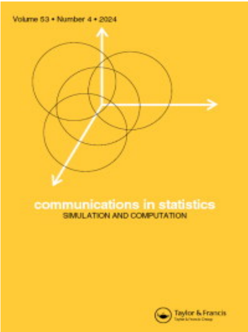

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Kalimuthu Krishnamoorthy & Shanshan Lv

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LETTER TO THE EDITOR

A report on the paper "xia cai, feng siman & yan liang (2022): generalized fiducial inference for the lower confidence limit of reliability based on weibull distribution, communications in statistics - simulation and computation, DOI: 10.1080/03610918.2022.2067873"

Consider the two-parameter Weibull(b, c) distribution with the probability density function (pdf) given by

$$f(x|b,c) = \frac{c}{b}\left(\frac{x}{b}\right)^{c-1} \exp\left\{-\left[\frac{x}{b}\right]^c\right\}, x>0,\ b>0,\ c>0.$$

Cai, Siman and Liang (2022) provided generalized fiducial inference for finding confidence intervals (CIs) for the parameters b and c and for the reliability $S_{t_0} = P(X > t_0)$, where t_0 is a specified value. They proposed this fiducial inference claiming that small sample accurate confidence intervals are not available. On the basis of their limited simulation studies, they have concluded that the generalized fiducial CIs for the parameters and the reliability are better than the frequentist classical CIs based on the likelihood method. However, we find that their claims are NOT true. There are much simpler exact methods are available to find CIs for the parameters, tolerance intervals and confidence limits for S_{t_0}. We also note that fiducial CIs given in Krishnamoorthy et al. (2009) and Krishnamoorthy and Lin (2010) are simple, accurate and straightforward to implement compared to the generalized fiducial inference proposed in Cai, Siman and Liang (2022).

In the following, we present available pivotal-based approach to find CIs for the Weibull parameters and a fiducial approach to find a CI for the reliability $P(X > t_0)$, where t_0 is a specified value. Let $(\hat{b}, \hat{c})$ denote the maximum likelihood estimate (MLE) of (b, c) based on a sample of size n from a Weibull(b, c) distribution. The MLEs are not in closed-from and they can be obtained only numerically. See Cohen (1965) or Krishnamoorthy et al. (2009). The MLEs can be readily obtained using the following R function in the package "survival."

```
model = survreg(Surv(x, rep(1, length(x)))~1, dist = "weibull")
```

Here x is the vector of sample data, MLE $\hat{c}$ = 1/(model$scale) and MLE $\hat{b}$ = exp(model$coef).

Exact Confidence Intervals for b and c

Noting that the Weibull distribution is log-location-scale distribution, it can be shown that

$$\hat{c}/c \quad \text{and} \quad \hat{c}\ln\left(\hat{b}/b\right) \tag{1}$$

are pivotal quantities; see Krishnamoorthy (2015, Chapter 25). This means that

$$\hat{c}/c \sim \hat{c}^* \quad \text{and} \quad \hat{c}\ln\left(\hat{b}/b\right) \sim \hat{c}^*\ln(\hat{b}^*), \tag{2}$$

where $\hat{c}^*$ and $\hat{b}^*$ are MLEs based on a sample $Z_1, \ldots, Z_n$ from a Weibull (1) distribution. Since the distributions of $\hat{c}^*$ and $\hat{b}^*$ do not depend on any parameter, their distributions can be obtained

CONTACT Kalimuthu Krishnamoorthy krishna@louisiana.edu Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA.

empirically via Monte Carlo simulation. Let $\hat{c}^*_\alpha$ denote the 100 α percentile of $\hat{c}^*$. Note that $\hat{c}^*_\alpha$ can be estimated using Monte Carlo simulation. Then

$$(\hat{c}/\hat{c}^*_{1-\alpha},\ \hat{c}/\hat{c}^*_\alpha) \tag{3}$$

is a $100(1-2\alpha)\%$ CI for c.

Letting Q^*_α to denote the 100α percentile of $\hat{c}^*\ln(\hat{b}^*)$, a $100(1-2\alpha)\%$ CI can be obtained using the distributional result (1) as

$$\left(\hat{b}\exp\left(-Q^*_{1-\alpha}/\hat{c},\ \hat{b}\exp\left(-Q^*_\alpha/\hat{c}\right)\right)\right). \tag{4}$$

Note that the above two CIs are based on the pivotal quantities and they are exact.

Fiducial CIs for a survival probability

Let $(\hat{b}_0, \hat{c}_0)$ be an observed value of $(\hat{b}, \hat{c})$. Solving the "equations" in (2) for c and b, and then replacing $(\hat{b}, \hat{c})$ with $(\hat{b}_0, \hat{c}_0)$, we can find the fiducial quantities for c and b as

$$Q_c = \hat{c}_0/\hat{c}^* \quad \text{and} \quad Q_b = \hat{b}_0\left(1/\hat{b}^*\right)^{\hat{c}^*/\hat{c}_0},$$

respectively. See Equations (25.10) and (25.11) in the book by Krishnamoorthy (2015).

Noting that the survival probability $S_{t_0} = P(X > t_0) = \exp\left(-(t_0/b)^c\right)$, a fiducial quantity for S_t is given by

$$Q_{S_{t_0}} = \exp\left(-(t_0/Q_b)^{Q_c}\right).$$

For a given $(\hat{b}_0, \hat{c}_0)$, the distribution of $Q_{S_{t_0}}$ does not depend on any parameter, and so its percentiles can be estimated using Monte Carlo simulation. Appropriate lower and upper percentiles of $Q_{S_{t_0}}$ form a CI for S_{t_0}.

Comparison of generalized fiducial confidence intervals (GFCIs) and the pivotal-based confidence intervals (PCIs)

The pivotal-based CIs (PCIs) for b and c are exact and so no coverage study is needed to assess the coverage levels. However, in order to compare them with the generalized fiducial confidence intervals (GFCIs), we estimated the coverage probabilities and average lengths using Monte Carlo simulation with 100,000 runs. For ease of comparison studies, we used the same parameter and sample size configurations as given in Cai, Siman and Liang (2022). Coverage probabilities and average lengths of the CIs for c are given in Table 1 and for b in Table 2. In Table 3, we presented coverage probabilities and expectations of generalized fiducial lower confidence limits (GFCL) and fiducial lower confidence limits (FCL, Krishnamoorthy et al. 2009) of S_{t_0}. We observe from all three tables that pivotal-based CIs have coverage probabilities very close to the nominal level. Average lengths of the both CIs (for each case) are in agreement in most cases. In some cases, the average length of the GFCI is smaller than the pivotal-based CI because GFCI has lower coverage probabilities in those cases. For example, see the results for the case $(n, c, b) = (10, 3, 5)$ in Table 2.

Table 1. Coverage probabilities and average lengths of 95% CIs for c.

	GFCI		PCI	
(n, c, b)	CP	AL	CP	AL
(2, 3, 10)	0.953	3.364	0.950	3.388
(3, 5, 10)	0.955	3.329	0.950	3.369
(2, 3, 20)	0.959	2.233	0.949	2.205
(3, 5, 20)	0.961	2.196	0.950	2.203

Table 2. Coverage probabilities and average lengths of 95% CIs for b.

	GFCI		PCI	
(n, c, b)	CP	AL	CP	AL
(2, 3, 10)	0.939	0.999	0.949	0.996
(3, 5, 10)	0.935	1.984	0.950	2.501
(2, 3, 20)	0.953	0.654	0.950	0.652
(3, 5, 20)	0.953	1.636	0.950	1.631

Table 3. Coverage probabilities and average lengths of 95% CIs for S_{t_0}.

	GFCL		FCL	
(n, c, b, t_0, S_{t_0})	CP	Average	CP	Average
(10, 3.0, 2.0, 2.5, 0.142)	0.957	0.051	0.950	0.050
(10, 3.0, 2.0, 1.0, 0.882)	0.954	0.699	0.949	0.700
(10, 3.0, 5.0, 6.0, 0.178)	0.953	0.070	0.948	0.071
(10, 3.0, 5.0, 2.0, 0.938)	0.950	0.791	0.949	0.789
(20, 3.0, 2.0, 2.5, 0.142)	0.944	0.067	0.948	0.067
(20, 3.0, 2.0, 1.0, 0.882)	0.958	0.766	0.949	0.764
(20, 3.0, 5.0, 6.0, 0.178)	0.951	0.091	0.953	0.089
(20, 3.0, 5.0, 2.0, 0.938)	0.941	0.850	0.947	0.848

Conclusion

The generalized fiducial approach may produce accurate estimates in the present problem. However, the approach involves Metropolis-Hasting steps and Gibbs sampler, and quite complex. The existing pivotal-based classical approach is simple and exact. Furthermore, the generalized fiducial approach is in no way superior to the alternative simple fiducial approach given in Krishnamoorthy et al. (2009) and Krishnamoorthy and Lin (2010). The proposed fiducial approach in these papers have been used to find CIs for parameters and survival probability, prediction intervals and tolerance intervals.

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Kalimuthu Krishnamoorthy
Department of Mathematics
University of Louisiana at Lafayette
Lafayette, LA 70504-1010, USA
krishna@louisiana.edu

Shanshan Lv [Database]
Department of Statistics
Truman State University
Kirksville, MO 63501, USA