\bar{x}-x_0)^2\frac{(\sum_{i=1}^k u_i(u_i-\mu_i)v_i^{-1})^2}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right)$$

$$= c_n S_{xx} E\left[\frac{\sum_{i=1}^k u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right] + (\bar{x}-x_0)^2 \times E\left[1 + \frac{\sum_{i=1}^k \mu_i^2 u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2} + \frac{\sum_{i=1}^k \sum_{i\neq j}\mu_i\mu_j u_i u_j v_i^{-1} v_j^{-1}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2} - 2\frac{\sum_{i=1}^k \mu_i u_i v_i^{-1}}{\sum_{j=1}^k u_j^2 v_j^{-1}}\right]. \quad \text{(A.11)}$$

Now

$$E\left(\frac{\sum_{i=1}^k u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right) = E\left(\frac{\sum_{i=1}^k u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2)^2(\sum_{j=1}^k p_j v_j^{-1})^2}\right), \quad \text{(A.12)}$$

where the p_j's are defined as in (A.6). Thus, conditioning on the u_j's and applying Lemma A.4, we get

$$\frac{(n-8)^2}{(n-4)(n-6)} E\left(\sum_{j=1}^k u_j^2\right)^{-1} \le E\left(\frac{\sum_{i=1}^k u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right) \le \frac{n(n-2)}{(n-4)(n-6)} E\left(\sum_{j=1}^k u_j^2\right)^{-1}.$$

Using Lemma A.3a and the fact that $E(k+2Z-2)^{-1} < 1$ for $k \ge 3$, we get from the foregoing expression that

$$E\left(\sum_{i=1}^k u_i^2 v_i^{-2} \Big/ \left(\sum_{j=1}^k u_j^2 v_j^{-1}\right)^2\right) = E(k+2Z-2)^{-1} + O(n^{-1}). \quad \text{(A.13)}$$

Similarly, using Lemma A.4, it can be shown that

$$\frac{(n-8)^2}{(n-4)(n-6)} E\left(\frac{\sum_{i=1}^k \mu_i^2 u_i^2}{(\sum_{j=1}^k u_j^2)^2}\right) \le E\left(\frac{\sum_{i=1}^k \mu_i^2 u_i^2 v_i^{-2}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right) \le \frac{n(n-2)}{(n-4)(n-6)} E\left(\frac{\sum_{i=1}^k \mu_i^2 u_i^2}{(\sum_{j=1}^k u_j^2)^2}\right). \quad \text{(A.14)}$$

It follows from Lemma A.4 and Lemma A.2b that the third expectation in (A.11) is absolutely convergent. To find bounds for it, we can write as in (A.7) that

$$E\left(\frac{\mu_1\mu_2 u_1 u_2 v_1^{-1} v_2^{-1}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right) = \frac{\exp(-(\mu_1^2+\mu_2^2)/2)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\mu_1^2 u_{01}^2)^{k+1} \times (\mu_2^2 u_{02}^2)^{m+1} v_1^{-1} v_2^{-1}}{(2k+1)!(2m+1)!(u_{01}^2+u_{02}^2+\sum_{j\neq 1,2}u_j^2)^2 \times (\sum_{j=1}^k p_{0j} v_j^{-1})^2}, \quad \text{(A.15)}$$

where u_{01} and u_{02} are independent standard normal random variables, $p_{01} = u_{01}^2/(u_{01}^2+u_{02}^2+\sum_{j\neq 1,2}u_j^2)$, and p_{02} and the p_{0j}'s are defined similarly so that $\sum_{j=1}^k p_{0j} = 1$. Thus, conditional on all the u's in (A.15) and applying Lemma A.4, we get

$$\frac{(n-6)^2}{(n-4)^2} E\frac{\mu_1\mu_2 u_1 u_2}{(\sum_{j=1}^k u_j^2)^2} \le E\left(\frac{\mu_1\mu_2 u_1 u_2 v_1^{-1} v_2^{-1}}{(\sum_{j=1}^k u_j^2 v_j^{-1})^2}\right) \le \frac{n(n-2)}{(n-4)^2} E\frac{\mu_1\mu_2 u_1 u_2}{(\sum_{j=1}^k u_j^2)^2}. \quad \text{(A.16)}$$

Because $2\lambda E[(2Z+1)/((k+2Z)(k+2Z-2))] < 2\lambda E(k+2Z)^{-1} < 1$ for $k \ge 3$, it follows from (A.14), (A.16) and Lemma A.3c that

$$E\left(\sum_{i=1}^k \mu_i u_i v_i^{-1} \Big/ \sum_{j=1}^k u_j^2 v_j^{-1}\right)^2 = E\left(\sum_{i=1}^k \mu_i u_i \Big/ \sum_{j=1}^k u_j^2\right)^2 + O(n^{-1}) = 2\lambda E\left(\frac{2Z+1}{(k+2Z)(k+2Z-2)}\right) + O(n^{-1}). \quad \text{(A.17)}$$

We see from these derivations that all of the expectations given in (A.11) are finite provided that $n \ge 7$ and $k \ge 3$, and hence $\text{MSE}(\hat{x}_c)$ is finite under the stated conditions in Theorem 2.1. Finally, substituting (A.10), (A.13), and (A.17) in (A.11), we get the desired expression for the MSE.

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