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MINIMUM VARIANCE UNBIASED ESTIMATION IN SOME NONREGULAR FAMILIES

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ABSTRACT

Explicit expressions for the UMVUEs of the means, survival functions, quantiles of order p, and density functions are obtained for some one- and two-truncation parameter probability density functions. The UMVUE of the hazard function whenever it exists is also derived. Variances of some of these estimators are also computed.

1. INTRODUCTION

A distribution is said to be *nonregular* if its support depends on the unknown parameter(s). One- or two-truncation parameter probability distributions have been the subject of investigation by many authors. See for example Davis (1951), Hogg and Craig

(1956), Tate (1959), and more recently Guenther (1978), and Bar-Lev and Boukai (1985). For a reasonably complete account of the work done till 1970 see Zacks (1971), sections 3.5 and 3.6. Typically these authors obtain general expressions for uniformly minimum variance unbiased estimators (UMVUE) of U-estimable functions, that is, parametric functions which admit unbiased estimators. Some (such as Tate (1959) and Washio, Morimoto and Ikeda (1956)) rely on the use of integral transforms while others (Davis (1951) as well as Tate (1959)) obtain UMVUEs as a solution of a certain integral equation. See Guenther (1978) for a review of some of these methods.

In this paper we give explicit expresions for the UMVUEs of the means, survival functions, quantiles of order p, $0 < p < 1$, hazard functions (of f_1) and pdfs of the following three truncation parameter families of pdfs.

$$f_1(x;\theta) = q_1(\theta)\ h_1(x), \quad a < \theta < x < b, \tag{1.1}$$

$$f_2(x;\theta) = q_2(\theta)\ h_2(x), \quad a < x < \theta < b, \tag{1.2}$$

and

$$f(x;\theta_1,\theta_2) = q(\theta_1,\theta_2)\ h(x), \quad a < \theta_1 < x < \theta_2 < b, \tag{1.3}$$

where $-\infty \le a < b \le \infty$ are known, h_1, h_2, and h are positive absolutely continuous functions, q_1, q_2 are everywhere differentiable and $q(\theta_1,\theta_2)$ possesses continuous first order partial derivatives with respect to θ_1 and θ_2. We will also indicate how the variance of these estimators may be computed. We do not claim any originality in this work since we will be using the general expressions for UMVUEs derived by earlier authors. Our objective here is to provide a source of reference for easy access.

Section 2 deals with one truncation parameter pdfs and in section 3 we consider the two parameter case.

2. ONE-TRUCATION PARAMETER

Let $X_1, X_2, \ldots, X_n$ be a random sample on pdf f_i given in (1.1) or (1.2) and let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the set of ordered observations. It is well known (Tate (1959)) that $X_{1:n}$ is a complete sufficient statistic for the family $\{f_1(x;\theta)\}$ and $X_{n:n}$ is a complete sufficient statistic for the family $\{f_2(x:\theta)\}$. Let $g(\theta)$ be an absolutely continuous U-estimable function. Tate (1959) showed that the UMVUE for $g(\theta)$ for the family $\{f_1\}$ is given by

$$\varphi(X_{1:n}) = g(X_{1:n}) - \frac{g'(X_{1:n})}{nq_1(X_{1:n})\, h_1(X_{1:n})}, \tag{2.1}$$

and that for $\{f_2\}$ is given by

$$\varphi(X_{n:n} = g(X_{n:n}) + \frac{g'(X_{n:n})}{nq_2(X_{n:n})\, h_2(X_{n:n})} \tag{2.2}$$

where $g'(\theta) = \partial g(\theta)/\partial\theta$.

Let

$$\mu_i(\theta) = \int_a^b x f_i(x;\theta)dx, \quad i = 1,2, \tag{2.3}$$

$$\overline{F}_1(t;\theta) = \begin{cases} \int_t^b f_1(x;\theta)dx & \text{if } a < \theta < t < b \\ 1 & \text{if } a < t \le \theta < b, \end{cases} \tag{2.4}$$

$$\overline{F}_2(t;\theta) = \begin{cases} 0 & \text{if } a < \theta \le t < b \\ \int_t^\theta f_2(x;\theta)dx & \text{if } a < t < \theta < b, \end{cases} \tag{2.5}$$

and for $a < t < b$

$$\Lambda_i(t;\theta) = -\ln \overline{F}_i(t;\theta), \quad i = 1,2. \tag{2.6}$$

We also let $Q_{p,i} \equiv Q_{p,i}(\theta)$ to be the quantile of order p, $0 < p < 1$ for f_i, $i = 1,2$. Then

$$p = \int_{\theta}^{Q_{p,1}} f_1(x;\theta)dx, \quad \text{and} \quad p = \int_{a}^{Q_{p,2}} f_2(x;\theta)dx. \tag{2.7}$$

Then simple computations using (2.1) and (2.2) show that the UMVUEs of μ_1, $\overline{F}_1$, Λ_1, $Q_{p,1}$ and f_1 are given, respectively, by

$$\hat{\mu}_1(\theta) = \frac{n-1}{n}\mu_1(X_{1:n}) + \frac{X_{1:n}}{n} \tag{2.8}$$

$$\hat{\overline{F}}_1(t;\theta) = \begin{cases} 1, & \text{if } a < t \leq X_{1:n} < b \\ \frac{n-1}{n}\overline{F}_1(t;X_{1:n}), & \text{if } a < X_{1:n} < t < b, \end{cases} \tag{2.9}$$

$$\hat{\Lambda}_1(t;\theta) = \begin{cases} \Lambda_1(t;X_{1:n}) + \frac{1}{n}, & \text{if } a < X_{1:n} < t < b \\ 0, & \text{if } a < t \leq X_{1:n} < b, \end{cases} \tag{2.10}$$

$$\hat{Q}_{p,1}(\theta) = Q_{p,1}(X_{1:n}) - \frac{1-p}{nq_1(X_{1:n})\,h_1(Q_{p,1}(X_{1:n}))}, \tag{2.11}$$

and

$$\hat{f}_1(x;\theta) = \begin{cases} \left(\frac{n-1}{n}\right) f_1(x;X_{1:n}), & \text{if } X_{1:n} < x < b \\ \frac{1}{n}, & \text{if } x = X_{1:n} \\ 0, & \text{elsewhere.} \end{cases} \tag{2.12}$$

The corresponding UMVUEs of μ_2, $\overline{F}_2$, $Q_{p,2}$ and F_2 are given by

$$\hat{\mu}_2(\theta) = \frac{n-1}{n}\mu_2(X_{n:n}) + \frac{X_{n:n}}{n} \tag{2.13}$$

$$\hat{\overline{F}}_2(t;\theta) = \begin{cases} 0, & a < X_{n:n} \leq t < b, \\ \frac{1}{n} + \left(\frac{n-1}{n}\right)\overline{F}(t;X_{n:n}), & a < t < X_{n:n} < b, \end{cases} \tag{2.14}$$

$$\hat{Q}_{p,2}(\theta) = Q_{p,2}(X_{n:n}) + \frac{p}{nq_2(X_{n:n})\,h_2(Q_{p,2}(X_{n:n}))}. \tag{2.15}$$

and

$$\hat{f}_2(x;\theta) = \begin{cases} \left(\dfrac{n-1}{n}\right) f_2(x;X_{n:n}), & a < x < X_{n:n} \\ \dfrac{1}{n}, & x = X_{n:n} \\ 0, & \text{elsewhere}. \end{cases} \tag{2.16}$$

Since $\Lambda_2(t;\theta) = -\ln \overline{F}_2(t;\theta)$ it is clear that Λ_2 is not U-estimable.

The variance of the UMVUE in (2.1) (or (2.2)) can be computed directly but the expression is complicated. It is, of course, easier to compute the variance of the specific estimators directly. We indicate this computation by computing the variance $\hat{\overline{F}}_1(t;\theta)$. Since the pdf of $X_{1:n}$ is given by

$$f_{1:n}(x;\,\theta) = nq_1^n(\theta)\, h_1(x) \left\{\int_x^b h_1(u)\, du\right\}^{n-1}, \quad \theta < x < b, \tag{2.17}$$

we have for $\theta < t < b$

$$\mathcal{E}[\hat{\overline{F}}_1(t;\,\theta)]^2 = P\{X_{1:n} \geq t\} + \left(\frac{n-1}{n}\right)^2 \int_\theta^t \overline{F}_1{}^2(t;x)\; nq_1^n(\theta)\, h_1(x) \cdot \left\{\int_x^b h_1(u)\, du\right\}^{n-1} dx. \tag{2.18}$$

Substituting $\overline{F}_1(t;x) = q_1(x)/q_1(t)$ we see that the integral on the right side of (2.18) is given by

$$\begin{aligned} I(t;\,\theta) &= \int_\theta^t \frac{q_1^2(x)}{q_1^2(t)}\, nq_1^n(\theta)\, h_1(x) \left\{\int_x^b h_1(u)du\right\}^{n-1} dx \\ &= \frac{nq_1^n(\theta)}{q_1^2(t)} \int_\theta^t h_1(x) \left\{\int_x^b h_1(u)du\right\}^{n-3} dx \quad \left(q_1(x) = \left\{\int_x^b h_1(u)\, du\right\}^{-1}\right) \\ &= \frac{n}{n-2}\,[\overline{F}^{\,2}(t;\theta) - \overline{F}^{\,n}(t;\theta)], \end{aligned}$$

provided $n > 2$.

It follows that for $n > 2$,

$$\mathrm{var}[\hat{\bar{F}}_1(t;\theta)] = \frac{1}{n(n-2)}\{\bar{F}_1{}^2(t;\theta) - \bar{F}_1{}^n(t;\theta)\}. \qquad (2.19)$$

For $n = 2$ on the otherhand

$$I(t;\theta) = \frac{2q_1^2(\theta)}{q_1^2(t)}\int_\theta^t h_1(x)\left\{\int_x^b h_1(u)\,du\right\}^{-1} dx = -2\bar{F}_1{}^2(t;\theta)\ \ln\bar{F}_1(t;\theta)$$

so that

$$\mathrm{var}(\hat{\bar{F}}_1(t;\theta)] = -\frac{1}{2}\bar{F}_1{}^2(t;\theta)\ \ln \bar{F}_1(t;\theta). \qquad (2.20)$$

Similar computations show that

$$\mathrm{var}[\hat{\Lambda}_1(t;\theta)]\ [1 - q_1^n(\theta)/q_1^n(t)]/n^2 \qquad (2.21)$$

$$\mathrm{var}[\hat{f}_i(x;\theta)] = f_i(x;\theta)/[n(n-2)],\ \ n > 2,\ \ i = 1,2, \qquad (2.22)$$

and

$$\mathrm{var}[\hat{\bar{F}}_2(t;\theta)] = \begin{cases}\bar{F}_2{}^2(t;\theta) - \bar{F}_2{}^n(t;\theta)]/[n(n-2)], & n > 2\\ -\{\bar{F}_2{}^2(t;\theta)\ \ln \bar{F}_2(t;\theta)\}/2, & n = 2.\end{cases} \qquad (2.23)$$

EXAMPLE 2.1. *Let* $f_1(x;\theta) = (e^{-\theta} - e^{-1})e^{-x}$, $\theta < x < 1$. *Then the UMVUEs of* μ_1, $\bar{F}_1$, Λ_1, Q_p *and* f_1 *are given by*

$$\hat{\mu}_1(\theta) = \frac{X_{1:n}}{n} + \left(\frac{n-1}{n}\right)\left(1 + \frac{X_{1:n}e^{-X_{1:n}} - e^{-1}}{e^{-X_{1:n}} - e^{-1}}\right).$$

$$\hat{\bar{F}}_1(t;\theta) = \begin{cases}1, & t \le X_{1:n} < 1\\ \left(\frac{n-1}{n}\right)\left(\frac{e^{-t} - e^{-1}}{e^{-X_{1:n}} - e^{-1}}\right), & X_{1:n} < t < 1,\end{cases}$$

$$\hat{\Lambda}_1(t;\theta) = \begin{cases}0, & t \le X_{1:n} < 1\\ \frac{1}{n} + \ln(e^{-X_{1:n}} - e^{-1}) - \ln(e^{-t} - e^{-1}), & X_{1:n} < t < 1,\end{cases}$$

$$\hat{Q}_{p,1}(\theta) = -\ln\{e^{-X_{1:n}}(1-p) + pe^{-1}\} - \frac{(1-p)(e^{-X_{1:n}} - e^{-1})}{n[(1-p)e^{-X_{1:n}} + pe^{-1}]},$$

and

$$f_1(x;\theta) = \begin{cases} \frac{n-1}{n} e^{-x}(e^{-X_{1:n}} - e^{-1})^{-1}, & X_{1:n} < x < 1 \\ \frac{1}{n}, & x = X_{1:n} \\ 0, & \text{elsewhere.} \end{cases}$$

Also

$$\mathrm{var}[\hat{\bar{F}}_1(t;\theta)] = \begin{cases} \frac{1}{n(n-2)} \left\{ \left(\frac{e^{-t} - e^{-1}}{e^{-\theta} - e^{-1}}\right)^2 - \left(\frac{e^{-t} - e^{-1}}{e^{-\theta} - e^{-1}}\right)^n \right\}, & n > 2 \\ -\frac{1}{2}\left(\frac{e^{-t} - e^{-1}}{e^{-\theta} - e^{-1}}\right)^2 \ln\left(\frac{e^{-t} - e^{-1}}{e^{-\theta} - e^{-1}}\right), & n = 2, \end{cases}$$

and

$$\mathrm{var}[\hat{\Lambda}_1(t;\theta)] = \frac{1}{n^2}\left\{1 - \left(\frac{e^{-t} - e^{-1}}{e^{-\theta} - e^{-1}}\right)^n\right\}.$$

3. TWO-TRUNCATION PARAMETERS

In this section we assume that $X_1, X_2, \ldots, X_n$ is a random sample from pdf f given in (1.3). Hogg and Craig (1956) showed that $(X_{1:n}, X_{n:n})$ is a complete sufficient statistic for the family $\{f(x;\theta_1,\theta_2), (\theta_1,\theta_2) \subset (a,b)\}$. Let $g(\theta_1,\theta_2)$ be a U-estimable function. According to Bar-Lev and Boukai (1985), the UMVUE of $g(\theta_1,\theta_2)$ is given by

$$\begin{aligned}\hat{g}(\theta_1,\theta_2) = g(X_{1:n}, X_{n:n}) &+ \frac{g_2(X_{1:n},X_{n:n})}{(n-1)\,h(X_{n:n})\,q(X_{1:n},X_{n:n})} \\ &- \frac{g_1(X_{1:n},X_{n:n})}{(n-1)\,h(X_{1:n})\,q(X_{1:n},X_{n:n})} \\ &- \frac{g_{12}(X_{1:n},X_{n:n})}{n(n-1)\,h(X_{1:n})\,h(X_{n:n})\,q^2(X_{1:n},X_{n:n})}, \qquad (3.1)\end{aligned}$$

where $g_j(\theta_1,\theta_2) = \partial g(\theta_1,\theta_2)/\partial\theta_j$, $j = 1,2$ and $g_{12} = \partial^2 g(\theta_1,\theta_2)/\partial\theta_1\theta_2$. Let $\mu(\theta_1,\theta_2) = \int_{\theta_1}^{\theta_2} x f(x;\theta_1,\theta_2)dx$, $\overline{F_1}(t;\theta_1,\theta_2) = P(X_1 > t)$, and $Q_p \equiv Q_p(\theta_1,\theta_2)$ be the quantile of order p given by

$$p = q(\theta_1,\theta_2)\int_{\theta_1}^{Q_p} h(x)\,dx.$$

Then simple computations using (3.1) lead to the following UMVUEs of μ, $\overline{F}$ for $n \geq 3$

$$\hat{\mu}(\theta_1,\theta_2) = \frac{n-2}{n}\,\mu(X_{1:n},X_{n:n}) + \frac{X_{1:n}+X_{n:n}}{n}, \tag{3.2}$$

$$\hat{\overline{F}}(t;\theta_1,\theta_2) = \begin{cases} 1, & a < t < X_{1:n} \\ \frac{n-2}{n}\,\overline{F}(t;X_{1:n},X_{n:n}) + \frac{1}{n}, & X_{1:n} < t < X_{n:n} \\ 0, & X_{n:n} \leq t < b, \end{cases} \tag{3.3}$$

and that of Q_p for $n > 2$ by

$$\hat{Q}_p(\theta_1,\theta_2) = Q_p(X_{1:n},X_{n:n}) + \frac{2p-1}{(n-1)\,q(X_{1:n},X_{n:n})\,h(Q_p(X_{1:n},X_{n:n}))} + \frac{p(1-p)\dfrac{\partial h(Q_p)}{\partial Q_p}}{n(n-1)\,q^2(X_{1:n},X_{n:n})\,h^3(Q_p(X_{1:n},X_{n:n}))}. \tag{3.4}$$

Clearly Λ does not admit an unbiased estimator. The UMVUE of f for $n > 2$ is given by

$$\hat{f}(x;\theta_1,\theta_2) = \begin{cases} \frac{1}{n}, & x = X_{1:n}, \text{ or } x = X_{n:n} \\ \frac{n-2}{n}\,f(x;X_{1:n},X_{n:n}), & X_{1:n} < x < X_{n:n} \\ 0, & \text{elsewhere.} \end{cases} \tag{3.5}$$

Bar-Lev and Boukai (1985) compute the variance of $\hat{\bar{F}}(t;\theta_1,\theta_2)$. The same procedure can be used to compute the variances of the other estimators considered above.

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