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Confidence regions for the common mean vector of several multivariate normal populations

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ABSTRACT

Suppose that there are independent samples available from several multivariate normal populations with the same mean vector $\boldsymbol{\mu}$ but possibly different covariance matrices. The problem of developing a confidence region for the common mean vector based on all the samples is considered. An exact confidence region centered at a generalized version of the well-known Graybill-Deal estimator of $\boldsymbol{\mu}$ is developed, and a multiple comparison procedure based on this confidence region is outlined. Necessary percentile points for constructing the confidence region are given for the two-sample case. For more than two samples, a convenient method of approximating the percentile points is suggested. Also, a numerical example is presented to illustrate the methods. Further, for the bivariate case, the proposed confidence region and the ones based on individual samples are compared numerically with respect to their expected areas. The numerical results indicate that the new confidence region is preferable to the single-sample versions for practical use.

RÉSUMÉ

Supposons qu'il y a des échantillons indépendants provenant de plusieurs populations à plusieurs variables normales avec le même vecteur moyen $\boldsymbol{\mu}$ mais possiblement différentes matrices des covariances. Cette article considère le problème de développer une région de confiance pour le vecteur moyen commun basée sur tous les échantillons. Une région de confiance exacte centrée sur un version généralisée du fameux estimateur de $\boldsymbol{\mu}$ de Graybill-Deal est développée et une procédure de comparaisons multiples basée sur cette région de confiance est évoquée. Les points percentiles nécessaires à la construction de la région de confiance sont donnés pour le cas où il y a deux échantillons. Dans le cas où il y a plus de deux échantillons, une méthode pratique d'approximation des points percentiles est suggérée. Aussi, un exemple numérique est présenté afin d'illustrer les méthodes. De plus, pour le cas bidimensionnel, la région de confiance proposée et celles basées sur des échantillons individuels sont comparées numériquement quant à leur aires probables. Les résultats numériques indiquent que la nouvelle région de confiance est préférable aux versions d'échantillon unique pour un usage pratique.

1. INTRODUCTION

Suppose that there are k p-variate normal populations with common mean vector $\boldsymbol{\mu}$ and unknown covariance matrices $\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_k$. Further, let $\mathbf{X}_{i1}, \ldots, \mathbf{X}_{in_i}$ be independent vector observations from the ith population, $i = 1, \ldots, k$. Assume that the X_{ij}'s are independent for all i and j. Let

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i}\mathbf{X}_{ij} \quad \text{and} \quad \mathbf{S}_i = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(\mathbf{X}_{ij}-\bar{\mathbf{X}}_i)(\mathbf{X}_{ij}-\bar{\mathbf{X}}_i)^{\mathsf{T}} \tag{1.1}$$

for $i = 1,\dots,k$. We here consider the problem of estimating the common mean vector $\boldsymbol{\mu}$ based on the minimal sufficient statistics $(\bar{\mathbf{X}}_1,\dots,\bar{\mathbf{X}}_k, \mathbf{S}_1,\dots,\mathbf{S}_k)$. These statistics are mutually independent with the following distributions:

$$\bar{\mathbf{X}}_i \sim \mathsf{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_i/n_i) \quad \text{and} \quad m_i\mathbf{S}_i \sim W_p(\boldsymbol{\Sigma}_i, m_i),$$

where $m_i = n_i - 1$, $\mathsf{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_i/n_i)$ denotes the p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}_i/n_i$, and $W_p(\boldsymbol{\Sigma}_i, m_i)$ denotes the Wishart distribution with parameter matrix $\mathcal{E}S_i = \boldsymbol{\Sigma}_i$ and degrees of freedom m_i, $i = 1,\dots,k$.

This problem, when $p = 1$, has received considerable attention in the statistical literature. A landmark paper in this area is due to Graybill and Deal (1959), who first proved that, when $k = 2$, the combined estimator

$$\hat{\boldsymbol{\mu}}_{\mathrm{GD}} = \left(\sum_{i=1}^{k} n_i\mathbf{S}_i^{-1}\right)^{-1}\sum_{j=1}^{k} n_j\mathbf{S}_j^{-1}\bar{\mathbf{X}}_j \tag{1.2}$$

has lower variance than either sample mean, provided both sample sizes are larger than 10. This result is further generalized and extended to the k-sample case by Khatri and Shah (1974), Norwood and Hinkelmann (1977), Bhattacharya (1984) and a host of others. Further, several confidence intervals have been proposed for $\boldsymbol{\mu}$ in the present context as well as in the context of interblock analysis of a balanced incomplete block design (BIBD). We refer the reader to Brown and Cohen (1974), Maric and Graybill (1979), Khatri and Shah (1981), and Jordan and Krishnamoorthy (1994) and the references therein.

In the multivariate case, Chiou and Cohen (1985) considered point estimation of $\boldsymbol{\mu}$ and showed that, when $k = 2$, $\hat{\boldsymbol{\mu}}_{\mathrm{GD}}$ dominates neither $\bar{\mathbf{X}}_1$ nor $\bar{\mathbf{X}}_2$ with respect to the covariance criterion. Loh (1991) suggested some combined estimators of $\boldsymbol{\mu}$ as alternatives to $\hat{\boldsymbol{\mu}}_{\mathrm{GD}}$ under a symmetric loss function. For other related work in point estimation of $\boldsymbol{\mu}$, we refer the reader to Kubokawa (1989), George (1991) and Krishnamoorthy (1991).

Recently, Zhou and Mathew (1993, 1994) proposed several combined tests for testing $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ which have better power properties than the traditional Fisher test. However, the problem of multiple comparison, which arises when the null hypothesis is rejected, is not addressed in the literature. Therefore, in order to understand the location of the unknown $\boldsymbol{\mu}$ and also for multiple comparison purposes, one needs a confidence region for $\boldsymbol{\mu}$. Since, as a point estimator, $\hat{\boldsymbol{\mu}}_{\mathrm{GD}}$ has some appealing features such as being unbiased, symmetric and affine invariant, developing a confidence region centered at an estimator of this form seems to be a worthwhile objective. This is what we accomplish in this article.

In the next section, we develop first a confidence region for $\boldsymbol{\mu}$ which is centered at

$$\hat{\boldsymbol{\mu}} = \left(\sum_{i=1}^{k} c_i n_i S_i^{-1}\right)^{-1}\sum_{j=1}^{k} c_j n_j S_j^{-1}\bar{X}_j, \tag{1.3}$$

where the c_i's are some positive constants such that $\sum_{i=1}^{k} c_i = 1$. This is a multivariate extension of the univariate result by Jordan and Krishnamoorthy (1994). Incidentally, we also find a conservative test for the multivariate Behrens-Fisher problem while developing this confidence region. The percentile points that are needed to construct the confidence region for the cases $k = 2, p = 2, 3, 4$ are given. For $k \geq 3$, we suggest a simple method to approximate these percentile points. In Section 3, we outline a multiple comparison procedure based on this confidence region. The statistical procedures developed in this article are demonstrated by a numerical example in Section 4. The confidence regions centered at the estimators (1.3) and the ones based on the individual samples are compared numerically, for the case $k = 2$ and $p = 2$, with respect to their expected surface areas. This is the content of Section 5.

2. CONFIDENCE REGION FOR $\boldsymbol{\mu}$

We first note that the estimator (1.3) is the weighted least-squares estimator in the sense that it minimizes

$$\sum_{i=1}^{k} c_i n_i (\bar{\mathbf{X}}_i - \boldsymbol{\mu})^{\mathsf{T}} \mathbf{S}_i^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}) \tag{2.1}$$

with respect to $\boldsymbol{\mu}$. Therefore, if we can find a number a such that

$$\sum_{i=1}^{k} c_i n_i (\bar{\mathbf{X}}_i - \boldsymbol{\mu})^{\mathsf{T}} \mathbf{S}_i^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}) \leq a \tag{2.2}$$

holds with probability $1 - \alpha$, then the values of $\boldsymbol{\mu}$ that satisfying the inequality (2.2) constitute a $100(1 - \alpha)\%$ confidence region with center $\hat{\boldsymbol{\mu}}$. Using this fact, we deduce from (2.2) an exact confidence region for $\boldsymbol{\mu}$ in the following theorem.

The following lemma, which is needed to prove Theorem 2.1, can be easily verified.

LEMMA 2.1. *Let $A_1, \ldots, A_k$ be square matrices and $C = \sum_{i=1}^{k} A_i$. If C^{-1} exists, then*

$$A_i C^{-1} \sum_{j \neq i}^{k} A_j = \sum_{j \neq i}^{k} A_j C^{-1} A_i.$$

For notational convenience in the following, we write $\mathbf{W}_i^{-1} = c_i n_i \mathbf{S}_i^{-1}$ and $\mathbf{V} = \sum_{i=1}^{k} \mathbf{W}_i^{-1}$. Also, let $F_{m,n}$ denote the F random variable with m and n degrees of freedom.

THEOREM 2.1. *Let a be the number such that the inequality $\sum_{i=1}^{k} c_i T_i^2 \leq a$ holds with probability $1 - \alpha$, where $T_1^2, \ldots, T_k^2$ are independent Hotelling -T^2 statistics with $T_i^2 \sim \{m_i p/(m_i - p + 1)\} F_{p, m_i - p + 1}$. Then the set of values of $\boldsymbol{\mu}$ satisfying the inequality*

$$(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathsf{T}} \mathbf{V} (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})$$
$$\leq a - \sum_{i=1}^{k} \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \left(\sum_{j \neq i}^{k} \mathbf{W}_j^{-1} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j) \right) \tag{2.3}$$

is a $100(1 - \alpha)\%$ confidence region for μ.

Proof. It is easy to see that $\hat{\boldsymbol{\mu}}$ in (1.3) can be written as

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}}_i - \mathbf{V}^{-1} \sum_{j \neq i}^{k} \mathbf{W}_j^{-1} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j), \qquad i = 1, \ldots, k. \tag{2.4}$$

Using this expression, we can write

$$
\begin{aligned}
&(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathsf{T}} \mathbf{V} (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}) \\
&\quad = \sum_{i=1}^{k} \Bigg\{ (\bar{\mathbf{X}}_i - \boldsymbol{\mu})^{\mathsf{T}} \mathbf{W}_i^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}) \\
&\qquad - 2 \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} (\bar{\mathbf{X}}_i - \boldsymbol{\mu}) \\
&\qquad + \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \left(\sum_{j \neq i}^{k} \mathbf{W}_j^{-1} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j) \right) \Bigg\}. \qquad (2.5)
\end{aligned}
$$

The coefficient of $\boldsymbol{\mu}$ in the second term of (2.5) is

$$
\begin{aligned}
&\sum_{i=1}^{k} \sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \mathbf{V}^{-1} \mathbf{W}_i^{-1} \\
&\qquad = \sum_{i=1}^{k} \sum_{j \neq i}^{k} \left(\bar{\mathbf{X}}_i^{\mathsf{T}} \mathbf{W}_j^{-1} \mathbf{V}^{-1} \mathbf{W}_i^{-1} - \sum_{j \neq i}^{k} \bar{\mathbf{X}}_j^{\mathsf{T}} \mathbf{W}_j^{-1} \mathbf{V}^{-1} \mathbf{W}_i^{-1} \right) \\
&\qquad = \sum_{i=1}^{k} \sum_{j \neq i}^{k} \left(\bar{\mathbf{X}}_i^{\mathsf{T}} \mathbf{W}_j^{-1} \mathbf{V}^{-1} \mathbf{W}_i^{-1} - \sum_{j \neq i}^{k} \bar{\mathbf{X}}_i^{\mathsf{T}} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \mathbf{W}_j^{-1} \right) \\
&\qquad = \sum_{i=1}^{k} \left(\bar{\mathbf{X}}_i^{\mathsf{T}} \sum_{j \neq i}^{k} \mathbf{W}_j^{-1} \mathbf{V}^{-1} \mathbf{W}_i^{-1} - \bar{\mathbf{X}}_i^{\mathsf{T}} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \sum_{j \neq i}^{k} \mathbf{W}_j^{-1} \right), \qquad (2.6)
\end{aligned}
$$

which is equal to the zero vector by Lemma 2.1. Next, using the relation that $\mathbf{V}\bar{\mathbf{X}}_i = \sum_{j \neq i}^{k} \mathbf{W}_j^{-1}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j) + \sum_{j=1}^{k} \mathbf{W}_j^{-1}\bar{\mathbf{X}}_j$, we see that

$$
\begin{aligned}
&\sum_{i=1}^{k} \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} \bar{\mathbf{X}}_i \\
&\quad = \sum_{i=1}^{k} \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \left(\sum_{j \neq i}^{k} \mathbf{W}_j^{-1} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j) \right) \\
&\qquad + \sum_{i=1}^{k} \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \left(\sum_{j=1}^{k} \mathbf{W}_j^{-1} \bar{\mathbf{X}}_j \right) \\
&\quad = \sum_{i=1}^{k} \left(\sum_{j \neq i}^{k} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}} \mathbf{W}_j^{-1} \right) \mathbf{V}^{-1} \mathbf{W}_i^{-1} \mathbf{V}^{-1} \left(\sum_{j \neq i}^{k} \mathbf{W}_j^{-1} (\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j) \right).
\end{aligned}
$$

This last equality follows from the fact that $\sum_{i=1}^{k} \{\sum_{j \neq i}^{k} (\bar{X}_i - \bar{X}_j)^{\mathsf{T}} W_j^{-1}\} V^{-1} W_i^{-1} = 0$, as shown in (2.6). Thus we have shown that the second term in the rhs of (2.5) is equal to -2 times the third term in the rhs of (2.5), and hence

$$(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathsf{T}}\mathbf{V}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})$$

$$= \sum_{i=1}^{k}(\bar{\mathbf{X}}_i - \boldsymbol{\mu})^{\mathsf{T}}\mathbf{W}_i^{-1}(\bar{\mathbf{X}}_i - \boldsymbol{\mu})$$

$$- \sum_{i=1}^{k}\left(\sum_{j\neq i}^{k}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}}\mathbf{W}_j^{-1}\right)\mathbf{V}^{-1}\mathbf{W}_i^{-1}\mathbf{V}^{-1}\left(\sum_{j\neq i}^{k}\mathbf{W}_j^{-1}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)\right). \quad (2.7)$$

We now complete the proof by noting that $\sum_{i=1}^{k}(\bar{\mathbf{X}}_i - \boldsymbol{\mu})^{\mathsf{T}}\mathbf{W}_i^{-1}(\bar{\mathbf{X}}_i - \boldsymbol{\mu})$ is distributed as $\sum_{i=1}^{k} c_i T_i^2$. □

The weights c_i can be chosen as inversely proportional to $\mathcal{V}ar(T_i^2) = 2pm_i^2(m_i - 1)/\{(m_i - p - 1)^2(m_i - p - 3)\}$, $i = 1, \ldots, k$. Again, to satisfy the restriction that $\sum_{i=1}^{k} c_i = 1$, a reasonable choice of c_i is

$$c_i = \frac{\{\mathcal{V}ar(T_i^2)\}^{-1}}{\sum_{j=1}^{k}\{\mathcal{V}ar(T_j^2)\}^{-1}}. \quad (2.8)$$

For $k = 2$, the percentile point a can be obtained numerically as the solution of the equation

$$\Pr(c_{10}F_{p,m_1-p+1} + c_{20}F_{p,m_2-p+1} \leq a) = 1 - \alpha, \quad (2.9)$$

where $c_{i0} = c_i m_i p/(m_i - p + 1)$, $i = 1, 2$. The exact 95th and 99th percentile points of $c_1T_1^2 + c_2T_2^2$ with c_i given in (2.8) are presented respectively in Tables 1 and 2 for $p = 2$, 3 and 4. For $k \geq 3$, it is difficult to compute the exact percentile points; however, as suggested in Jordan and Krishnamoorthy (1994), they can be approximated by the distribution of $dF_{kp,\nu}$. The unknown positive constants d and ν can be estimated in the usual way by equating the first two moments of $\sum_{i=1}^{k} c_i T_i^2$ to those of $dF_{kp,\nu}$. If $\min\{m_i\} > p + 3$, then the estimated values are

$$\nu = \frac{4M_2kp - 2M_1^2(kp+2)}{M_2kp - M_1^2(kp+2)} \quad \text{and} \quad d = M_1\frac{\nu - 2}{\nu}, \quad (2.10)$$

where

$$M_1 = p\sum_{i=1}^{k}\frac{c_i m_i}{m_i - p - 1}$$

and

$$M_2 = p(p+2)\sum_{i=1}^{k}\frac{c_i^2 m_i^2}{(m_i - p - 1)(m_i - p - 3)} + 2p^2\sum_{i>j}\frac{c_i c_j m_i m_j}{(m_i - p - 1)(m_j - p - 1)}.$$

This approximation is simple to use and also gives very satisfactory results (see Figure 1) except in the cases where one of the sample sizes is very small compared to the other. In these cases, the estimated points are usually larger than the exact percentile points (Jordan and Krishnamoorthy 1994).

REMARK 2.1. The confidence region is nonempty provided the rhs of (2.3) is positive. That is,

$$\sum_{i=1}^{k}\left(\sum_{j\neq i}^{k}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathsf{T}}\mathbf{W}_j^{-1}\right)\mathbf{V}^{-1}\mathbf{W}_i^{-1}\mathbf{V}^{-1}\left(\sum_{j\neq i}^{k}\mathbf{W}_j^{-1}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)\right) \leq a. \quad (2.11)$$

TABLE 1: 95th-percentile points of $c_1 T_1^2 + c_2 T_2^2$.

$p = 2$

m_1	$m_2 = 6$	7	8	9	10	11	12	13
6	11.824							
7	10.795	10.038						
8	10.023	9.439	8.957					
9	9.443	8.972	8.568	8.235				
10	8.999	8.603	8.255	7.693	7.721			
11	8.652	8.309	8.001	7.740	7.521	7.338		
12	8.373	8.069	7.792	7.554	7.352	7.183	7.041	
13	8.144	7.871	7.618	7.398	7.211	7.052	6.918	6.804
14	7.954	7.705	7.471	7.265	7.090	6.941	6.814	6.705
15	7.794	7.563	7.345	7.152	6.985	6.844	6.724	6.619
20	7.265	7.092	6.919	6.763	6.627	6.509	6.407	6.319
25	6.971	6.824	6.676	6.538	6.418	6.313	6.221	6.142
30	6.784	6.653	6.518	6.393	6.282	6.184	6.099	6.025
40	6.560	6.448	6.328	6.215	6.115	6.026	5.949	5.881
50	6.431	6.328	6.217	6.112	6.017	5.933	5.860	5.796
∞	5.949	5.877	5.794	5.714	5.641	5.574	5.516	5.464

m_1	$m_2 = 14$	15	20	25	30	40	50	∞
14	6.611							
15	6.530	6.451						
20	6.243	6.176	5.938					
25	6.072	6.012	5.794	5.661				
30	5.960	5.903	5.698	5.573	5.489			
40	5.821	5.769	5.579	5.462	5.384	5.285		
50	5.739	5.689	5.508	5.396	5.321	5.226	5.169	
∞	5.418	5.377	5.228	5.134	5.070	4.990	4.941	4.745

$p = 3$

m_1	$m_2 = 7$	8	9	10	11	12	13
7	19.371						
8	17.324	15.962					
9	15.757	14.802	13.923				
10	14.585	13.875	13.184	12.581			
11	13.691	13.137	12.576	12.069	11.631		
12	12.998	12.549	12.078	11.641	11.257	10.927	
13	12.446	12.070	11.665	11.282	10.940	10.642	10.385
14	11.997	11.676	11.320	10.978	10.669	10.397	10.161
15	11.627	11.345	11.028	10.718	10.436	10.186	9.967
20	10.448	10.277	10.065	9.847	9.643	9.457	9.291
25	9.825	9.700	9.534	9.359	9.191	9.036	8.895
30	9.441	9.339	9.199	9.048	8.901	8.763	8.639
40	8.993	8.916	8.802	8.677	8.552	8.435	8.328
50	8.739	8.675	8.575	8.463	8.351	8.245	8.146
∞	7.831	7.803	7.746	7.675	7.601	7.528	7.460

TABLE 1: (*Concluded*).

	$m_2 = 14$	15	20	25	30	40	50
14	9.955						
15	9.774	9.606					
20	9.142	9.010	8.533				
25	8.770	8.657	8.242	7.984			
30	8.526	8.424	8.047	7.811	7.651		
40	8.229	8.140	7.806	7.594	7.450	7.267	
50	8.055	7.974	7.664	7.465	7.329	7.157	7.053
∞	7.396	7.336	7.108	6.957	6.852	6.717	6.635

∞	$m_2 = \infty$
	6.295

$p = 4$

m_1	$m_2 = 8$	9	10	11	12	13	14
8	28.384						
9	24.990	22.881					
10	22.363	20.991	19.626				
11	20.405	19.453	18.433	17.493			
12	18.926	18.230	17.439	16.677	15.996		
13	17.781	17.251	16.615	15.983	15.403	14.890	
14	16.877	16.458	15.933	15.396	14.893	14.441	14.041
15	16.145	15.804	15.360	14.895	14.453	14.050	13.690
20	13.927	13.770	13.527	13.250	12.971	12.705	12.461
25	12.816	12.723	12.556	12.356	12.146	11.942	11.750
30	12.152	12.090	11.962	11.801	11.627	11.456	11.294
40	11.398	11.364	11.273	11.151	11.017	10.880	10.748
50	10.981	10.962	10.889	10.786	10.670	10.551	10.435
∞	9.543	9.555	9.530	9.481	9.420	9.353	9.285

	$m_2 = 15$	20	25	30	40	50	∞
15	13.370						
20	12.239	11.407					
25	11.572	10.892	10.457				
30	11.143	10.549	10.162	9.897			
40	10.624	10.126	9.794	9.563	9.269		
50	10.325	9.878	9.575	9.363	9.091	8.926	
∞	9.218	8.931	8.726	8.577	8.381	8.261	7.755

TABLE 2: 99th-percentile points of $c_1 T_1^2 + c_2 T_2^2$.

$p = 2$

m_1	$m_2 = 6$	7	8	9	10	11	12	13
6	24.121							
7	21.201	18.936						
8	19.072	17.319	16.023					
9	17.524	16.111	15.034	14.192				
10	16.338	15.181	14.258	13.525	12.939			
11	15.437	14.448	13.635	12.991	12.463	12.031		
12	14.719	13.855	13.137	12.551	12.072	11.680	11.350	
13	14.141	13.379	12.720	12.185	11.746	11.379	11.075	10.819
14	13.672	12.976	12.375	11.877	11.467	11.130	10.841	10.599
15	13.276	12.639	12.083	11.621	11.233	10.911	10.640	10.413
20	12.006	11.533	11.108	10.746	10.435	10.175	9.951	9.764
25	11.321	10.925	10.566	10.251	9.980	9.750	9.556	9.387
30	10.889	10.544	10.219	9.933	9.688	9.475	9.296	9.142
40	10.383	10.087	9.805	9.552	9.332	9.146	8.984	8.842
50	10.101	9.830	9.570	9.336	9.131	8.955	8.801	8.669
∞	9.053	8.867	8.679	8.506	8.352	8.218	8.101	7.998
	$m_2 = 14$	15	20	25	30	40	50	∞
14	10.394							
15	10.215	10.046						
20	9.600	9.457	8.962					
25	9.241	9.113	8.666	8.398				
30	9.006	8.889	8.472	8.219	8.054			
40	8.721	8.615	8.232	7.999	7.847	7.655		
50	8.552	8.453	8.091	7.871	7.725	7.542	7.432	
∞	7.908	7.827	7.539	7.361	7.241	7.092	7.002	6.640

$p = 3$

m_1	$m_2 = 7$	8	9	10	11	12	13
7	38.486						
8	33.047	29.199					
9	29.063	26.289	24.082				
10	26.152	24.082	22.344	20.918			
11	23.984	22.373	20.957	19.766	18.779		
12	22.324	21.035	19.844	18.818	17.959	17.246	
13	21.035	19.961	18.936	18.047	17.285	16.641	16.094
14	20.010	19.082	18.193	17.393	16.709	16.133	15.635
15	19.170	18.359	17.568	16.846	16.230	15.693	15.234
20	16.582	16.079	15.552	15.059	14.614	14.219	13.882
25	15.264	14.883	14.473	14.077	13.716	13.394	13.110
30	14.463	14.150	13.804	13.467	13.154	12.871	12.617
40	13.555	13.311	13.027	12.744	12.485	12.246	12.031
50	13.047	12.837	12.588	12.339	12.099	11.885	11.689
∞	11.287	11.172	11.021	10.864	10.713	10.566	10.435

TABLE 2: (*Concluded*).

	$m_2 = 14$	15	20	25	30	40	50
14	15.205						
15	14.839	14.497					
20	13.584	13.320	12.397				
25	12.861	12.642	11.846	11.365			
30	12.397	12.197	11.484	11.047	10.757		
40	11.838	11.667	11.040	10.654	10.396	10.071	
50	11.514	11.357	10.781	10.422	10.181	9.878	9.697
∞	10.315	10.205	9.792	9.526	9.346	9.116	8.977

∞	$m_2 = \infty$
	8.405

$p = 4$

m_1	$m_2 = 8$	9	10	11	12	13	14
8	55.518						
9	46.875	41.113					
10	40.488	36.602	33.301				
11	35.859	33.105	30.605	28.477			
12	32.451	30.410	28.457	26.729	25.264		
13	29.873	28.301	26.709	25.273	24.023	22.969	
14	27.871	26.602	25.293	24.072	22.988	22.061	21.260
15	26.289	25.244	24.121	23.066	22.109	21.279	20.566
20	21.641	21.104	20.479	19.844	19.248	18.701	18.218
25	19.414	19.053	18.613	18.149	17.700	17.285	16.909
30	18.115	17.842	17.495	17.119	16.748	16.401	16.084
40	16.670	16.484	16.221	15.938	15.644	15.371	15.112
50	15.889	15.737	15.522	15.278	15.029	14.785	14.561
∞	13.283	13.226	13.125	12.997	12.861	12.724	12.592

	$m_2 = 15$	20	25	30	40	50	∞
15	19.932						
20	17.783	16.230					
25	16.567	15.308	14.536				
30	15.791	14.702	14.023	13.564			
40	14.873	13.970	13.389	12.993	12.495		
50	14.351	13.542	13.018	12.656	12.197	11.931	
∞	12.466	11.956	11.609	11.362	11.041	10.851	10.045

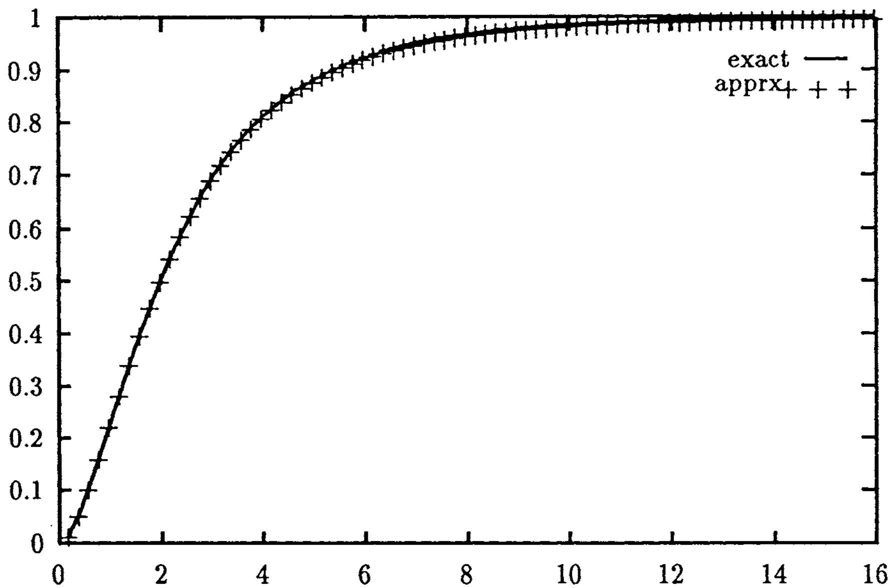

FIGURE 1: Exact and approximate distribution functions of $c_1T_1^2 + c_2T_2^2$ when $p = 2$, $m_1 = 10$ and $m_2 = 15$.

Since the lhs of (2.11) is a measure of difference among the sample mean vectors, it is expected to be small under the model assumption of equal means. Thus, the failure of (2.3) to provide a nonempty region is an indication that all the population means are not really equal. Further, since the confidence region (2.3) is exact, it is expected that (2.11) holds with probability at least $1 - \alpha$ under the model assumption. This means that one can use the expression in the lhs of (2.11) as a test statistic for testing the null hypothesis of equal mean in the multivariate Behrens-Fisher problem. The test that rejects the hypothesis of equal mean whenever (2.11) does not hold has size less than or equal to α.

REMARK 2.2. The confidence region (2.3) has some desirable properties like the classical ones based on the individual samples. It is invariant under nonsingular transformations and is the ellipsoid centered at $\hat{\boldsymbol{\mu}}$, and beginning at the center, the axes of the ellipsoid are $\pm\sqrt{Gl_i}$ in the direction of $\mathbf{e}_i$, where

$$G = a - \sum_{i=1}^{k}\left(\sum_{j\neq i}^{k}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)^{\mathrm{T}}\mathbf{W}_j^{-1}\right)\mathbf{V}^{-1}\mathbf{W}_i^{-1}\mathbf{V}^{-1}\left(\sum_{j\neq i}^{k}\mathbf{W}_j^{-1}(\bar{\mathbf{X}}_i - \bar{\mathbf{X}}_j)\right), \qquad (2.12)$$

the l_i's are the eigenvalues of $\mathbf{V}^{-1}$, and the $\mathbf{e}_i$'s are the eigenvectors corresponding to the l_i's, $i = 1, \ldots, p$.

3. MULTIPLE COMPARISONS

Consider the problem of testing

$$\mathcal{H}_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{vs.} \quad \mathcal{H}_a : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0. \qquad (3.1)$$

Then the test that rejects $\mathcal{H}_0$ whenever $\boldsymbol{\mu}_0$ does not fall in the confidence region (2.3) or the confidence region is empty has exact size α. If the null hypothesis is rejected,

TABLE 3: Simulated vector samples.

Sample 1		Sample 2	
x_{1i}	x_{2i}	x_{1i}	x_{2i}
7.00	11.92	5.56	6.77
3.68	8.37	6.57	9.22
4.56	5.37	8.05	10.33
3.07	4.08	3.60	8.61
6.02	8.83	4.69	9.51
4.65	4.93	1.50	6.48
6.62	14.38	6.92	7.52
3.93	6.47	5.45	10.14
2.54	6.41	5.78	12.65
6.59	9.78	9.15	8.69
5.71	10.68	2.20	8.20
2.34	3.96	3.05	8.52

then the problem of interest would be to identify the components of $\boldsymbol{\mu}_0$ which caused the rejection. Further, one may want to test a set of linear functions of the components of $\boldsymbol{\mu}$. This can be addressed by examining the simultaneous confidence intervals for the linear functions of $\boldsymbol{\mu}$ which are of interest. Simultaneous confidence intervals can be developed, from consideration of confidence intervals for $\mathbf{l}^{\mathsf{T}}\boldsymbol{\mu}$ for various choices of nonzero $p \times 1$ vectors $\mathbf{l}$, as follows: The inequality (2.3) holds if and only if

$$\{\mathbf{l}^{\mathsf{T}}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\}^2/\mathbf{l}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{l} \le G \qquad \text{for every} \quad \mathbf{l} \neq \mathbf{0}, \tag{3.2}$$

where G is given in (2.12). This implies that

$$\Pr(\mathbf{l}^{\mathsf{T}}\hat{\boldsymbol{\mu}} - \sqrt{G\mathbf{l}'\mathbf{V}^{-1}\mathbf{l}} \le \mathbf{l}^{\mathsf{T}}\boldsymbol{\mu} \le \mathbf{l}^{\mathsf{T}}\hat{\boldsymbol{\mu}} - \sqrt{G\mathbf{l}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{l}} \text{ for all } \mathbf{l}) = 1 - \alpha. \tag{3.3}$$

Thus, for particular choices of $\mathbf{l}$, the lhs of (3.3) is at least $1 - \alpha$. In particular, we have

$$\Pr(\hat{\boldsymbol{\mu}}_i - \sqrt{Gv^{ii}} \le \mu_i \le \hat{\mu}_i - \sqrt{Gv^{ii}} \text{ for all } i) \ge 1 - \alpha, \tag{3.4}$$

where v^{ii} denotes the (i, i)th element of $\mathbf{V}^{-1}$.

4. NUMERICAL EXAMPLE

As we do not have any real data sets, we here illustrate the statistical procedures developed in this article using simulated data points. Therefore, this example is for demonstration purpose only. We simulated samples of 12 vectors each from $\mathsf{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}_1)$ and $\mathsf{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}_2)$ with $\boldsymbol{\mu}' = (5, 8)$,

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 4.5 & 3.0 \\ 3.0 & 7.9 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 5.5 & 3.8 \\ 3.8 & 6.3 \end{pmatrix},$$

as shown in Table 3. The summary statistics are $\bar{\mathbf{X}}_1^{\mathsf{T}} = (4.73, 7.93)$, $\bar{\mathbf{X}}_2^{\mathsf{T}} = (5.21, 8.89)$, $c_1 = c_2 = 0.50$,

$$\mathbf{S}_1 = \begin{pmatrix} 2.71 & 4.46 \\ 4.46 & 10.91 \end{pmatrix}, \qquad \mathbf{S}_2 = \begin{pmatrix} 5.39 & 1.37 \\ 1.37 & 2.84 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} 8.07 & -3.39 \\ -3.39 & 4.10 \end{pmatrix},$$

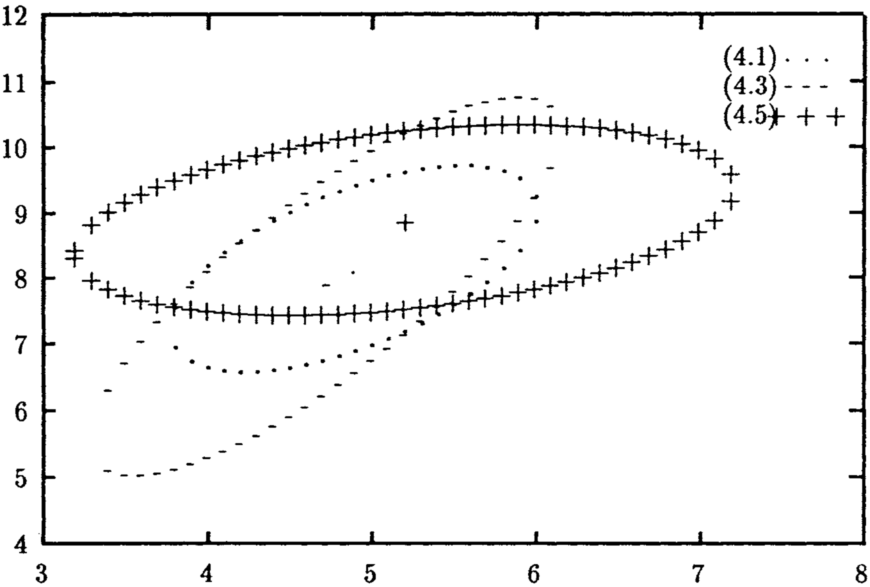

FIGURE 2: 95% confidence ellipses (4.1), (4.3) and (4.5).

and $\hat{\boldsymbol{\mu}}^T = [\mathbf{W}_1(\mathbf{W}_1 + \mathbf{W}_2)^{-1}\bar{\mathbf{X}}_1 + \mathbf{W}_2(\mathbf{W}_1 + \mathbf{W}_2)^{-1}\bar{\mathbf{X}}_2]^T = (4.89, 8.12)$. For $k = 2$, the rhs of (2.3) simplifies to $a - (\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2)^T(\mathbf{W}_1 + \mathbf{W}_2)^{-1}(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_2) = 7.04 - 0.40 = 6.64$ when $\alpha = 0.05$. Using these values in (2.3), we get a 95% confidence region for $\boldsymbol{\mu} = (\mu_1, \mu_2)^T$ which is the set of vectors (μ_1, μ_2) satisfying the inequality

$$8.07(4.89 - \mu_1)^2 + 4.10(8.12 - \mu_2)^2 - 6.78(4.89 - \mu_1)(8.12 - \mu_2) \le 6.64. \qquad (4.1)$$

The population mean $\boldsymbol{\mu}$ satisfies the inequality (4.1). The simultaneous confidence intervals for μ_1 and μ_2, computed using (3.4) with $\alpha = 0.05$, are

$$3.77 \le \mu_1 \le 6.01 \quad \text{and} \quad 6.55 \le \mu_2 \le 9.69. \qquad (4.2)$$

The 95% confidence region based on sample 1 alone is given by the set of (μ_1, μ_2)'s satisfying the inequality

$$13.60(4.73 - \mu_1)^2 + 3.38(7.93 - \mu_2)^2 - 11.14(4.73 - \mu_1)(7.93 - \mu_2) \le 9.03. \qquad (4.3)$$

In this case, the simultaneous confidence intervals for μ_1 and μ_2 are

$$3.30 \le \mu_1 \le 6.16 \quad \text{and} \quad 5.06 \le \mu_2 \le 10.80. \qquad (4.4)$$

Using Sample 2 alone, we get a 95% confidence region as the set of (μ_1, μ_2)'s satisfying the inequality

$$2.54(5.21 - \mu_1)^2 + 4.82(8.89 - \mu_2)^2 - 2.44(5.21 - \mu_1)(8.89 - \mu_2) \le 9.03 \qquad (4.5)$$

and the simultaneous confidence intervals

$$3.20 \le \boldsymbol{\mu}_1 \le 7.22 \quad \text{and} \quad 7.43 \le \boldsymbol{\mu}_2 \le 10.35. \qquad (4.6)$$

In order to understand the differences between the ellipsoids (4.1), (4.3) and (4.5), we have plotted them in Figure 2. We see that (4.1) yields a shorter confidence region than (4.3) and (4.5).

TABLE 4: Expected areas of 95% confidence regions.

$(\sigma_{11}^{(1)}, \sigma_{12}^{(1)}, \sigma_{22}^{(1)}), (\sigma_{11}^{(2)}, \sigma_{12}^{(2)}, \sigma_{22}^{(2)})$	C.R. (1)	(2)	(3)
$p = 2$, $m_1 = 8$ and $m_2 = 14$			
(1, 0, 1), (1, 0, 1)	1.32	3.29	2.96
(1, 0, 1), (1, 0, 2)	1.67	3.31	4.17
(1, 0, 1), (1, 0, 3)	1.88	3.30	5.12
(1, 0, 1), (1, 0, 5)	2.12	3.31	6.61
(1, 0, 1), (2, 0, 4)	2.55	3.28	8.34
(1, 0, 1), (5, 0, 6)	3.55	3.30	16.17
(1, 0, 1), (100, 0, 50)	5.84	3.31	208.67
(1, 0.1, 1), (1, 0.1, 1)	1.31	3.28	2.94
(1, 0.1, 1), (1, 0.2, 1)	1.30	3.30	2.91
(1, 0.1, 1), (1, 0.3, 1)	1.27	3.29	2.82
(1, 0.1, 1), (1, 0.5, 1)	1.17	3.28	2.56
(1, 0.1, 1), (1, 0.6, 1)	1.08	3.29	2.36
(1, 0.1, 1), (1, 0.7, 1)	0.96	3.29	2.10
(1, 0.1, 1), (1, 0.9, 1)	0.60	3.28	1.29
(2, 4, 16), (2, 2, 16)	6.35	13.24	15.69
(2, 2, 16), (2, 4, 16)	5.53	17.50	11.86
(2, 4, 16), (2, 4, 16)	5.29	13.21	11.82
(2, 4, 16), (2, 5, 16)	3.74	13.26	7.83
(2, 5, 16), (2, 4, 16)	4.57	8.73	11.84
(2, 4, 16), (2, 5.5, 16)	1.96	13.21	3.92
(2, 5.5, 16), (2, 4, 16)	3.01	4.37	11.85
$p = 2$, $m_1 = 20$ and $m_2 = 20$			
(4, 0, 6), (4, 0, 6)	3.25	5.16	5.16
(4, 0, 6), (5, 0, 6)	3.40	5.15	5.78
(4, 0, 6), (8, 0, 6)	3.76	5.16	7.30
(4, 0, 6), (12, 0, 6)	3.98	5.16	8.93
(4, 0, 6), (8, 0, 12)	4.33	5.15	10.32
(4, 0, 6), (15, 0, 16)	4.93	5.15	16.32
(4, 0, 6), (20, 0, 25)	5.37	5.15	23.52
(2, 1, 3), (2, 1, 3)	1.47	2.35	2.35
(2, 1, 3), (3, 2, 4)	1.64	2.35	2.98
(2, 1, 3), (4, 2, 6)	1.97	2.36	4.71
(2, 1, 3), (10, 7, 9)	2.12	2.35	6.75
(2, 1, 3), (15, 10, 25)	2.65	2.36	17.47
(2, 1, 3), (50, 20, 65)	2.91	2.35	56.25
(2, 1, 3), (50, 55, 65)	2.25	2.35	15.81
(4, 6, 11), (4, 0, 11)	2.11	2.98	6.99
(4, 6, 11), (4, 1, 11)	2.19	2.97	6.91
(4, 6, 11), (4, 2, 11)	2.24	2.98	6.66
(4, 6, 11), (4, 3, 11)	2.28	2.97	6.23
(4, 6, 11), (4, 4, 11)	2.26	2.98	6.66
(4, 6, 11), (4, 5, 11)	2.19	2.97	6.19
(4, 6, 11), (4, 6, 11)	1.82	2.98	2.98

5. SIMULATION RESULTS

In this section, for the case $k = 2$ and $p = 2$, we estimate the expected surface areas of the confidence region proposed in this article and the ones based on individual samples using Monte Carlo simulation. The normal random vectors are generated by the *IMSL* subroutine RNMVN, and the Wishart random matrices are generated using the Fortran subroutine WISHRT given in Smith and Hocking (1972). Each simulation consists of 100,000 runs. The estimated areas are presented in Table 4 for the following confidence regions:

(1) the confidence region given in (2.3),
(2) the confidence region based on the first sample alone,
(3) the confidence region based on the second sample alone.

We see from Remark 2.2 that, for $p = 2$, the expected area of confidence region 1 is given by $\pi\mathcal{E}(G/\sqrt{|V|})$. The single-sample confidence regions are based on Hotelling-T^2 statistics. We note here that, for the case $p = 2$, the exact expected area of the $100(1-\alpha)\%$ confidence region based on the first sample alone, say, is given by

$$c\pi\mathcal{E}(|\mathbf{S}_1|^{\frac{1}{2}}) = c\pi\frac{2\Gamma\big((m_1+1)/2\big)}{m_1\Gamma\big((m_1-1)/2\big)}|\mathbf{\Sigma}_1|^{\frac{1}{2}}, \tag{5.1}$$

where $c = pm_1\{(m_1+1)(m_1-p+1)\}^{-1}F_{p,m_1-p+1}(\alpha)$. (See Johnson and Wichern 1992, p. 191). However, in order to understand the validity of the simulation results and also to have a fair comparison, we estimate the areas of all three regions using simulation. For example, when $m_1 = 8$ and $\mathbf{\Sigma}_1 = (\sigma_{11}^{(1)}, \sigma_{12}^{(1)}, \sigma_{22}^{(1)}) = (1, 0, 1)$, the exact value of (5.1) is 3.3074; when $m_2 = 14$ and $\mathbf{\Sigma}_2 = (2, 4, 16)$ it is 13.2294; when $m_1 = 20$ and $\mathbf{\Sigma}_1 = (4, 6, 11)$ it is 2.9804. Comparison of these exact values with the corresponding estimated values indicates that the estimated values are accurate at least up to the first decimal place.

From the simulation results given in Table 4, we observe that, in general, confidence region 1 is smaller than 2 and 3 provided the "variances" between the sample mean vectors are not too large. This may be the situation in many practical problems. For example, when different instruments are used to measure several characteristics of like products to estimate the average quality or when different pharmaceutical companies estimate the effects of the same drug, we expect that the underlying "variances" are not far away from each other. Another situation, which is of theoretical interest, where confidence region 1 is smaller than 2 and 3 is when the variables in one of the populations are highly correlated compared to those in the other population. We also computed the areas of the regions at different parameter points. They are not reported here, since they all exhibited similar patterns as the numbers in Table 2.

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