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Confidence intervals for a ratio of percentiles of location-scale distributions

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The problem of estimating a ratio of percentiles of two independent location-scale distributions is considered. A fiducial approach is proposed and described in details for the normal, lognormal, two-parameter exponential and Weibull distributions. For the normal case, the fiducial confidence intervals (CIs) turn out to be exact when the variances are equal. Procedures for constructing CIs for ratio of percentiles involving two-parameter exponential distributions and Weibull distributions are given with computational details. The fiducial methods can be readily extended to the case where the samples are type II censored. The methods are illustrated using real-world data sets.

1. Introduction

In most applications, two populations are compared using the means or the medians. Also, ratio of means or percentiles is used to compare two populations of positive data. Since the ratio of means or of percentiles is free of units of measurements, a ratio can be used to combine related but different outcomes. Applications of a ratio of percentiles in the wood industry have been noted in Huang and Johnson (2006). These authors have noted that it is common practice to compare two different strength properties of lumber of the same dimension, grade and species. Wood technology engineers often make a comparison in terms of the ratio of two strength properties such as the mean bending strengths. As the lumber standards are measured in terms of population fifth percentiles, the ratio of the fifth percentiles of the two strength distributions is used to compare the strengths.

For a while, normal models are commonly postulated for lumber strength properties. However, as the lumber strength data are not in general normally distributed, nonparametric methods were also used to analyze the data. In some situations, however, a parametric estimate of lumber properties for reliability-based design is required. Based on empirical evidence, many authors noted that a three-parameter Weibull model emerges as a serious candidate; see Aplin et al. (1986), Bodig (1977), and Pierce (1976). To use this parametric approach, one needs to develop procedures for finding confidence intervals for ratios of percentiles from two Weibull populations.

Huang and Johnson (2006) have proposed an exact method of finding CIs for the ratio of normal percentiles when the variances are equal. The proposed CIs are not in closed-form and an iterative method is required to find the CIs. Although it is common to compare the same percentiles of two different distributions, in this article we propose methods that can be used to find CIs for the ratio of different percentiles from two different distributions from the same family. Specifically, we propose a fiducial method that can be used to find CIs for a ratio of percentiles of any two location-scale distributions. Furthermore, we provide closed-form approximate fiducial CIs for the normal and lognormal cases.

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The concepts of fiducial probability and fiducial inference were introduced by Fisher (1930, 1935). There are some criticisms concerning the interpretation of fiducial distribution (Zabell, 1992; Efron, 1998). Efron (1998) has interpreted the fiducial distribution as the posterior distribution of a parameter without assuming a prior distribution. He has concluded in Section 8 of his paper that "maybe Fishers' biggest blunder will become a big hit in the 21st century!" Fiducial inference have made a resurgence under the label of generalized inference by Tsui and Weerahandi (1989) and Weerahandi (1993). Hannig et al. (2006) have noted that the generalized variable procedures are a special case of fiducial inference procedures, and are asymptotically exact in many situations. For more details and applications of the fiducial/generalized inference, see the books by Weerahandi (1995), Hannig et al. (2006), Hannig (2009) and Li and Hannig (2020). For the continuous case, the fiducial approach has been used successfully to estimate or to test a function of parameters where ordinary pivotal quantities are available for individual parameters (e.g., location-scale family, log-location-scale family and quantiles in one-way random model). Hannig et al. (2006) and Hannig (2009) have introduced the concept of generalized fiducial inference which can be used to find inference for continuous as well as discrete models such as the binomial and Poisson. Even though there are several methods of obtaining fiducial distributions for parameters, for location-scale or log-location-scale family of distributions, Dawid and Stone's (1982) approach is simple and easy to describe, and we use this approach in the sequel to find fiducial distributions of parameters. The rest of the article is organized as follows. In the following section, we describe a method of finding fiducial distributions for the parameters of a location-scale family of distributions. Using these fiducial distributions, we obtain fiducial quantities for a ratio of percentiles of two independent location-scale distributions. In Section 3, we apply the fiducial method to find CIs for a ratio of normal percentiles. We show that the fiducial CIs are exact when the variances of the normal distributions are equal. We also provide a closed-form approximate CI for a ratio of two normal percentiles when the variances are unknown and arbitrary. Accuracy of the CIs are evaluated in terms of coverage probability estimated by Monte Carlo simulation. Similar studies are carried out for the lognormal case (Section 4), two-parameter exponential distributions (Section 5) and for Weibull distributions (Section 6). For each case, an illustrative example is used to illustrate the methods. In Section 7, we provide some useful references to apply the fiducial methods to logistic distributions, Laplace distributions and to type II censored case. Some concluding remarks are given in Section 8.

2. Fiducial inference

Let $X_1, \ldots, X_n$ be a sample from a location-scale distribution with the probability density function (PDF) of the form

$$f(x|\mu, \sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < \mu < \infty, \quad \sigma > 0. \tag{1}$$

Let $\widehat{\mu}$ and $\widehat{\sigma}$ be equivariant estimators of μ and σ, respectively, based on $X_1, \ldots, X_n$. Then $(\widehat{\mu} - \mu)/\sigma$ and $\widehat{\sigma}/\sigma$ are pivotal quantities (see Lawless (2003), Theorem E2). As a consequence,

$$\frac{\widehat{\mu} - \mu}{\sigma} \stackrel{d}{=} \widehat{\mu}^* \quad \text{and} \quad \frac{\widehat{\sigma}}{\sigma} \stackrel{d}{=} \widehat{\sigma}^*, \tag{2}$$

where the notation "$X \stackrel{d}{=} Y$" means X and Y are identically distributed and $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are the equivariant estimators based on a sample of size n from the location-scale distribution with $\mu = 0$ and $\sigma = 1$.

Let $q_p(\mu, \sigma)$ denote the $100p$ percentile of the location-scale distribution with the pdf $f(x|\mu, \sigma)$. It can be easily verified that $q_p(\mu, \sigma) = \mu + q_p(0, 1)\sigma$. Using this relation and the results in (2), it can be readily checked that

$$\frac{q_p(\mu, \sigma) - \widehat{\mu}}{\widehat{\sigma}} \stackrel{d}{=} \frac{q_p(0, 1) - \widehat{\mu}^*}{\widehat{\sigma}^*}, \tag{3}$$

and so $\left(q_p(\mu, \sigma) - \widehat{\mu}\right)/\widehat{\sigma}$ is a pivotal quantity. Therefore, percentiles of $(q_p(0, 1) - \widehat{\mu}^*)/\widehat{\sigma}^*$, which can be estimated by Monte Carlo simulation or calculated numerically, can be used to find confidence intervals for $q_p(\mu, \sigma)$.

Fiducial Quantity

We can find the fiducial distributions of the parameters using Dawid and Stone's (1982) functional model stochastic relation. Let $(\widehat{\mu}_0, \widehat{\sigma}_0)$ be an observed value of $(\widehat{\mu}, \widehat{\sigma})$. Solving the "equations" in (2) for μ and σ and then replacing $(\widehat{\mu}, \widehat{\sigma})$ with $(\widehat{\mu}_0, \widehat{\sigma}_0)$, we find the fiducial quantities as

$$W_\mu = \widehat{\mu}_0 - \frac{\widehat{\mu}^*}{\widehat{\sigma}^*}\widehat{\sigma}_0 \quad \text{and} \quad W_\sigma = \frac{\widehat{\sigma}_0}{\widehat{\sigma}^*} \tag{4}$$

for μ and σ, respectively. For a given $(\widehat{\mu}_0, \widehat{\sigma}_0)$, the distributions of W_μ and W_σ are called fiducial distributions of μ and σ, respectively. Thus, the fiducial distributions can be regarded as the posterior distributions of the parameters without assuming a prior distribution.

Fiducial distribution for a real-valued function $h(\mu, \sigma)$ of parameters can be obtained by substitution. For example, a fiducial distribution for $q_p(\mu, \sigma) = \mu + q_p(0, 1)\sigma$ is given by

$$q_p(W_\mu, W_\sigma) = W_\mu + q_p(0, 1)W_\sigma = \widehat{\mu}_0 + \frac{q_p(0, 1) - \widehat{\mu}^*}{\widehat{\sigma}^*}\widehat{\sigma}_0. \tag{5}$$

The above fiducial quantity (FQ) $q_p(W_\mu, W_\sigma)$ can also be deduced from the classical pivotal quantity (3), and so the fiducial inference based on the above fiducial quantity and the inference based on the pivotal quantity in (3) are the same. In other words, the fiducial confidence interval for $q_p(\mu, \sigma)$ is exact in the frequentist sense.

2.1. Fiducial confidence intervals for the ratio of percentiles

Let $X_{i1}, \ldots, X_{in_i}$ be a sample from a location-scale distribution with pdf $f(x|\mu_i, \sigma_i)$, $i = 1, 2$. Assume that the samples $\{X_{11}, \ldots, X_{1n_1}\}$ and $\{X_{21}, \ldots, X_{2n_2}\}$ are independent. Let $(\widehat{\mu}_i, \widehat{\sigma}_i)$ denote the equivariant estimator of (μ_i, σ_i) based on the ith sample, $i = 1, 2$. Further, let $(\widehat{\mu}_i^*, \widehat{\sigma}_i^*)$ denote the equivariant estimator based on the sample $X_{i1}^*, \ldots, X_{in_i}^*$ from the same location-scale distribution with pdf $f(x|0, 1)$, $i = 1, 2$. Then the fiducial quantity for the ratio $R = q_{p_1}(\mu_1, \sigma_1)/q_{p_2}(\mu_2, \sigma_2)$ of percentiles can be obtained from (5) as

$$W_R = \frac{q_{p_1}(W_{\mu_1}, W_{\sigma_1})}{q_{p_2}(W_{\mu_2}, W_{\sigma_2})} = \frac{\widehat{\mu}_{10} + \frac{q_{p_1}(0,1)-\widehat{\mu}_1^*}{\widehat{\sigma}_1^*}\widehat{\sigma}_{10}}{\widehat{\mu}_{20} + \frac{q_{p_2}(0,1)-\widehat{\mu}_2^*}{\widehat{\sigma}_2^*}\widehat{\sigma}_{20}}, \tag{6}$$

where $(\widehat{\mu}_{i0}, \widehat{\sigma}_{i0})$ is an observed value of $(\widehat{\mu}_i, \widehat{\sigma}_i)$, $i = 1, 2$. For an observed value $(\widehat{\mu}_{10}, \widehat{\sigma}_{10}, \widehat{\mu}_{20}, \widehat{\sigma}_{20})$, let $W_{R;\alpha}$ denote the 100α percentile of W_R. Then $(W_{R;\alpha},\ W_{R;1-\alpha})$ is a $100(1-2\alpha)\%$ CI for the ratio R. These percentiles can be estimated by Monte Carlo simulation because the fiducial distribution of W_R, when $(\widehat{\mu}_{10}, \widehat{\sigma}_{10}, \widehat{\mu}_{20}, \widehat{\sigma}_{20})$ is fixed, does not depend on any parameter.

Approximate percentiles of W_R can also be obtained using the following approximation to the percentiles of the ratio of two independent positive random variables.

An Approximation for the Ratio of Percentiles of Independent Random Variables

Let X and Y be independent positive random variables with $E(X) = \mu_x$ and $E(Y) = \mu_y$. Let X_α denote the 100α percentile of X, and define Y_α similarly. An approximation to the 100α percentile of the ratio $R = X/Y$ is given by

$$R_\alpha \simeq \begin{cases} \dfrac{\mu_x\mu_y - \left\{(\mu_x\mu_y)^2 - \left[\mu_y^2-(\mu_y-Y_{1-\alpha})^2\right]\left[\mu_x^2-(\mu_x-X_\alpha)^2\right]\right\}^{\frac{1}{2}}}{\left[\mu_y^2-(\mu_y-Y_{1-\alpha})^2\right]}, & 0<\alpha\le .5, \\[2ex] \dfrac{\mu_x\mu_y + \left\{(\mu_x\mu_y)^2 - \left[\mu_y^2-(\mu_y-Y_{1-\alpha})^2\right]\left[\mu_x^2-(\mu_x-X_\alpha)^2\right]\right\}^{\frac{1}{2}}}{\left[\mu_y^2-(\mu_y-Y_{1-\alpha})^2\right]}, & .5<\alpha<1. \end{cases} \tag{7}$$

This approximation is given in Krishnamoorthy and Wang (2016) and is a generalization of the one by Li et al. (2010). This approximation is referred to as the modified normal-based approximation and it can be used to find the percentiles of W_R in (6).

3. Normal distributions

In general, ratio is used as a measure of difference only when the parameters are positive. For example, CIs for the ratio of variances and CIs for the ratio of Poisson means are constructed for inference purpose. Confidence intervals for a ratio of normal percentiles may have meaningful interpretation only when normal models are postulated for positive random variables. So we assume that the percentiles of both normal populations are positive.

For the normal case, the usual equivariant estimators are given by the sample mean $\widehat{\mu}_i = \bar{X}_i$ and the sample variance $\widehat{\sigma}_i^2 = S_i^2$, where the variance is defined with divisor $m_i = n_i - 1$ and n_i is the size of the sample from the ith population. Let $(\bar{x}_i, s_i^2)$ be an observed value of $(\bar{X}_i, S_i^2)$. Let $(\widehat{\mu}_i^*, \widehat{\sigma}_i^{*2}) = (\bar{X}_i^*, S_i^{*2}) = $ (mean, variance) based on a sample of size n_i from the standard normal distribution, $N(0, 1)$. Let z_p denote the $100p$ percentile of the standard normal distribution. Noting that the $100p_i$ percentile of a $N(\mu_i, \sigma_i^2)$ distribution is given by $q_{p_i}(\mu_i, \sigma_i) = \mu_i + z_{p_i}\sigma_i$ and using (5), we find the FQ for $\mu_i + z_{p_i}\sigma_i$ as $W_{q_{p_i}(\mu_i,\sigma_i)} = \bar{x}_i + [(z_{p_i} - \bar{X}_i^*)/S_i^*]s_i$. Noting that $\bar{X}_i^* \sim N(0, 1/n_i)$ independently of $m_iS_i^{*2} \sim \chi^2_{m_i}$, where $m_i = n_i - 1$, we can write

$$W_{q_p(\mu_i,\sigma_i)} = \bar{x}_i + \frac{z_{p_i}\sqrt{n_i} - Z_i}{\sqrt{\chi^2_{m_i}/m_i}}\frac{s_i}{\sqrt{n_i}},\ \ i = 1, 2. \tag{8}$$

where $t_m(\delta)$ denotes the noncentral random variable with degrees of freedom (df) m and the noncentrality parameter δ. If Z is a standard normal random variable, then Z and $-Z$ are identically distributed. Furthermore, if Z is independent of a chi-square random variable U^2 with df $= m$, then $(Z + \delta)/(U/\sqrt{m})$ follows a noncentral t distribution with df $= m$ and the noncentrality parameter δ, $t_m(\delta)$ (for example, see Section 20.1 of Krishnamoorthy (2015)). Using this result, we can write

$$W_{q_p(\mu_i,\sigma_i)} = \bar{x}_i + t_{m_i}(z_{p_i}\sqrt{n_i})\frac{s_i}{\sqrt{n_i}},\ \ i = 1, 2. \tag{9}$$

Krishnamoorthy and Mathew (2009) have developed the similar fiducial quantity (9) using the generalized variable approach by Weerahandi (1993). The generalized variable approach is a special case of fiducial inference; see Hannig (2009).

Table 1
The probability in (11) when $\mu = 3\sigma$.

p	.01	.05	.1	.25	.5	.75	.90	.95	.99
n	120	35	25	10	6	3	3	3	3
Prob (11)	.993	.998	.999	.999	.999	.998	.9999	1	1

3.1. Confidence intervals when the variances are unknown and arbitrary

A FQ for $R_N = q_{p_1}(\mu_1, \sigma_1)/q_{p_2}(\mu_2, \sigma_2) = (\mu_1 + z_{p_1}\sigma_1)/(\mu_2 + z_{p_2}\sigma_2)$ can be obtained by using individual FQs in (9) as

$$Q_{R_N} = \frac{\bar{x}_1 + t_{m_1}(z_{p_1}\sqrt{n_1})\frac{s_1}{\sqrt{n_1}}}{\bar{x}_2 + t_{m_2}(z_{p_2}\sqrt{n_2})\frac{s_2}{\sqrt{n_2}}}, \tag{10}$$

where the random variables $t_{m_1}(z_{p_1}\sqrt{n_1})$ an $t_{m_2}(z_{p_2}\sqrt{n_2})$ are independent. For a given $(\bar{x}_1, \bar{x}_2, s_1, s_2)$ and sample sizes, the following R code computes the 95% fiducial CI based on (10).

```
***********************************************************
M1 = xb1 + rt(10^5,n1-1,qnorm(p1)*sqrt(n1))*s1/sqrt(n1)
M2 = xb2 + rt(10^5,n2-1,qnorm(p2)*sqrt(n2))*s2/sqrt(n2)
quantile(M1/M2, c(.025, .975))
***********************************************************
```

Remark 1. The CI based on the above fiducial quantity (10) is a bona fide positive CI only when both numerator and the denominator in (10) are positive. As shown in Appendix A, these quantities are more likely to be positive under the assumption that all percentiles are positive. In the following Table 1, we provide minimum sample sizes so that the

$$P\left(\bar{x} + \frac{Z + z_p\sqrt{n}}{U}\frac{s}{\sqrt{n}} > 0\right) \simeq 1, \tag{11}$$

where $Z \sim N(0, 1)$ independently of $U^2 \sim \chi^2_m/m$, $m = n - 1$ and $(\bar{x}, s^2)$ is an observed values of (mean, variance) based on a sample of size n. For example, when $p_1 = p_2 = 0.05$, the numerator and the denominator of (10) are more likely to be positive if both n_1 and n_2 are greater than or equal to 35.

Remark 2. Alternatively, we can define the fiducial quantity for a ratio of percentiles as

$$Q^+_{R_N} = \frac{\left[\bar{x}_1 + t_{m_1}(z_{p_1}\sqrt{n_1})\frac{s_1}{\sqrt{n_1}}\right]_+}{\left[\bar{x}_2 + t_{m_2}(z_{p_2}\sqrt{n_2})\frac{s_2}{\sqrt{n_2}}\right]_+}, \tag{12}$$

where $[x]_+ = x$ if x is positive, and is zero if $x < 0$. Such truncated FQ produces CIs which are very similar to those based on Q_{R_N} for all sample sizes parameter configurations considered in Table 6. However, for some cases the expected widths of the CIs based on $Q^+_{R_N}$ are infinite or undefined because both numerator and denominator are zeros. For this reason, we do not include this CI for coverage studies in Section 6.4.

An approximation

An approximation for the percentile of Q_{R_N} in (10) can be obtained using the modified normal-based approximation in (7). Malekzadeh and Mahmoudi (2020) have already provided such closed-form approximate CI for a ratio of percentiles using the approximate percentiles of (10). However, this approximate CI is a bona fide positive CI only when both numerator and the denominator of (10) are positive. For a given p, the minimum sample size required, so that the numerator and denominator of (10) are positive with a high probability, is given in Table 1.

3.2. Confidence interval for a ratio of percentiles when $\sigma_1^2 = \sigma_2^2$

Let $(\bar{X}_i, S_i^2)$ denote the (mean, variance) based on a sample of n_i observations from a $N(\mu_i, \sigma_i^2)$ distribution, $i = 1, 2$. Let $(\bar{x}_i, s_i^2)$ be an observed value of $(\bar{X}_i, S_i^2)$, $i = 1, 2$. Let us assume that the population percentiles $\mu_i + z_{p_i}\sigma$ are positive. Consider testing $H_0 : R_N = R_0$ vs. $H_a : R_N < R_0$, where R_0 is a specified value of the ratio of percentiles. Note that, under the assumption that the percentiles are positive, the above hypotheses can be written as

$$H_0 : \mu_1 + z_{p_1}\sigma - R_0(\mu_2 + z_{p_2}\sigma) = 0 \quad \text{vs.} \quad H_a : \mu_1 + z_{p_1}\sigma - R_0(\mu_2 + z_{p_2}\sigma) < 0. \tag{13}$$

When the variances are equal, the common unknown variance is estimated by the sample pooled variance $S_p^2 = [m_1S_1^2 + m_2S_2^2]/(m_1 + m_2)$, which has the $\sigma^2\chi^2_f/f$ distribution, where $f = m_1 + m_2$. Let s_p^2 be an observed value of S_p^2. Using the pooled variance estimate, a fiducial quantity for $D = (\mu_1 + z_{p_1}\sigma) - R_0(\mu_2 + z_{p_2}\sigma)$ can be obtained from (9), and is given by

$$Q_D = \left(\bar{x}_1 + \frac{Z_1 + z_{p_1}\sqrt{n_1}}{U_f}\frac{s_p}{\sqrt{n_1}}\right) - R_0\left(\bar{x}_2 + \frac{Z_2 + z_{p_2}\sqrt{n_2}}{U_f}\frac{s_p}{\sqrt{n_2}}\right), \tag{14}$$

where Z_1, Z_2 and U_f are independent random variables with $Z_i \sim N(0, 1)$ and $fU_f^2 \sim \chi_f^2$ distribution. The fiducial test rejects H_0 whenever the fiducial p-value $P(Q_D < 0|\bar{x}_1, \bar{x}_2, s_p) \le \alpha$. As shown in Appendix B, this fiducial test is exact in the frequentist sense, and so the confidence limit obtained by inverting the fiducial test is also exact.

Let $t_{m;q}(\delta)$ denote the qth quantile of $t_m(\delta)$ and let

$$\delta(R_0) = \frac{z_{p_1} - R_0 z_{p_2}}{\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}}. \tag{15}$$

As shown in Appendix B, by solving the equation

$$(\bar{x}_1 - R_0\bar{x}_2) + t_{f;1-\alpha}(\delta(R_0))s_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}} = 0 \tag{16}$$

for R_0, we find a $100(1-\alpha)\%$ upper confidence limit for R_N. Similarly, by considering a right-tailed test, we arrive at the equation

$$(\bar{x}_1 - R_0\bar{x}_2) + t_{f;\alpha}(\delta(R_0))s_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}} = 0, \tag{17}$$

and solving the equation for R_0, we can find the $100(1-\alpha)\%$ lower confidence limit.

As shown in Appendix B, the p-value for testing hypotheses in (13) has a uniform (0, 1) distribution and hence the test described in Appendix B is exact. So the CI formed by (16) and (17) is obtained by inverting an exact test is also exact.

Remark 3. The exact CI given above is the same as the exact CI given in Huang and Johnson (2006). Using the frequentist approach, Huang and Johnson have shown that the $100(1-\alpha)\%$ upper confidence limit for the ratio of percentiles is the value of R_0 for which the probability

$$P\left(t_f(\delta(R_0)) > \frac{R_0\bar{x}_2 - \bar{x}_1}{s_p\sqrt{1/n_1 + R_0^2/n_2}}\right) = \alpha.$$

Note that the value R_0 that satisfies the above equation also satisfies our Eq. (16).

Remark 4. When $p_1 = p_2 = 0.5$, the exact CI for the ratio of percentiles determined by (16) and (17) is the same as the exact CI for the ratio of means based on Fieller's (1954) results. That is, the CI for $\theta = \mu_1/\mu_2$ based on the result that $(\bar{X}_1 - \theta\bar{X}_2)/\sqrt{S_p^2(1/n_1 + \theta^2/n_2)} \sim t_f$.

A fiducial quantity when $\sigma_1^2 = \sigma_2^2$ can be obtained using the pooled variance s_p^2. Replacing s_i in (9) with s_p and using the result that $(m_1 + m_2)S_p^2 \sim \chi^2_{m_1+m_2}$,

$$Q_{RE_N} = \frac{\bar{x}_1 + \frac{Z_1 + z_{p_1}\sqrt{n_1}}{U_f}\frac{s_p}{\sqrt{n_1}}}{\bar{x}_2 + \frac{Z_2 + z_{p_2}\sqrt{n_2}}{U_f}\frac{s_p}{\sqrt{n_2}}}, \tag{18}$$

where s_p^2 is the pooled variance, U_f and f are as defined in (14). This FQ is also valid when both sample sizes are greater than or equal to the values of n in Table 1. Percentiles of Q_{RE_N} can be estimated using Monte Carlo simulation, and the estimates can be used as initial values to find the roots of Eqs. (16) and (17).

To provide some numerical evidence to show that the CIs based on simulation of (18) and the exact ones formed by (16) and (17) are the same, we computed 95% CIs for ratio of normal percentiles using some simulated samples of sizes (n_1, n_2) and reported them in Table 2. The values of the means and variances along with some values of (p_1, p_2) are also shown in the table. On the basis of numerical results in Table 2, we see that the CIs based on Monte Carlo estimates of Q_{RE_N} and the exact ones formed by the roots of (16) and (17) are practically the same for all the cases.

3.3. Example

Modulus of rupture (MOR) is a measure of a specimen's strength before rupture, and it is commonly used to determine a wood species' overall strength. The data on the modulus of rupture (MOR) of Douglas fir specimens are presented in

Table 2
Confidence intervals for the ratio of normal percentiles based on simulation of Q_{RE_N} and the exact ones formed by the roots of (16) and (17).

(p_1, p_2)	(n_1, n_2)	$\bar{x}_1$	$\bar{x}_2$	s_1^2	s_2^2	Simulation of (18) with 10^5 runs	Exact
(.05, .05)	(20, 10)	17.746	14.624	3.535	16.988	(1.080, 1.678)	(1.080, 1.679)
(.15, .05)	(20, 20)	18.458	14.868	7.833	10.009	(1.324, 1.908)	(1.324, 1.908)
(.15, .15)	(20, 20)	18.448	14.681	5.726	8.284	(1.163, 1.506)	(1.164, 1.506)
(.75, .25)	(20, 20)	18.338	15.594	2.628	6.934	(1.290, 1.563)	(1.290, 1.563)
(.75, .25)	(20, 10)	18.172	13.933	2.294	6.402	(1.391, 1.761)	(1.391, 1.761)
(.75, .25)	(30, 10)	18.375	16.618	2.646	14.234	(1.196, 1.512)	(1.197, 1.511)
(.95, .05)	(30, 10)	17.716	15.892	3.145	4.660	(1.455, 1.911)	(1.455, 1.911)

Table 3
Modulus of rupture data.

Data 1: MOR (lb/in^2) of 2 × 4in Grade 2, Green (30% moisture content)									
5418.6	4795.9	7061.8	6307.9	6964.0	6674.1	8153.4	6843.5	7011.3	5817.3
6617.3	6136.7	7529.2	6357.9	7643.6	7311.8	6997.6	4533.1	5691.8	6245.9
6455.7	6082.7	5511.1	5976.8	4607.6	4414.5	5268.4	8145.4	4616.1	3508.1
5231.6	5851.4	4281.8	9213.0	6051.4	4050.5	5677.2	5531.8	4872.4	3677.3
4230.9	2524.8	4896.3	4161.0	4818.4	5325.4	5818.8	4787.2	5988.6	5530.9
6351.2	2764.8	4432.9	5325.4	5651.6	3917.7	4429.6	3938.0	5143.3	4044.5
5128.6	5681.0	4690.8	5894.0	2350.2	2822.7	5465.8	3770.9	3168.1	4994.6
2327.6	2642.7	4760.3	3022.1	4187.7	4420.0	3095.2	5289.3	3440.7	4533.1
3340.9	3207.7	5270.6	4274.4	3854.5	2748.2	4461.2	4892.6	5078.2	5278.7
1420.3	3015.2	4451.7	1931.2	3556.0	3396.4	6767.2	2566.1	1228.4	5883.6
2444.5	3747.0	3879.0	4392.0	2105.5	5161.1	4658.4			
Data 2: MOR (lb/in^2) of 2 × 6in select structural, Green (30% moisture content)									
9579.3	6475.0	8374.9	9364.2	9265.1	10653.4	8051.2	9828.1	7930.1	8304.1
7896.3	8041.5	9024.7	7614.5	6720.6	8685.8	11077.9	9178.9	8375.9	10342.8
8793.8	6836.1	7774.3	6724.0	7887.9	5045.7	6556.9	6574.8	5576.5	7194.2
8018.8	7078.2	9015.2	8935.7	8388.6	7684.5	6800.9	8254.7	6700.2	7859.9
7271.5	7776.8	7962.0	7806.5	7418.5	6575.0	5495.9	7820.9	6122.8	5345.8
7762.6	5169.9	8267.5	4597.8	7445.3	6533.7	3911.9	6768.4	7865.4	9499.1
6598.4	5253.9	8643.2	7323.3	7099.8	6842.3	7853.9	6594.6	6724.0	3981.0
6835.0	7449.2	5477.9	7288.0	7803.5	5616.3	7137.9	8925.9	7182.6	5208.5
6244.2	6893.2	10100.9	5548.2	5783.9	4606.4	5679.9	7984.7	6014.0	6159.1
6922.1	5120.6	6067.5	4639.9	3680.3	5399.4	4917.3	5398.0	4173.1	7418.5

Table 2 of Huang and Johnson (2006). The normal QQ-plots suggest that data are normally distributed. The summary statistics provided in their paper are based on $n_1 = 107$ data points and $n_2 = 103$ data points. However, we found that the second data set given Table 2 of Huang and Johnson's paper includes only 100 measurements, not 103. It appears that the statistics $\bar{x}_2$ and s_2^2 given in their paper are based on all 103 measurements. So, to avoid any confusion, we shall use the data in Huang and Johnson's paper (reproduced here in Table 3) to find 95% confidence intervals for the ratio of 5th percentiles and 95% CI for the ratio of means. The statistics based on the data in Table 3 are

$$(n_1, \bar{x}_1, s_1^2) = (107, 4840.3, 2354470), \quad (n_2, \bar{x}_2, s_2^2) = (100, 7144.9, 2414487), \quad \text{and} \quad s_p = 1543.8.$$

Assuming equal variance, the 95% fiducial CI (18) for the ratio of the 5th percentiles based on 10^6 simulation runs is (**0.420**, **0.575**). The exact one formed by the roots of Eqs. (16) and (17) is also (**.420**, **.575**). In this case, Huang and Johnson computed the exact CI as **(0.410, 0.583)**. The slight difference between their CI and our CI is due to fact that Huang and Johnson used extra three measurements in the second sample which are not reported in their paper. If no assumption is made on the variances, then the fiducial CI based on the FQ (10) with 10^5 simulation runs is **(0.390, 0.618)**. The approximate CI obtained using (7) is also **(0.390, 0.618)**. All the CIs indicate that the 5th percentiles of MOR for 2 × 4 in Grade 2 wood is smaller than that for 2 × 6 in select structural wood.

We also computed 95% CI for the ratio of means by taking $p_1 = p_2 = 0.5$ in (33). Under the assumption of equal variance, the exact one formed by the roots of Eqs. (16) and (17) is **(0.628, 0.729)**. The 95% fiducial CI (18) is also **(0.628, 0.729)**. When the variances are arbitrary, then the fiducial CI based on the fiducial quantity is also **(0.628, 0.729)**.

4. Lognormal distributions

Recall that Y is said to have a lognormal distribution with parameters μ and σ^2, say, LN(μ, σ) if $X = \ln Y$ follows a normal distribution with mean μ and variance σ^2. This definition implies that $\exp(\mu + z_p\sigma)$ is the $100p$ percentile of a LN(μ, σ) distribution. So the ratio of the $100p_1$ percentile of a LN(μ_1, σ_1) distribution and the $100p_2$ percentile of a

Table 4
TDDB data (in min) from furnaces A and B.

Furnace A				Furnace B			
33.4	47.5	56.1	65.1	37.4	58.8	66.0	77.2
35.8	48.5	56.4	67.0	45.3	59.0	66.2	81.0
39.8	48.8	56.7	67.3	48.6	59.3	67.6	84.0
41.1	49.6	59.7	68.0	49.1	60.3	68.8	85.7
41.8	50.6	60.4	68.3	50.4	61.4	69.8	86.2
42.2	50.7	60.7	74.1	51.0	62.1	70.4	90.2
45.5	51.5	61.3	74.7	51.7	63.0	71.0	93.9
46.0	51.7	61.8	80.6	55.9	63.3	71.9	97.4

$\mathrm{LN}(\mu_2, \sigma_2)$ distribution is given by

$$R = \frac{\exp(\mu_1 + z_{p_1}\sigma_1)}{\exp(\mu_2 + z_{p_2}\sigma_2)} = \exp[(\mu_1 + z_{p_1}\sigma_1) - (\mu_2 + z_{p_2}\sigma_2)]. \tag{19}$$

4.1. Lognormal: Fiducial confidence intervals

We see from (19) that the problem of estimating the ratio of lognormal percentiles simplifies to estimating the difference between two normal percentiles. In particular, let $(\bar{x}_i, s_i^2)$ be an observed value of the (mean, variance) based on a log-transformed sample of size n_i from a $\mathrm{LN}(\mu_i, \sigma_i)$ distribution, $i = 1, 2$. Then, following (9), a FQ for the difference between percentiles can be expressed as

$$W_D = \left(\bar{x}_1 + t_{n_1-1}(z_{p_1}\sqrt{n_1})\frac{s_1}{\sqrt{n_1}}\right) - \left(\bar{x}_2 + t_{n_2-1}(z_{p_2}\sqrt{n_2})\frac{s_2}{\sqrt{n_2}}\right), \tag{20}$$

where the noncentral random variables $t_{n_i-1}(z_{p_i}\sqrt{n_i})$ are independent. For a given $(\bar{x}_1, s_1, \bar{x}_2, s_2)$, the percentiles of W_D can be used to find a CI for the difference $D = (\mu_1 + z_{p_1}\sigma_1) - (\mu_2 + z_{p_2}\sigma_2)$. The percentiles of W_D can be estimated using Monte Carlo simulation or by approximation as shown below.

Approximation to the Percentiles of W_D

The mean of a noncentral random variable with df $= m$ and the noncentrality parameter δ is given by $E\left(t_m(\delta)\right) = [\delta\Gamma((m-1)/2)\sqrt{m/2}]/\Gamma(m/2)$. For writing convenience, let $t_{i;\alpha} = t_{n_i-1;\alpha}(z_{p_i}\sqrt{n_i})$ and $M_i = E(t_{n_i-1}(z_{p_i}\sqrt{n_i}))$, $i = 1, 2$. Using the modified normal-based approximation by Krishnamoorthy and Wang (2016), the lower 100α percentile of W_D is approximated by

$$W_{D;\alpha} \simeq \bar{x}_1 - \bar{x}_2 + \left(\frac{s_1}{\sqrt{n_1}}M_1 - \frac{s_2}{\sqrt{n_2}}M_2\right) - \sqrt{\frac{s_1^2}{n_1}\left(M_1 - t_{1;\alpha}\right)^2 + \frac{s_2^2}{n_2}\left(M_2 - t_{2;1-\alpha}\right)^2} \tag{21}$$

and

$$W_{D;1-\alpha} \simeq \bar{x}_1 - \bar{x}_2 + \left(\frac{s_1}{\sqrt{n_1}}M_1 - \frac{s_2}{\sqrt{n_2}}M_2\right) + \sqrt{\frac{s_1^2}{n_1}\left(M_1 - t_{1;1-\alpha}\right)^2 + \frac{s_2^2}{n_2}\left(M_2 - t_{2;\alpha}\right)^2}. \tag{22}$$

The interval $(W_{D;\alpha}, W_{D;1-\alpha})$ is a $100(1-2\alpha)\%$ CI for the difference $D = (\mu_1 + z_{p_1}\sigma_1) - (\mu_2 + z_{p_2}\sigma_2)$ and $\left(\exp(W_{D;\alpha}), \exp(W_{D;1-\alpha})\right)$ is a $100(1-2\alpha)\%$ CI for the ratio of lognormal percentiles in (19).

4.2. Lognormal: Example

The data for this example were taken from Doganaksoy (2021) and were used to illustrate a statistical method of comparing means of two lifetime distributions. The data are given in Doganaksoy (2021) and they are reproduced here in Table 4. In reality, such data were collected from semiconductor manufacturing plants for the purpose of reliability assurance. For performance and reliability of semiconductor devices, integrity of gate oxide is crucial as it serves as a dielectric layer between the gate and the substrate. To estimate oxide life, a time-dependent dielectric breakdown (TDDB) test is used on special test structures during device development and manufacturing process. The results of the test are useful to assess the time it takes for the oxide to break which in turn leading to the failure of the device. In order to save time, such testing is conducted under accelerated voltage and temperature.

Suppose an engineer wanted to compare the oxide TDDB distributions of devices built with two nominally identical oxidation furnaces, A and B, by comparing the 95th percentiles of the two life distributions. A simulated sample of $n_1 = 32$ test specimens built using furnace A and a simulated sample of $n_2 = 32$ specimens built using furnace B were obtained and presented in Table 4.

To find a 95% CI for the ratio of 95th percentiles, we computed the following statistics based log-transformed data: $\bar{x}_1 = 3.986$, $\bar{x}_2 = 4.171$, $s_1^2 = .04796$, $s_2^2 = .05093$. Using simulation with 100,000 runs, we computed the 95% fiducial CI as **(0.703, 1.006)**. We computed various quantities to find the approximate fiducial CI as follows:

$$M_1 = M_2 = -9.530, \quad t_{31;.025}(-9.305) = -13.133, \quad \text{and} \quad t_{39;.975}(-9.305) = -6.759.$$

Substituting these values in (21) and (22), we find the 95% CI for $(\mu_1 + z_{.05}\sigma_1) - (\mu_2 + z_{.05}\sigma_2)$ as $(-0.3523, 0.0060)$. By taking exponentiation, we find the 95% approximate fiducial CI as **(0.703, 1.006)**, which is the same as the one based on simulation. This CI indicates that the 95th percentile of lifetimes of gate oxide built using furnace B is greater than the 95th percentile of lifetimes of gate oxide built using furnace A.

To compare the medians of lognormal distributions, we take $(p_1, p_2) = (0.5, 0.5)$, and find $M_1 = M_2 = 0$, $t_{31;.025}(0) = -2.0395$, $t_{31;.975}(0) = 2.0395$. Using these values in (21) and (22), we find the 95% CI for $\mu_1 - \mu_2$ as $(-0.29807, -0.07206)$. By taking exponentiation, we find the 95% approximate fiducial CI as **(0.742, 0.931)**. The 95% CI based on simulation is also **(0.742, 0.931)**. These CIs indicate that the median lifetime of gate oxide built using furnace B is greater than the median lifetime of gate oxide built using furnace A.

5. Two-parameter exponential distributions

Let $X_1, \ldots, X_n$ be a sample from a two-parameter exponential distribution with the probability density function

$$f(x|\mu, \sigma) = \frac{1}{\sigma}\exp(-(x-\mu)/\sigma), \quad x > \mu, \quad \sigma > 0. \tag{23}$$

The MLEs of μ and σ are given by $\widehat{\mu} = X_{(1)}$ and $\widehat{\sigma} = \bar{X} - X_{(1)}$, where $X_{(1)}$ is the smallest of the X_i's. Let $(\widehat{\mu}^*, \widehat{\sigma}^*)$ denote the MLEs based on a sample of n observations from the standard exponential distribution. It is known that $\widehat{\mu}^* \sim \frac{1}{2n}\chi_2^2$ independently of $\widehat{\sigma}^* \sim \frac{1}{2n}\chi_{2n-2}^2$.

The 100p percentile of a two-parameter exponential distribution is given by $q_p(\mu, \sigma) = \mu + q_p(0, 1)\sigma$, where $q_p(0, 1) = -\ln(1-p)$ is the 100p percentile of the standard exponential distribution. Let $(\widehat{\mu}_0, \widehat{\sigma}_0)$ be an observed value of $(\widehat{\mu}, \widehat{\sigma})$. Using (5), we write the FQ for $q_p(\mu, \sigma)$ as

$$\widehat{\mu}_0 + \frac{2nq_p(0,1) - \widehat{\mu}^*}{\widehat{\sigma}^*}\widehat{\sigma}_0 \sim \widehat{\mu}_0 + f_{n,p}(U, V)\widehat{\sigma}_0, \text{ say,} \tag{24}$$

where $f_{n,p}(U, V) = \frac{2nq_p(0,1)-U}{V}$ and $U \sim \chi_2^2$ independently of $V \sim \chi_{2n-2}^2$. Krishnamoorthy and Xia (2018) have provided an exact method of computing the percentiles of $f_{n;p}(U, V)$.

5.1. Exponential: Fiducial confidence intervals

To express the FQ for the ratio of percentiles, let $(\widehat{\mu}_i, \widehat{\sigma}_i)$ be the MLEs based on a sample of size n_i from a two-parameter exponential distribution with parameters μ_i and σ_i, $i = 1, 2$. Let $U_i \sim \chi_2^2$ independently of $V_i \sim \chi_{2n_i-2}^2$, $i = 1, 2$. The random variables U_1, V_1, U_2 and V_2 are all mutually independent. Then a FQ for the ratio of percentiles is given by

$$R_E = \frac{\widehat{\mu}_{10} + f_{n_1,p_1}(U_1, V_1)\widehat{\sigma}_{10}}{\widehat{\mu}_{20} + f_{n_2,p_2}(U_2, V_2)\widehat{\sigma}_{20}}, \tag{25}$$

For a given $(\widehat{\mu}_{10}, \widehat{\sigma}_{10}, \widehat{\mu}_{20}, \widehat{\sigma}_{10})$, the percentiles of R_E can be estimated using Monte Carlo simulation, and the lower and upper 100α percentiles form a $1-2\alpha$ CI for the ratio of percentiles.

Approximation to the Percentiles of R_E

Let $X = \widehat{\mu}_{10} + f_{n_1,p_1}(U_1, V_1)\widehat{\sigma}_{10}$ and $Y = \widehat{\mu}_{20} + f_{n_2,p_2}(U_2, V_2)\widehat{\sigma}_{20}$. Let

$$\mu_x = E(X) = \widehat{\mu}_{10} - \frac{2n_1\ln(1-p_1)+2}{2n_1-4}\widehat{\sigma}_{10} \text{ and } \mu_y = E(Y) = \widehat{\mu}_{20} - \frac{2n_2\ln(1-p_2)+2}{2n_2-4}\widehat{\sigma}_{20}$$

Note that $X_\alpha = \widehat{\mu}_{10} + f_{n_1,p_1;\alpha}(U_1, V_1)\widehat{\sigma}_{10}$ and $Y_\alpha = \widehat{\mu}_{20} + f_{n_2,p_2;\alpha}(U_2, V_2)\widehat{\sigma}_{20}$. Using these expressions in (7), the percentiles of R_E can be obtained. R code to compute the percentiles $f_{n_i,p_i;\alpha}(U_i, V_i)$ is given in a supplementary file. Let $R_{E;\alpha}$ denote the 100α percentile of R_E. Then $(R_{E;\alpha},\ R_{E;1-\alpha})$ is an approximate $1-2\alpha$ CI for the ratio of percentiles of two independent exponential distributions.

5.2. Exponential: Example

The rolling contact fatigue lives (measured in millions of revolutions) in samples of size 10 using specimens made from each of five different types of steel are given in Table 8.2 of McCool (2012). For illustrative purpose we consider only the data from Type A and Type C steel, referred here as Type 1 and Type 2, respectively. These data are reported in Table 5. Using the data, we shall use the methods of this section to find CIs for the ratio of median fatigue lives and for the ratio of 95th percentiles of fatigue lives of Type 1 and Type 2 steel.

Table 5
Rolling contact fatigue data for two steel compositions 10^6 stress cycles.

Type 1:	3.46	5.22	5.69	6.54	9.16	9.40	10.19	10.71	12.58	13.41
Type 2:	3.03	5.53	5.60	9.30	9.92	12.51	12.95	15.21	16.04	16.84

The MLEs are $\widehat{a}_1 = 3.46$, $\widehat{b}_1 = 5.176$, $\widehat{a}_2 = 3.03$ and $\widehat{b}_2 = 7.663$. The 95% CI for the ratio of the medians using Monte Carlo simulation is (0.440, 1.61). To compute the approximate ones, we found $\mu_x = 7.298$, $\mu_y = 8.712$, $f_{10,.5;.025}(U_1, V_1) = f_{10,.5;.025}(U_2, V_2) = 0.3033$ and $f_{10,.5;.975}(U_1, V_1) = f_{10,.5;.975}(U_2, V_2) = 1.5004$. Substituting these quantities in (7), we computed the 95% CI as (0.444, 1.60).

The 95% CI for the ratio of the 95 percentiles using Monte Carlo simulation is (0.314, 1.67). To compute the approximate CI, we found $\mu_x = 22.195$, $\mu_y = 36.768$, $f_{10,.95;.025}(U_1, V_1) = f_{10,.95;.025}(U_2, V_2) = 1.827$ and $f_{10,.95;.975}(U_1, V_1) = f_{10,.95;.975}(U_2, V_2) = 7.052$. Substituting these quantities in (7), we computed the 95% CI as (0.316, 1.66).

6. Weibull distributions

The pdf of a Weibull distribution with shape parameter c and the scale parameter b is given by

$$f(x|b,c) = \frac{c}{b}\left(\frac{x}{b}\right)^{c-1} \exp\left\{-\left[\frac{x}{b}\right]^c\right\},\ x > 0,\ b > 0,\ c > 0.$$

Let $X_1, \ldots, X_n$ be a sample from a Weibull distribution, and let $Y_i = \ln(X_i)$, $i = 1, \ldots, n$. The maximum likelihood estimate (MLE) $\widehat{c}$ of c is the solution of the equation

$$\frac{1}{\widehat{c}} - \left(\sum_{i=1}^n X_i^{\widehat{c}} Y_i\right)\left(\sum_{i=1}^n X_i^{\widehat{c}}\right)^{-1} + \frac{1}{n}\sum_{i=1}^n Y_i = 0, \tag{26}$$

and the MLE of b is given by $\widehat{b} = \left(\frac{1}{n}\sum_{i=1}^n X_i^{\widehat{c}}\right)^{1/\widehat{c}}$. See Cohen (1965) or Krishnamoorthy et al. (2009). In the Newton–Raphson iterative scheme, the estimator $\widehat{c} = \frac{\pi}{\sqrt{6}S_y}$, where $S_y^2 = \sum_{i=1}^n (Y_i - \bar{Y})^2/(n-1)$ can be used as a starting value. The following R code can be used to compute the MLEs.

```
#####################################################################
# package "survival" is needed; x = vector of sample data
model = survreg(Surv(x, rep(1, length(x)))~1, dist="weibull")
c.hat = 1/unname(model$scale); b.hat = exp(unname(model$coef))
#####################################################################
```

6.1. Weibull: Fiducial quantities

Let $(\widehat{b}_0, \widehat{c}_0)$ be an observed value of $(\widehat{b}, \widehat{c})$. Furthermore, let $(\widehat{b}^*, \widehat{c}^*)$ denote the MLEs based on a sample of size n from a Weibull(1, 1) distribution. Using the result that the log-transformed sample from a Weibull distribution follows a location-scale distribution, Krishnamoorthy et al. (2009) have developed the FQs for c and b, which are given by $W_c = \widehat{c}_0/\widehat{c}^*$ and $W_b = \widehat{b}_0 \left(1/\widehat{b}^*\right)^{\widehat{c}^*/\widehat{c}_0}$, respectively.

The pth quantile of a Weibull(b, c) distribution is given by $q_p(b,c) = b(-\ln(1-p))^{\frac{1}{c}}$. A fiducial quantity of $q_p(b,c)$ can be obtained by substitution as $q_p(W_b, W_c)$. Letting

$$w_p = \widehat{c}^*\left(-\ln(\widehat{b}^*) + \ln(-\ln(1-p))\right), \tag{27}$$

we can express the FQ for $q_p(b,c)$ as

$$W_{q_p(b,c)} = q_p(W_b, W_c) = \widehat{b}_0 \exp(w_p/\widehat{c}_0). \tag{28}$$

6.2. Weibull: Fiducial confidence intervals

Let $(\widehat{b}_i, \widehat{c}_i)$ denote the MLE of (b_i, c_i) based on a sample of size n_i from a Weibull(b_i, c_i) distribution, $i = 1, 2$. Let $(\widehat{b}_i^*, \widehat{c}_i^*)$ be the MLEs based on a sample of size n_i from a Weibull(1, 1) distribution, $i = 1, 2$. Recall that the p_i quantile of the Weibull(b_i, c_i) distribution is given $q_{p_i}(b_i, c_i) = b_i\left(-\ln(1-p_i)\right)^{1/c_i}$. The fiducial quantity for the ratio $q_{p_1}(b_1, c_1)/q_{p_2}(b_2, c_2)$ is given by

$$R_W = \frac{q_{p_1}(W_{b_1}, W_{c_1})}{q_{p_2}(W_{b_2}, W_{c_2})} = \frac{\widehat{b}_{10}\exp(w_{p_1}/\widehat{c}_{10})}{\widehat{b}_{20}\exp(w_{p_2}/\widehat{c}_{20})}, \tag{29}$$

where $w_{p_i} = \widehat{c}_i^*\left[-\ln(\widehat{b}_i^*) + \ln(-\ln(1-p_i))\right]$ and $(\widehat{b}_{i0}, \widehat{c}_{i0})$ is an observed value of $(\widehat{b}_i, \widehat{c}_i)$, $i = 1, 2$.

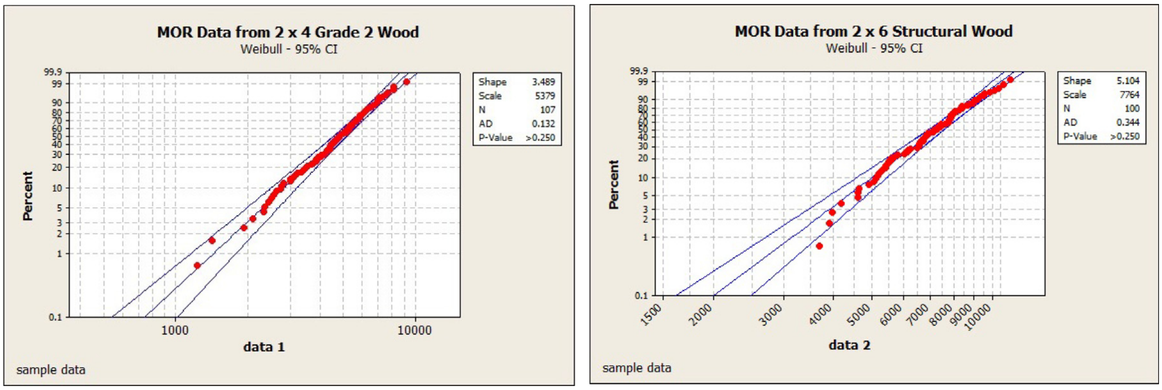

Fig. 1. Weibull probability plots of modus of rupture data in Table 3.

For given sample sizes and $(\widehat{b}_{10}, \widehat{c}_{10}, \widehat{b}_{20}, \widehat{c}_{20})$, the distribution of R_W does not depend on any parameter and so its percentiles can be estimated using Monte Carlo simulation as shown in the following algorithm. The lower and upper 100α percentiles of R_W form a $1-2\alpha$ CI for the ratio $q_{p_1}(b_1, c_1)/q_{p_2}(b_2, c_2)$.

Algorithm 1.

1. Compute the MLEs $(\widehat{b}_{i0}, \widehat{c}_{i0})$ based on a sample of size n_i from a Weibull(b_i, c_i) distribution, $i = 1, 2$.
2. Generate independent samples $X^*_{11}, \ldots, X^*_{1n_1}$ and $X^*_{21}, \ldots, X^*_{2n_2}$ from a Weibull(1, 1) distribution.
3. Compute the MLEs $(\widehat{b}^*_i, \widehat{c}^*_i)$ based on the sample $X^*_{i1}, \ldots, X_{in^*_i}$, $i = 1, 2$.
4. Compute $w_{p_i} = \widehat{c}^*_i \left[-\ln(\widehat{b}^*_i) + \ln(-\ln(1-p_i))\right]$, $i = 1, 2$
5. Compute $R_W = \frac{\widehat{b}_{10}\exp(w_{p_1}/\widehat{c}_{10})}{\widehat{b}_{20}\exp(w_{p_2}/\widehat{c}_{20})}$
6. Repeat steps 2–5 for a large number of times, say, 100,000
7. The lower and upper 100α percentiles of these 100,000 R_W's form a $1-2\alpha$ fiducial CI for the ratio of percentiles.

6.3. Weibull: Example

To illustrate the construction of CI for the ratio of percentiles of two Weibull distributions, we shall use the modus of rupture data given in Table 2 of Huang and Johnson (2006). Recall that the data were used to illustrate the interval estimation methods for the ratio of normal percentiles in Section 3.3. The probability plots of the data indicate that Weibull distributions also fit the data well; see Fig. 1.

The MLEs are $\widehat{c}_{10} = 3.4888$, $\widehat{b}_{10} = 5378.6$, $\widehat{c}_{20} = 5.1042$, and $\widehat{b}_{20} = 7763.7$. The sample sizes are $n_1 = 107$ and $n_2 = 100$. Using Algorithm 1 with 100,000 runs, we estimated 95% CI for the ratio of the 5th percentile as **(.433, .637)**. Note that the CI based on normal models is **(.390, .618)**; see Section 3.3. The CIs based on Weibull models is shifted to the right of the normal-based CI.

We also computed 95% confidence interval for the ratio of medians as (**0.618**, **0.725**). Recall that the one based on normal models is **(0.628, 0.729)**. These two CIs for the ratio of medians are in good agreement.

6.4. Coverage and precision studies

The fiducial CIs are not exact in the frequentist sense except for the normal case with variances are equal. In all other cases, the fiducial CIs approximate. The fiducial CIs are also obtained using simulation or by approximations of percentiles of fiducial quantities. To judge the accuracy of the fiducial CIs and approximate CIs for ratios of percentiles, we estimated the coverage probabilities for some values of (p_1, p_2) and sample sizes. For the normal case, the coverage probabilities of the fiducial CI in (10) are estimated as follows. We first generated 10,000 values of $(\bar{x}_1, \bar{x}_2, s^2_1, s^2_2)$, for each set of values, we used simulation with 10,000 runs to estimate the CI. The proportion of the 10,000 CIs that include the true ratio of the percentiles is a Monte Carlo estimate of the coverage probability. We used simulation with 100,000 runs to estimate the coverage probabilities of the approximate fiducial CI based on (7). The coverage probabilities of fiducial CIs for the lognormal, exponential and Weibull distributions are estimated similarly.

Normal Distributions

For the normal case, the estimated coverage probabilities and the expected widths of 95% CIs for $(p_1, p_2) = (0.75, 0.75), (0.80, 0.90), (0.25, 0.15)$ and $(0.05, 0.05)$ and $(n_1, n_2) = (10, 10), (20, 20), (15, 20), (25, 30), (25, 20), (30,$

Table 6
Coverage probabilities and expected widths of 95% CIs for the ratio of normal percentiles when the variances are unknown and arbitrary.

$\sigma_1 = 1$; $\mu_1 = 4$;										
$\mu_2 = 4\sigma_2$		(p_1, p_2)								
		(.75, .75)		(.80, .90)			(.25, .15)		(.05, .05)	
(n_1, n_2)	σ_2	Siml.	Apprx	Siml.	Apprx	(n_1, n_2)	Siml.	Apprx	Siml.	Apprx
(10, 10)	.05	.958(9.540)	.959(9.532)	.958(9.159)	.956(9.107)	(20, 20)	.955(13.32)	.955(13.16)	.957(26.57)	.951(23.71)
	.10	.961(4.701)	.959(4.766)	.951(4.500)	.956(4.554)		.952(6.545)	.953(6.583)	.956(13.29)	.951(14.56)
	.30	.956(1.555)	.959(1.589)	.955(1.509)	.956(1.518)		.955(2.204)	.953(2.197)	.950(4.335)	.951(4.504)
	.50	.955(0.948)	.959(0.953)	.955(0.903)	.956(0.911)		.956(1.304)	.954(1.320)	.953(2.482)	.951(2.510L)
	.70	.958(0.679)	.959(0.681)	.954(0.650)	.956(0.651)		.954(0.958)	.953(0.941)	.953(1.817)	.951(1.760)
	.80	.961(0.600)	.960(0.596)	.954(0.557)	.957(0.569)		.952(0.845)	.953(0.823)	.958(1.671)	.950(1.511)
	.90	.960(0.534)	.959(0.529)	.960(0.506)	.956(0.506)		.949(0.718)	.955(0.733)	.955(1.441)	.951(1.663)
	1.0	.963(0.476)	.960(0.476)	.955(0.451)	.956(0.455)		.955(0.669)	.954(0.658)	.955(1.364)	.950(1.287)
(15, 20)	.05	.953(6.798)	.955(6.875)	.952(6.523)	.954(6.565)	(25, 30)	.954(10.30)	.953(10.25)	.955(18.33)	.950(18.02)
	.10	.955(3.401)	.956(3.438)	.954(3.224)	.954(3.282)		.953(5.093)	.953(5.130)	.952(9.104)	.951(9.024)
	.30	.954(1.151)	.956(1.146)	.959(1.101)	.954(1.094)		.957(1.724)	.953(1.709)	.949(3.048)	.952(3.008)
	.50	.957(0.684)	.956(0.688)	.952(0.654)	.954(0.656)		.953(1.037)	.953(1.027)	.952(1.825)	.952(1.807)
	.70	.956(0.491)	.956(0.491)	.951(0.467)	.954(0.469)		.953(0.717)	.954(0.732)	.948(1.298)	.952(1.291)
	.80	.951(0.427)	.955(0.430)	.960(0.416)	.954(0.410)		.960(0.655)	.953(0.641)	.950(1.129)	.952(1.127)
	.90	.957(0.386)	.956(0.382)	.953(0.363)	.954(0.365)		.951(0.568)	.952(0.569)	.956(0.993)	.952(1.004)
	1.0	.953(0.339)	.955(0.344)	.956(0.327)	.954(0.328)		.953(0.502)	.954(0.512)	.951(0.913)	.951(0.904)
(25, 20)	.05	.951(5.846)	.954(5.889)	.949(5.585)	.953(5.658)	(30, 25)	.951(10.94)	.952(10.79)	.949(19.81)	.950(19.98)
	.10	.943(2.898)	.954(2.945)	.956(2.848)	.953(2.829)		.955(5.590)	.954(5.397)	.946(10.01)	.950(10.04)
	.30	.957(0.975)	.954(0.982)	.956(0.949)	.953(0.943)		.947(1.769)	.953(1.801)	.955(3.360)	.952(3.295)
	.50	.948(0.584)	.954(0.589)	.945(0.563)	.953(0.566)		.951(1.046)	.952(1.081)	.958(1.998)	.951(1.990)
	.70	.947(0.413)	.955(0.421)	.953(0.406)	.953(0.404)		.951(0.770)	.953(0.771)	.942(1.400)	.952(1.417)
	.80	.955(0.368)	.954(0.368)	.953(0.349)	.953(0.354)		.955(0.680)	.953(0.676)	.952(1.235)	.951(1.246)
	.90	.950(0.323)	.954(0.327)	.951(0.312)	.953(0.314)		.955(0.608)	.952(0.600)	.951(1.080)	.952(1.097)
	1.0	.948(0.283)	.955(0.294)	.955(0.281)	.953(0.283)		.949(0.548)	.952(0.541)	.950(0.982)	.951(1.005)
(25, 30)	.05	.953(5.279)	.954(5.276)	.951(5.023)	.952(5.045)	(30, 40)	.949(8.646)	.953(8.755)	.947(14.72)	.951(14.82)
	.10	.954(2.633)	.953(2.637)	.955(2.499)	.952(2.522)		.954(4.430)	.951(4.386)	.947(7.274)	.952(7.419)
	.30	.953(0.872)	.953(0.879)	.951(0.848)	.952(0.841)		.950(1.449)	.953(1.460)	.951(2.524)	.951(2.475)
	.50	.951(0.528)	.954(0.527)	.952(0.499)	.953(0.504)		.952(0.868)	.952(0.875)	.953(1.492)	.951(1.483)
	.70	.945(0.369)	.954(0.377)	.951(0.361)	.952(0.360)		.953(0.628)	.953(0.626)	.957(1.081)	.951(1.061)
	.80	.949(0.327)	.954(0.330)	.956(0.325)	.953(0.315)		.951(0.555)	.952(0.547)	.952(0.930)	.950(0.928)
	.90	.954(0.291)	.954(0.293)	.950(0.281)	.952(0.280)		.951(0.489)	.952(0.487)	.946(0.807)	.951(0.824)
	1.0	.953(0.261)	.954(0.264)	.957(0.255)	.952(0.252)		.953(0.436)	.954(0.438)	.949(0.754)	.951(0.742)

25), (25, 30) and (30, 40) are reported in Table 6. The estimated values clearly indicate that the fiducial CIs based on simulation and the approximation are very similar in terms of coverage probabilities and expected widths. For all cases considered, the coverage probabilities are very close to or little larger than the nominal level 0.95. Specifically, the coverage probabilities are very close to the nominal level 0.95 when both sample sizes are 20 or more. These findings suggest that, in applications, the approximate fiducial CIs can be safely used and thereby simulation can be avoided.

Lognormal Distributions

We estimated the coverage probabilities and expected widths of CIs for ratios of percentiles of lognormal distributions for some values of (p_1, p_2) and sample sizes and reported them in Table 7. We used simulation with 100,000 runs to estimate the coverage probabilities of the approximate fiducial CIs in (21) and (22). The estimated coverage probabilities and the expected widths clearly indicate that the fiducial CIs based on simulation and the approximation are very similar in terms of coverage probabilities and expected widths. For all cases considered, the coverage probabilities are very close to or little larger than the nominal level 0.95. Thus, the fiducial CIs or the approximate one formed by (21) and (22) can be safely used even when sample sizes are small.

Exponential Distributions

For the exponential case, the estimated coverage probabilities and expected widths are reported in Table 8. Parameter values for simulation studies were chosen by considering equivariant property of the confidence intervals. The coverage probabilities and expected widths of the fiducial CI based on (25) and the approximate fiducial CI show that they are practically very similar. Both CIs control the coverage probabilities very close to the nominal level for most cases. The coverage probabilities are seldom fall around 0.96. In general, both CIs perform like an exact confidence interval.

Weibull Distributions

Monte Carlo estimates of coverage probabilities of the CIs for ratio of percentiles of Weibull distributions are presented in Table 9. We chose $(n_1, n_2) = (5, 5), (15, 10), (15, 15), (20, 20), (30, 30)$, and $(p_1, p_2) = (.05, .05), (.15, .05), (.5, .5), (.90, .90), (.90, .05), (.25, .75)$. Examination of the coverage probabilities indicate that the fiducial CIs could be slightly conservative for small sample sizes; see the results for $(n_1, n_2) = (5, 5)$. Even for small samples of size 5, the coverage

Table 7
Coverage probabilities and expected widths of 95% CIs for the ratio of lognormal percentiles.

$\sigma_1 = 1$		(p_1, p_2)									
		(.05, .05)		(.15, .25)		(.25, .25)		(.75, .25)		(.95, .95)	
(n_1, n_2)	σ_2	Siml.	Apprx	Siml.	Apprx	Siml.	Apprx	Siml.	Apprx	Siml.	Apprx
(5, 5)	.05	.950(4.26)	.951(4.18)	.950(3.14)	0.950(3.17)	.948(2.65)	.951(2.72)	.952(2.68)	.950(2.71)	.950(4.18)	.951(4.17)
	.10	.962(4.21)	.953(4.21)	.950(3.29)	0.952(3.19)	.949(2.71)	.953(2.73)	.951(2.74)	.950(2.72)	.957(4.23)	.953(4.21)
	.30	.959(4.53)	.956(4.50)	.960(3.29)	0.957(3.34)	.960(2.83)	.960(2.90)	.957(2.87)	.957(2.86)	.952(4.44)	.955(4.52)
	.50	.955(5.05)	.955(4.96)	.963(3.59)	0.961(3.59)	.963(3.15)	.965(3.16)	.969(3.17)	.962(3.10)	.957(5.01)	.956(4.95)
	.70	.956(5.59)	.956(5.50)	.968(3.97)	0.963(3.89)	.969(3.45)	.968(3.48)	.973(3.45)	.964(3.41)	.957(5.63)	.954(5.51)
	.80	.951(5.75)	.955(5.80)	.968(4.11)	0.964(4.07)	.972(3.66)	.968(3.66)	.966(3.60)	.964(3.58)	.956(5.80)	.955(5.80)
	.90	.957(6.25)	.955(6.11)	.966(4.25)	0.965(4.25)	.968(3.87)	.968(3.86)	.976(3.88)	.965(3.76)	.952(6.15)	.955(6.11)
(10, 5)	.05	.950(2.25)	.952(2.29)	.958(1.85)	0.951(1.80)	.953(1.57)	.951(1.58)	.944(1.56)	.951(1.58)	.942(2.24)	.951(2.29)
	.10	.960(2.36)	.954(2.34)	.951(1.80)	0.953(1.82)	.952(1.60)	.953(1.60)	.952(1.62)	.951(1.60)	.948(2.26)	.953(2.34)
	.30	.952(2.73)	.955(2.72)	.955(2.00)	0.958(2.02)	.957(1.81)	.960(1.81)	.958(1.77)	.957(1.79)	.952(2.70)	.955(2.72)
	.50	.961(3.40)	.954(3.29)	.958(2.32)	0.963(2.34)	.965(2.15)	.964(2.16)	.960(2.10)	.960(2.13)	.955(3.34)	.954(3.29)
	.70	.959(4.00)	.952(3.93)	.961(2.78)	0.962(2.73)	.966(2.65)	.965(2.56)	.963(2.54)	.962(2.52)	.957(4.05)	.953(3.93)
	.80	.949(4.27)	.954(4.28)	.966(3.03)	0.962(2.94)	.961(2.77)	.963(2.78)	.962(2.67)	.961(2.74)	.956(4.24)	.953(4.27)
	.90	.956(4.60)	.953(4.63)	.955(3.09)	0.962(3.15)	.967(2.99)	.963(3.01)	.970(2.92)	.962(2.96)	.956(4.67)	.954(4.62)
(10, 15)	.05	.948(2.26)	.951(2.28)	.947(1.75)	0.950(1.79)	.956(1.60)	.951(1.57)	.951(1.57)	.951(1.57)	.953(2.30)	.950(2.28)
	.10	.955(2.33)	.952(2.28)	.948(1.81)	0.951(1.79)	.952(1.60)	.950(1.57)	.956(1.58)	.950(1.57)	.948(2.27)	.951(2.28)
	.30	.951(2.37)	.952(2.35)	.952(1.82)	0.952(1.83)	.948(1.59)	.953(1.62)	.950(1.62)	.951(1.61)	.953(2.39)	.952(2.35)
	.50	.945(2.43)	.952(2.47)	.953(1.90)	0.952(1.91)	.954(1.67)	.954(1.70)	.950(1.69)	.952(1.69)	.948(2.46)	.952(2.47)
	.70	.950(2.62)	.951(2.64)	.947(1.94)	0.953(2.01)	.950(1.79)	.957(1.81)	.957(1.78)	.955(1.80)	.940(2.62)	.951(2.64)
	.80	.952(2.78)	.952(2.74)	.960(2.10)	0.956(2.08)	.952(1.85)	.957(1.88)	.955(1.86)	.955(1.87)	.951(2.72)	.952(2.74)
	.90	.957(2.88)	.953(2.84)	.958(2.14)	0.955(2.14)	.953(1.91)	.958(1.95)	.957(1.93)	.956(1.93)	.955(2.89)	.952(2.84)
(20, 20)	.05	.943(1.44)	.951(1.46)	.945(1.15)	0.950(1.16)	.955(1.02)	.951(1.03)	.952(1.03)	.951(1.03)	.943(1.44)	.950(1.46)
	.10	.949(1.47)	.951(1.47)	.954(1.18)	0.951(1.17)	.951(1.03)	.949(1.03)	.947(1.03)	.951(1.03)	.938(1.46)	.950(1.47)
	.30	.953(1.53)	.952(1.53)	.947(1.19)	0.952(1.21)	.956(1.09)	.952(1.08)	.948(1.07)	.951(1.08)	.952(1.52)	.952(1.53)
	.50	.955(1.68)	.951(1.65)	.963(1.32)	0.954(1.28)	.951(1.14)	.954(1.16)	.948(1.15)	.951(1.16)	.951(1.67)	.952(1.65)
	.70	.948(1.84)	.952(1.81)	.952(1.36)	0.953(1.38)	.960(1.29)	.955(1.27)	.952(1.26)	.954(1.26)	.952(1.83)	.950(1.81)
	.80	.952(1.89)	.953(1.90)	.956(1.45)	0.952(1.44)	.952(1.32)	.955(1.33)	.951(1.31)	.953(1.33)	.949(1.91)	.951(1.90)
	.90	.948(2.00)	.951(2.00)	.944(1.46)	0.954(1.51)	.961(1.41)	.955(1.40)	.947(1.39)	.953(1.39)	.957(2.00)	.951(2.00)

probabilities are very close to the nominal level 0.95 for some cases; see the coverage probabilities for $(p_1, p_2) = (.05, .05)$ and $(.15, .05)$. Overall, we see that the fiducial CIs are very satisfactory when both sample sizes are 15 or more.

7. Some other continuous distributions

The fiducial method of finding CIs for the ratio of percentiles can be readily applied to other location-scale distributions such as the Laplace (double exponential) and the logistic. The MLEs for the parameters of a Laplace distribution can be found using the results of Childs and Balakrishnan (1997); also see Krishnamoorthy and Xie (2011). These MLEs are location-scale equivariant and so a FQ for the ratio of percentiles can be readily obtained. The MLEs of a logistic distribution cannot be expressed in closed-forms and they can be obtained only numerically. See Harter and Moore (1967) and Krishnamoorthy and Xie (2011). Using these MLEs, we can readily find FQs for the parameters of a logistic distribution.

Using the fiducial approach, approximate CIs for the ratio of percentiles of two gamma distributions can also be obtained. It is well-known that if Y has a gamma distribution with shape parameter a and the scale parameter b, then $X = Y^{1/3}$ has an approximate normal distribution (Wilson and Hilferty, 1931). Using this result, Krishnamoorthy et al. (2008) and Krishnamoorthy and Wang (2016) have obtained approximate FQs for a and b using the FQs of the normal mean and variance. Let $(\bar{x}_i, s_i^2)$ denote the (mean, variance) based on cube root transformed sample of size n_i from gamma(a_i, b_i) distribution, where a_i the shape parameter and b_i is the scale parameter, $i = 1, 2$. Then the FQ for the ratio of the $100p_1$ percentile of gamma(a_1, b_1) distribution and the $100p_2$ percentile of gamma(a_2, b_2) distribution is given by

$$Q_{R_G} = \frac{\left(\bar{x}_1 + \frac{Z_1 + z_{p_1}\sqrt{n_1}}{U_1}\frac{s_1}{\sqrt{n_1}}\right)^3}{\left(\bar{x}_2 + \frac{Z_2 + z_{p_2}\sqrt{n_2}}{U_2}\frac{s_2}{\sqrt{n_2}}\right)^3}, \tag{30}$$

where the random variables Z_1, Z_2, U_1 and U_2 are as defined in (10). For given $(\bar{x}_1, s_1, \bar{x}_2, s_2)$, appropriate percentiles of Q_{R_G} form a CI. Alternatively, an approximate CI for a ratio of percentiles of two gamma distributions can be readily obtained using the approximation in (7).

We have carried out some simulation studies to understand the properties of CIs based on the approximation in (7). The estimated coverage probabilities of CIs for the ratio of $100p_1$ and $100p_2$ percentiles are reported in Table 10. Our preliminary simulation studies (not reported here) indicated that the CIs for a ratio of gamma percentiles are unsatisfactory

Table 8
Coverage probabilities and expected widths of 95% CIs for the ratio of exponential percentiles.

$(\mu_1, \mu_2) = (1, 1)$; $\sigma_1 = 1$

		(p_1, p_2)									
		(.05, .05)		(.25, .25)		(.75, .25)		(.75, .75)		(.95, .75)	
(n_1, n_2)	σ_2	Siml.	Apprx	Siml.	Apprx	Siml.	Apprx	Siml.	Apprx	Siml.	Apprx
(10, 10)	.05	.956(.447)	.953(.447)	.951(.588)	.950(.584)	.951(2.09)	.951(2.09)	.949(2.00)	.951(1.99)	.950(4.45)	.951(4.38)
	.10	.960(.454)	.956(.452)	.949(.579)	.952(.581)	.950(2.07)	.951(2.07)	.952(1.91)	.951(1.90)	.953(4.11)	.953(4.13)
	.30	.953(.500)	.954(.496)	.961(.581)	.962(.583)	.954(1.99)	.953(2.00)	.952(1.67)	.952(1.65)	.953(3.52)	.951(3.49)
	.50	.954(.585)	.950(.570)	.972(.600)	.970(.604)	.953(1.93)	.956(1.96)	.956(1.49)	.951(1.48)	.950(3.11)	.951(3.07)
	.70	.951(.674)	.948(.668)	.971(.629)	.971(.634)	.955(1.94)	.958(1.95)	.953(1.37)	.951(1.35)	.953(2.80)	.950(2.75)
	.90	.952(.803)	.948(.804)	.969(.655)	.971(.665)	.957(1.93)	.961(1.95)	.953(1.25)	.950(1.23)	.950(2.53)	.950(2.50)
	1.0	.955(.879)	.946(.889)	.967(.671)	.971(.681)	.964(1.95)	.961(1.95)	.954(1.20)	.951(1.19)	.951(2.45)	.949(2.39)
(10, 15)	.05	.951(.452)	.951(.446)	.950(.587)	.950(.583)	.951(2.07)	.951(2.10)	.951(2.00)	.950(1.99)	.950(4.35)	.951(4.38)
	.10	.950(.442)	.953(.448)	.952(.573)	.950(.576)	.949(2.07)	.951(2.07)	.949(1.90)	.951(1.88)	.951(4.16)	.951(4.14)
	.30	.957(.465)	.955(.462)	.953(.563)	.955(.562)	.952(1.99)	.951(1.98)	.956(1.60)	.951(1.60)	.946(3.42)	.951(3.44)
	.50	.954(.503)	.952(.490)	.961(.556)	.960(.557)	.950(1.92)	.952(1.91)	.953(1.42)	.951(1.41)	.952(3.00)	.950(2.97)
	.70	.958(.542)	.951(.528)	.965(.552)	.966(.560)	.953(1.84)	.953(1.85)	.951(1.28)	.950(1.27)	.951(2.66)	.951(2.63)
	.90	.951(.585)	.949(.572)	.969(.557)	.969(.565)	.954(1.79)	.955(1.80)	.952(1.16)	.950(1.15)	.953(2.39)	.951(2.38)
	1.0	.955(.617)	.949(.598)	.970(.562)	.969(.567)	.956(1.78)	.956(1.78)	.952(1.11)	.951(1.10)	.949(2.27)	.950(2.26)
(15, 15)	.05	.955(.279)	.953(.281)	.949(.402)	.950(.407)	.949(1.58)	.951(1.58)	.948(1.51)	.951(1.50)	.954(3.29)	.951(3.30)
	.10	.957(.285)	.954(.283)	.949(.401)	.949(.403)	.947(1.57)	.950(1.56)	.952(1.42)	.951(1.43)	.947(3.10)	.949(3.11)
	.30	.960(.311)	.954(.305)	.952(.398)	.958(.404)	.953(1.50)	.952(1.50)	.953(1.28)	.951(1.25)	.950(2.65)	.950(2.64)
	.50	.950(.346)	.950(.340)	.961(.411)	.963(.416)	.952(1.46)	.952(1.46)	.951(1.14)	.951(1.13)	.953(2.34)	.950(2.32)
	.70	.955(.392)	.949(.384)	.967(.429)	.968(.432)	.950(1.41)	.955(1.43)	.951(1.04)	.949(1.03)	.949(2.10)	.950(2.08)
	.90	.955(.450)	.950(.436)	.968(.441)	.969(.448)	.956(1.40)	.957(1.41)	.952(0.97)	.950(0.95)	.953(1.91)	.950(1.89)
	1.0	.949(.471)	.948(.463)	.966(.452)	.968(.456)	.956(1.40)	.957(1.40)	.950(0.92)	.951(0.91)	.951(1.83)	.951(1.81)
(20, 20)	.05	.951(.207)	.951(.206)	.954(.326)	.950(.324)	.950(1.33)	.950(1.33)	.953(1.26)	.950(1.26)	.950(2.73)	.950(2.75)
	.10	.955(.209)	.953(.207)	.948(.322)	.950(.322)	.950(1.31)	.950(1.31)	.954(1.21)	.950(1.20)	.953(2.61)	.950(2.59)
	.30	.952(.221)	.953(.221)	.953(.318)	.955(.321)	.953(1.25)	.951(1.25)	.946(1.05)	.950(1.05)	.953(2.22)	.950(2.20)
	.50	.949(.247)	.951(.245)	.961(.326)	.960(.330)	.954(1.21)	.952(1.21)	.948(.963)	.950(.955)	.952(1.95)	.951(1.94)
	.70	.952(.279)	.951(.273)	.957(.338)	.963(.342)	.953(1.18)	.952(1.18)	.950(.881)	.949(.873)	.951(1.74)	.950(1.74)
	.90	.955(.312)	.949(.305)	.962(.352)	.964(.354)	.948(1.15)	.953(1.16)	.956(.814)	.949(.806)	.954(1.60)	.950(1.59)
	1.0	.954(.329)	.950(.322)	.962(.358)	.964(.360)	.957(1.15)	.954(1.15)	.949(.783)	.949(.774)	.950(1.53)	.949(1.52)
(30, 10)	.05	.960(.140)	.956(.140)	.947(.244)	.950(.245)	.950(1.04)	.949(1.04)	.949(1.00)	.951(1.01)	.949(2.19)	.950(2.17)
	.10	.959(.151)	.956(.150)	.955(.250)	.956(.250)	.951(1.03)	.952(1.03)	.950(1.01)	.950(.999)	.951(2.10)	.951(2.10)
	.30	.953(.227)	.949(.221)	.963(.299)	.966(.301)	.954(1.03)	.956(1.04)	.955(1.00)	.950(.994)	.951(1.97)	.949(1.95)
	.50	.953(.325)	.949(.318)	.962(.361)	.965(.365)	.961(1.08)	.961(1.08)	.954(.978)	.949(.965)	.951(1.85)	.949(1.83)
	.70	.953(.443)	.950(.437)	.965(.425)	.964(.428)	.961(1.12)	.964(1.14)	.948(.933)	.950(.922)	.951(1.74)	.949(1.71)
	.90	.958(.577)	.951(.585)	.957(.480)	.961(.486)	.964(1.17)	.966(1.20)	.956(.889)	.950(.875)	.947(1.63)	.950(1.61)
	1.0	.957(.645)	.951(.685)	.958(.502)	.958(.513)	.963(1.21)	.964(1.23)	.953(.865)	.951(.850)	.950(1.57)	.950(1.56)

for small values of (p_1, p_2). We reported the estimates of coverage probabilities for $(p_1, p_2) = (0.50, 0.50)$, $(0.75, 0.50)$, $(0.75, 0.75)$ and $(0.95, 0.95)$ in Table 10. These results indicate that the approximate CIs are satisfactory for practical use when both p_1 and p_2 are 0.5 or large.

8. Concluding remarks

In this article, we have proposed the fiducial approach to find CIs for a ratio of percentiles of two location-scale distributions. The fiducial approach is not only conceptually simple, but also accurate even for small samples in some cases. Since the fiducial approach is not necessarily exact in the frequentist sense, we carried out extensive simulation studies to judge the accuracy of the CIs in terms of coverage probability and precision. Our simulation studies and other theoretical comparisons showed that the fiducial CIs are exact for the normal case and they are very satisfactory for the lognormal, two-parameter exponential and Weibull distributions.

Recall that we have used the classical pivotal quantities based on the MLEs to get fiducial quantities for the location and scale parameters. The MLEs based on type II censored samples from a location-scale distribution are also equivariant, and so pivotal quantities can be readily obtained using the MLEs (see Section 4.1.2.1 of Lawelss, 2003). Thus, pivotal quantities and fiducial quantities are also valid if the samples are type II censored, and find CIs for the percentiles (see Krishnamoorthy and Xu (2011)) and for the ratio of percentiles of two location-scale distributions when the samples are type II censored. We are currently working on finding CIs for a ratio percentiles when the samples are type I or type II censored and plan to publish the results elsewhere.

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Table 9
Coverage probabilities and expected widths of 95% CIs for the ratio of Weibull percentiles.

$(b_1, b_2) = (1, 1)$						
	$(p_1, p_2) = (.05, .05)$					
	(n_1, n_2); $c_1 = 2$		(n_1, n_2); $c_1 = 5$		(n_1, n_2); $c_1 = 7$	
c_2	(10, 10)	(20, 10)	(10, 10)	(20, 10)	(10, 10)	(20, 10)
3	.950(3.14)	.951(2.86)	.951(4.05)	.950(3.71)	.952(4.23)	.952(4.67)
4	.954(1.72)	.950(1.52)	.954(2.11)	.947(1.91)	.944(2.18)	.955(2.26)
5	.953(1.21)	.949(1.04)	.956(1.43)	.951(1.20)	.950(1.48)	.955(1.45)
7	.954(.846)	.953(.685)	.954(.940)	.956(.788)	.950(.934)	.947(.870)
10	.951(.658)	.954(.509)	.956(.695)	.953(.545)	.947(.660)	.951(.566)
12	.953(.604)	.953(.458)	.956(.623)	.953(.474)	.949(.579)	.954(.479)
	$(p_1, p_2) = (.50, .25)$					
	(n_1, n_2); $c_1 = 2$		(n_1, n_2); $c_1 = 5$		(n_1, n_2); $c_1 = 7$	
c_2	(10, 10)	(20, 10)	(10, 10)	(20, 10)	(10, 10)	(30, 20)
3	.955(1.68)	.953(1.48)	.956(.342)	.951(.285)	.962(.254)	.952(.131)
4	.958(1.25)	.954(1.03)	.955(.253)	.951(.201)	.962(.186)	.951(.098)
5	.956(1.06)	.953(.833)	.953(.212)	.953(.163)	.959(.156)	.952(.083)
7	.955(.903)	.953(.663)	.950(.177)	.948(.130)	.955(.129)	.950(.069)
10	.952(.807)	.952(.564)	.946(.156)	.945(.112)	.955(.114)	.950(.061)
12	.953(.775)	.954(.535)	.944(.150)	.947(.106)	.953(.109)	.953(.059)
	$(p_1, p_2) = (.90, .90)$					
	(n_1, n_2); $c_1 = 2$		(n_1, n_2); $c_1 = 5$		(n_1, n_2); $c_1 = 7$	
c_2	(10, 10)	(20, 20)	(10, 10)	(20, 20)	(10, 10)	(30, 20)
3	.966(.679)	.955(.376)	.954(.255)	.959(.181)	.957(.190)	.949(.099)
4	.959(.682)	.954(.375)	.954(.257)	.961(.173)	.960(.192)	.948(.102)
5	.960(.686)	.955(.377)	.952(.260)	.961(.168)	.958(.193)	.950(.105)
7	.959(.699)	.951(.380)	.948(.265)	.961(.164)	.957(.196)	.946(.108)
10	.956(.713)	.947(.385)	.949(.270)	.958(.162)	.955(.200)	.949(.110)
12	.955(.720)	.950(.389)	.947(.273)	.956(.161)	.953(.202)	.948(.112)

Table 10
Coverage probabilities 95% CIs for the ratio of gamma percentiles.

	$a_1 = 0.5$															
a_2	$p_1 = p_2 = 0.50$				$p_1 = 0.75, p_2 = 0.50$				$p_1 = 0.75, p_2 = 0.75$				$p_1 = 0.95, p_2 = 0.95$			
	(10, 10)	(10, 20)	(20, 20)	(20, 30)	(10, 10)	(10, 20)	(20, 20)	(20, 30)	(10, 10)	(10, 20)	(20, 20)	(20, 30)	(10, 10)	(10, 20)	(20, 20)	(20, 30)
1	.960	.955	.953	.951	.951	.943	.945	.939	.944	.939	.939	.934	.939	.937	.934	.933
2	.955	.953	.948	.948	.946	.939	.944	.940	.942	.937	.939	.933	.939	.937	.934	.934
3	.953	.950	.947	.946	.942	.936	.945	.941	.941	.938	.938	.932	.940	.937	.934	.935
5	.950	.950	.946	.946	.940	.935	.944	.939	.937	.939	.938	.930	.940	.936	.933	.936
7	.950	.949	.945	.944	.938	.934	.946	.938	.937	.940	.939	.930	.941	.936	.937	.937
9	.949	.948	.945	.944	.937	.934	.943	.940	.936	.939	.940	.930	.938	.935	.937	.937
15	.947	.949	.944	.944	.936	.934	.945	.939	.934	.938	.938	.928	.939	.935	.934	.936
	$a_1 = 2.0$															
1	.963	.961	.955	.955	.960	.957	.955	.954	.954	.954	.949	.949	.951	.949	.950	.949
2	.963	.958	.957	.956	.959	.956	.954	.953	.959	.955	.953	.952	.954	.951	.951	.951
3	.962	.957	.957	.955	.960	.955	.954	.953	.958	.955	.953	.953	.953	.952	.952	.952
5	.961	.955	.956	.955	.959	.953	.953	.953	.958	.953	.953	.952	.954	.950	.953	.953
7	.960	.954	.955	.954	.957	.953	.951	.951	.956	.952	.952	.952	.954	.951	.953	.953
9	.958	.953	.954	.952	.956	.951	.950	.950	.955	.952	.952	.951	.953	.951	.951	.952
15	.956	.952	.953	.951	.955	.950	.950	.950	.955	.952	.950	.951	.952	.951	.951	.952

Appendix A

Let $(\bar{X}, S^2)$ denote the (mean, variance) based on a sample of size n from a $N(\mu, \sigma^2)$ distribution. Let $(\bar{x}, s^2)$ be an observed value of $(\bar{X}, S^2)$. Let $Z \sim N(0, 1)$ independently of $U \sim \chi^2_m/m$. To show that

$$\bar{x} + \frac{Z + z_p\sqrt{n}}{U}\frac{s}{\sqrt{n}} > 0 \quad \text{for all } \bar{x}, s \text{ and } p \in (0, 1), \tag{31}$$

we need to show that $\frac{\sqrt{n}\bar{X}}{S} > \frac{Z+z_p\sqrt{n}}{U}$ with practical certainty. Note that $\sqrt{n}\bar{X}/S \sim t_m(\sqrt{n}\mu/\sigma)$. If a normal model is postulated for a positive random variable X, then it is reasonable to assume that $\mu - 3\sigma > 0$ or $\mu/\sigma > 3$. As a result, for

(31) to hold, we should have

$$P\left(t_m(\mu\sqrt{n}/\sigma) > -t_m(z_p\sqrt{n})\right) > P\left(t_m(3\sqrt{n}) > t_m(-z_p\sqrt{n})\right) \simeq 1.$$

Let $F_m(\delta)$ denote the CDF of a noncentral t random variable with df $= m$ and the noncentrality parameter δ. Then, (31) holds if

$$P\left(t_m(3\sqrt{n}) > t_m(-z_p\sqrt{n})\right) = 1 - E_T\left(F_m\left(T|3\sqrt{n}\right)\right) \simeq 1, \tag{32}$$

where $T \sim t_m(-z_p\sqrt{n})$.

In Table 1, we give the minimum value of n for which the above probability is very close unity.

Appendix B

Recall that

$$Q_D = \left(\bar{x}_1 + \frac{Z_1 + z_{p_1}\sqrt{n_1}}{U_f}\frac{s_p}{\sqrt{n_1}}\right) - R_0\left(\bar{x}_2 + \frac{Z_2 + z_{p_2}\sqrt{n_2}}{U_f}\frac{s_p}{\sqrt{n_2}}\right),$$

where $U_f \sim \chi_f^2/f$ and $f = m_1 + m_2$. After rearranging the terms in Q_D, we can write the fiducial quantity as

$$Q_D = (\bar{x}_1 - R_0\bar{x}_2) - \frac{s_p}{U_f}\left[\frac{(Z_1 - \sqrt{n_1}z_{p_1})}{\sqrt{n_1}} - R_0\frac{Z_2 - z_{p_2}\sqrt{n_2}}{\sqrt{n_2}}\right].$$

It is easy to verify that the term within the square brackets is distributed as $(R_0 z_{p_2} - z_{p_1}) + Z\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}$, where Z is the standard normal random variable. Using this result and the definition of the noncentral t distribution, we see that

$$Q_D \overset{d}{=} (\bar{x}_1 - R_0\bar{x}_2) + t_f(\delta(R_0))s_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}, \text{ with } \delta(R_0) = \frac{z_{p_1} - R_0 z_{p_2}}{\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}}. \tag{33}$$

To get (33), we used the result that $t_m(-\delta)$ and $-t_m(\delta)$ are identically distributed. Thus, we can express the fiducial p-value for testing (13) as

$$P(Q_D > 0) = P\left(t_f(\delta(R_0)) > \frac{R_0\bar{x}_2 - \bar{x}_1}{s_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}}\right). \tag{34}$$

It is not difficult to check that the random quantity

$$\frac{R_0\bar{X}_2 - \bar{X}_1}{S_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}} \overset{d}{=} \frac{Z + \frac{z_{p_1} - z_{p_2}R_0}{\sqrt{1/n_1 + R_0^2/n_2}}}{\chi_f^2/f} \sim t_f(\delta(R_0)),$$

where $Z \sim N(0, 1)$ independently of χ_f^2, and hence the p-value in (34) is a realization of the random p-value

$$P\left(t_f(\delta(R_0)) > \frac{R_0\bar{X}_2 - \bar{X}_1}{S_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}}\right).$$

Using the probability integral transform, we see that the above p-value has the uniform (0, 1) distribution. So the test that rejects H_0 in (13) whenever the p-value in (34) is less than α is an exact level α test. Thus, the value of R_0 for which

$$P\left(t_f(\delta(R_0)) > \frac{R_0\bar{X}_2 - \bar{X}_1}{S_p\sqrt{\frac{1}{n_1} + \frac{R_0^2}{n_2}}}\right) = \alpha$$

is a $100(1-\alpha)\%$ upper confidence limit for the ratio of normal percentiles. Note that the value R_0 that satisfies the above equation can also be obtained as the root of Eq. (16).

Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jspi.2023.07.003.

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