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ORIGINAL ARTICLE

Statistical Intervals for Maxwell Distributions

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Abstract
The problem of constructing statistical intervals for two-parameter Maxwell distribution is considered. An appropriate method of finding the maximum likelihood estimators (MLEs) is proposed. Constructions of confidence intervals, prediction intervals and one-sided tolerance limits based on suitable pivotal quantities are described. Pivotal quantities based on the MLEs, moment estimators and the modified MLEs are proposed and compared the statistical intervals based on them in terms of expected widths. Comparison studies indicate that the statistical intervals based on the MLEs offer little improvements over other interval estimates when sample sizes are small, and all intervals are practically the same even for moderate sample sizes. R functions to compute various intervals are provided and the methods are illustrated using two examples involving real data sets.

Keywords Equivariant estimator · Gamma distribution · Location-scale family · Modified MLEs · Pivotal approach · Precision

1 Introduction

The Maxwell distribution, also known as the Maxwell–Boltzmann distribution, was first introduced by Maxwell [13] which describes the distribution of speeds of molecules in thermal equilibrium. In particular, this distribution is used to model the velocities among gas molecules. The probability density function (PDF) of the distribution is given by

$$f(\nu) = 4\pi \nu^2 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\nu^2}{2kT}\right), \quad \nu > 0, \tag{1}$$

where m is the molecular weight in kg/mol, T is the absolute temperature in Kelvin, k is the Boltzmann constant and υ denotes the speed of the molecule. After reparam-

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eterizing $\sqrt{2kT/m} = \sigma$ and then adding a location parameter μ, we obtain the PDF of the two-parameter Maxwell distribution as

$$f(x|\mu,\sigma) = \frac{4}{\sigma\Gamma(1/2)}\left(\frac{x-\mu}{\sigma}\right)^2 \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad x > \mu,\ \sigma > 0. \tag{2}$$

Let $G_{a,b}$ denote the gamma random variable with the shape parameter $a > 0$ and the scale parameter $b > 0$. Let us denote the distribution of $G_{a,b}$ by gamma(a, b). The cumulative distribution function (CDF) is given by

$$F(x|\mu,\sigma) = \frac{1}{\Gamma(3/2)}\int_0^{(x-\mu)^2/\sigma^2} t^{3/2-1}e^{-t}dt = P\left(G_{3/2,1} \le \frac{(x-\mu)^2}{\sigma^2}\right). \tag{3}$$

It is clear from the above CDF that the two-parameter Maxwell random variable has the following stochastic representation:

$$X \stackrel{d}{=} \mu + \sqrt{G_{3/2,1}}\,\sigma. \tag{4}$$

The Maxwell distribution (1) plays an important role in statistical mechanics. This distribution is getting popular as one of the lifetime distributions. It appears that Tyagi and Bhattacharya [18, 19] have first used the model (2) in lifetime data analysis. Many authors considered the distribution in the form (2) with $\mu = 0$ and addressed the problems of point estimation using the frequentist as well as the Bayesian approach. See the papers by Dey et al. [4], Arslan et al. [1] and the references therein.

In two-parameter Maxwell distribution, the location parameter (also known as the threshold parameter) represents the earliest time a failure may occur, and better suitable to model lifetime data. The threshold parameter locates the distribution along the time scale and has the same units of time or distance. The two-parameter Maxwell distribution fits many lifetime data better than other lifetime distributions. For example, Arslan et al. [1] have shown that the breaking strength of carbon fibers in Nicolas and Padgett [14] fit a Maxwell distribution better than some other lifetime distributions. In Example 1 of this paper, we find that a Maxwell distribution fits lifetimes of drills given in Chen et al. [3]. Dey at al. [4] have shown that a Maxwell distribution well fits the maximum flood level data; see Example 2. Our investigation showed that the alkalinity concentration data (Gibbons [6], p. 261) collected from groundwater fit a Maxwell distribution. Aryal et al. [2] and Krishnamoorthy et al. [10] have used a gamma distribution to analyse the alkalinity concentration data. The data that represent the number of million revolutions before failure for ball bearings in Thoman et al. [15] also fit a Maxwell distribution well. The data were analysed by Thoman et al. [15] and Krishnamoorthy et al. [11] using Weibull distributions.

Even though Maxwell distributions seem to be applicable to model data from various practical situations, only limited results are available on inference based on them. Most of the results are point estimation based on the classical approach and Bayesian approach. Dey et al. [4] and Arslan et al. [1] have considered the problem of estimating the parameters of Maxwell distributions. Dey et al. [4] have derived the maximum likelihood estimates (MLEs), moment estimates (MEs), and the least square estimate. Arslan et al. [1] have obtained the same and proposed modified MLEs that can be expressed in closed-form. However, statistical intervals such as the confidence intervals (CIs), prediction intervals (PIs) or tolerance intervals are not available for Maxwell distributions. These interval estimates are more important than point estimates and they are routinely used in applications.

In this article, we first review some available point estimates and note that the method proposed to find the MLEs is inaccurate, because it ignores the constraint that $\mu < x$. This means that, for a given sample, the MLE of μ produced by the existing method could be greater than $x_{(1)}$, the smallest order statistic for the sample. We propose a method to find the MLE subject to the constraint that $\mu < x_{(1)}$. We also describe the modified MLEs by Arslan et al. [1] and the moment estimates. In Sect. 3, we provide pivotal quantities to find CIs for the mean of a Maxwell distribution and to find CIs for quantiles or equivalently, one-sided tolerance limits. In Sect. 4, we compare the CIs based on the MLEs, modified MLEs (MMLEs) and moment estimates in terms of precision. Construction of one-sided tolerance limits and estimation of survival probability are given in Sect. 5, and the prediction intervals for the mean of a future sample are given in Sect. 6. For each problem, we compare the statistical intervals based on the MLEs, MEs and MMLEs with respect to precision. In Sect. 7, we outline numerical methods for computing tolerance intervals and equal-tailed tolerance intervals. The methods are illustrated using two examples in Sect. 8. Some concluding remarks are given in Sect. 9. R code to compute various statistical intervals are provided in the appendix.

2 Point Estimators

Let $\boldsymbol{X} = X_1, \ldots, X_n$ be a sample from a Maxwell(μ, σ) distribution. In the following sections, we shall describe the moment estimators (MEs), derive the maximum likelihood estimators (MLEs) and present the modified maximum likelihood estimates (MMLEs).

2.1 Moment Estimators

The moment estimators are obtained by equating the sample moments with the corresponding population moments. Let X be a random variable with Maxwell(μ, σ) distribution. Using the stochastic representation (4), we find

$$E(X) = \mu + \frac{2}{\sqrt{\pi}}\sigma \quad \text{and} \quad \mathrm{Var}(X) = \frac{3\pi - 8}{2\pi}\sigma^2. \tag{5}$$

Equating the above mean and variance to the sample mean $\bar{X}$ and variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, respectively, and solving the equations for the parameters, we find the moment estimators as

$$\widehat{\mu}_M = \bar{X} - \frac{2}{\sqrt{\pi}}\widehat{\sigma}_M \quad \text{and} \quad \widehat{\sigma}_M = \sqrt{\frac{2\pi}{3\pi - 8}}\, S. \tag{6}$$

2.2 Maximum Likelihood Estimators

The likelihood function (LF), given the sample $\boldsymbol{x} = x_1, \ldots, x_n$, can be written as

$$L(\mu, \sigma|\boldsymbol{x}) = \left[\frac{4}{\Gamma(1/2)}\right]^n \frac{1}{\sigma^{3n}} \prod_{i=1}^{n} (x_i - \mu)^2$$
$$\exp\left\{-\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\} I(x_{(1)} > \mu), \tag{7}$$

where $x_{(1)}$ is the smallest order statistics for the sample and $I(.)$ is the indicator function. The log-likelihood function, without the indicator function, is given by

$$\ln L(\mu, \sigma|\boldsymbol{x}) = C - 3n \ln\sigma + 2\sum_{i=1}^{n} \ln(x_i - \mu) - \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2,$$

where C is a constant term independent of the parameters. The partial derivative $\frac{\partial \ln L}{\partial \mu} = 0$ gives

$$n(\bar{x} - \mu) - \sigma^2 \sum_{i=1}^{n}(x_i - \mu)^{-1} = 0 \tag{8}$$

and the partial derivative $\frac{\partial \ln L}{\partial \sigma} = 0$ yields $\sigma^2 = \frac{2}{3n}\sum_{i=1}^{n}(x_i - \mu)^2$. Substituting this expression for σ^2 in (8) and simplifying, we get

$$n(\bar{x} - \mu) - \frac{2}{3}\left(\widehat{\sigma}_x^2 + (\bar{x} - \mu)^2\right)\sum_{i=1}^{n}(x_i - \mu)^{-1} = 0, \tag{9}$$

where $\widehat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

Since the parameter space depends on the sample space (see the PDF (1) and the LF (7)), the MLE of μ should be less than or equal to $x_{(1)}$. To find the MLE of μ with the constraint $\mu < x_{(1)}$, we follow the idea of Krishnamoorthy et al. (2019) who derived the MLEs for a two-parameter Rayleigh distribution. That is, we search for the value of μ that satisfies Eq. (9) in the root bracketing interval $(x_{(1)} - \widehat{t},\ x_{(1)})$, where

$$\widehat{t} = \widehat{\sigma}_M \sqrt{G^{-1}\left(1 - 10^{-3/n}|3/2\right)},$$

$G^{-1}(x|a)$ is the quantile function a gamma$(a, 1)$ distribution and $\widehat{\sigma}_M$ is the moment estimate of σ in (6). As shown in the "Appendix A", this interval includes μ with high probability. Furthermore, our extensive simulation studies in the sequel indicate that the MLE of μ always lies in the interval. Notice that the MLE $\widehat{\mu}$ of μ obtained using the bisection method satisfies the requirement that $\widehat{\mu} < x_{(1)}$.

Remark 1 Dey et al. [4] have derived likelihood equations and noted that they can be solved using some iterative methods. Later, in the example section, they have used a graphic method to compute the MLEs approximately. Arslan et al. [1] proposed a bivariate N-R method to find the MLEs. It should be noted that both papers ignored the fact that the sample space depends on the parameter space; that is $\mu < x_{(1)}$. Our approach here simplifies to finding the root of a single Eq. (9) and does not require the calculation of Hessian matrix. On the basis of our simulation studies, we found that for no more than 1.5% of simulated samples, the N-R method in Arslan et al. [1] has produced the MLE $\widehat{\mu}$ that is greater than $x_{(1)}$. As an example, consider the following data that are generated from Maxwell(2, 1) distribution.

```
2.99,3.28,3.29,2.21,3.21,2.69,2.76,3.21,2.95, 2.80
3.27,3.03,3.23,3.28,3.08,3.15,3.28,3.64,3.31,3.57
```

For the above data, $x_{(1)} = 2.21$. The MLEs based on our bisection root finding method, subject to the constraint $\mu < x_{(1)}$, are $\widehat{\mu} = 2.085$ and $\widehat{\sigma} = 0.877$; the MLEs based on the N-R method by Arslan et al. [1] are $\widehat{\mu} = 2.574$ and $\widehat{\sigma} = 0.508$. Note that this estimate $\widehat{\mu} = 2.574$ is larger than $x_{(1)}$, and so it is not a valid MLE.

2.3 Modified MLEs

Arslan et al. [1] have also derived approximate closed-form expressions for the MLEs using the modified maximum likelihood (MML) approach by Tiku [16, 17]. To write these modified maximum likelihood estimates (MMLEs), let $x_{(i)}$ denote the ith order statistic for a sample $x_1, \ldots, x_n$ from a Maxwell(μ, σ) distribution, and let

$$t_i = F^{-1}\left(\frac{i}{n+1}\right), \; i = 1, \ldots, n,$$

where F^{-1} is the quantile function of the standard Maxwell distribution. The MMLEs can be expressed as

$$\widehat{\mu}_{\text{mml}} = \bar{x}_w - \frac{\Delta}{m}\widehat{\sigma}_{\text{mml}} \quad \text{and} \quad \widehat{\sigma}_{\text{mml}} = \frac{1}{2\sqrt{n(n-1)}}\left(-B + \sqrt{B^2 + 4nC}\right) \tag{10}$$

where

$$\bar{x}_w = \frac{1}{m}\sum_{i=1}^{n} \delta_i x_{(i)}, \quad m = \sum_{i=1}^{n} \delta_i, \quad \delta_i = \beta_i + 1, \quad \beta_i = t_i^{-2}, \quad \Delta = \sum_{i=1}^{n} \alpha_i, \quad \alpha_i = 2t_i^{-2},$$

$$B = 2\sum_{i=1}^{n} \alpha_i \left(x_{(i)} - \bar{x}_w\right) \quad \text{and} \quad C = 2\sum_{i=1}^{n} \delta_i \left(x_{(i)} - \bar{x}_w\right)^2.$$

As this approximate method does not utilize the constraint that $\mu < x_{(1)}$, there is no guarantee that this method produces valid MLEs; that is, $\widehat{\mu} < x_{(1)}$. For the data in Remark 1, the approximate MLEs are $\widehat{\mu} = 2.231$ and $\widehat{\sigma} = .789$. Note that $\widehat{\mu}$ is greater than $x_{(1)} = 2.21$ and the exact MLEs $\widehat{\mu} = 2.085$ and $\widehat{\sigma} = 0.877$.

Remark 2 The least square estimates (LSEs) of μ and σ are obtained by minimizing the function $\sum_{i=1}^{n}\left(F(x_{(i)}|\mu,\sigma) - i/(n+1)\right)^2$, where $F(x|\mu,\sigma)$ is the CDF of the Maxwell(μ,σ) distribution, with respect to μ and σ. The LSEs can not be expressed in closed-forms and they have to be obtained only numerically. Simulation studies by Arslan et al. [1] indicated that the LSEs are worse than the MLEs and other estimates in terms of bias and the MSE, and so we will not consider the LSEs for constructing various statistical intervals in this paper.

3 Pivotal Quantities

Let $\widehat{\mu}_e$ and $\widehat{\sigma}_e$ be equivariant estimators based on a sample of size n from a Maxwell(μ,σ) distribution. Since Maxwell distributions are location-scale distributions, the MLEs are equivariant (see Theorem E1 in Lawless [12]), and $(\widehat{\mu}_e - \mu)/\sigma$ and $\widehat{\sigma}_e/\sigma$ are pivotal quantities. That is, the distributions of these quantities remain the same for all values of parameters, and hence their distributions can be found empirically assuming that $\mu = 0$ and $\sigma = 1$. In other words,

$$\frac{\widehat{\mu}_e - \mu}{\sigma} \stackrel{d}{=} \widehat{\mu}_e^* \quad \text{and} \quad \frac{\widehat{\sigma}_e}{\sigma} \stackrel{d}{=} \widehat{\sigma}_e^*, \tag{11}$$

where $\widehat{\mu}_e^*$ and $\widehat{\sigma}_e^*$ are the MLEs based on a sample of size n from the Maxwell(0, 1) distribution. That is, $\widehat{\mu}^*$ is the root of the Eq. (9) with respect to μ and $\widehat{\sigma}^* = \frac{2}{3n}\sum_{i=1}^{n}(x_i - \widehat{\mu}^*)^2$ and the samples $x_1, \ldots, x_n$ are from the Maxwell(0, 1) distribution. Furthermore, the notation "$X \stackrel{d}{=} Y$" means that X and Y are identically distributed.

To obtain a pivotal quantity for the mean or for a percentile of a Maxwell distribution, we note that these quantities are of the form $\mu + c\sigma$, where c is a known constant. A pivotal quantity for $\mu + c\sigma$ can be obtained as

$$\frac{\mu + c\sigma - \widehat{\mu}_e}{\widehat{\sigma}_e} \stackrel{d}{=} \frac{c - \widehat{\mu}_e^*}{\widehat{\sigma}_e^*}. \tag{12}$$

Let l and u denote the lower and upper 100α percentiles of $(c - \widehat{\mu}_e^*)/\widehat{\sigma}_e^*$. Then,

$$(\widehat{\mu}_e + l\widehat{\sigma}_e,\ \widehat{\mu}_e + u\widehat{\sigma}_e) \tag{13}$$

is a $100(1-2\alpha)\%$ CI for $\mu + c\sigma$. Note that the percentiles of $(c - \widehat{\mu}_e^*)/\widehat{\sigma}_e^*$ can be estimated using simulated samples from the standard Maxwell distribution.

Table 1 Coverage probabilities and (expected widths) of 95% confidence intervals for the mean

$\mu = 0$

	$n = 5$			$n = 10$		
σ	MLE	ME	MMLE	MLE	ME	MMLE
.5	.950 (0.57)	.951 (0.57)	.949 (0.57)	.952 (0.33)	.951 (0.34)	.950 (0.33)
1	.949 (1.14)	.949 (1.15)	.950 (1.14)	.951 (0.67)	.951 (0.68)	.951 (0.67)
2	.951 (2.27)	.949 (2.29)	.949 (2.27)	.949 (1.34)	.949 (1.35)	.947 (1.34)
4	.949 (4.53)	.948 (4.58)	.948 (4.54)	.951 (2.67)	.949 (2.70)	.950 (2.68)
5	.950 (5.66)	.949 (5.72)	.949 (5.68)	.950 (3.34)	.951 (3.38)	.950 (3.34)
8	.950 (9.06)	.949 (9.16)	.948 (9.09)	.949 (5.35)	.949 (5.41)	.950 (5.35)
10	.949 (11.3)	.952 (11.4)	.951 (11.3)	.950 (6.69)	.950 (6.76)	.952 (6.69)
	$n = 20$			$n = 30$		
σ	MLE	ME	MMLE	MLE	ME	MMLE
.5	.950 (0.22)	.948 (0.22)	.950 (0.22)	.951 (0.18)	.948 (0.18)	.951 (0.18)
1	.950 (0.44)	.948 (0.44)	.950 (0.44)	.950 (0.35)	.952 (0.36)	.949 (0.35)
2	.950 (0.88)	.949 (0.89)	.948 (0.88)	.950 (0.70)	.951 (0.71)	.952 (0.71)
4	.950 (1.76)	.950 (1.78)	.950 (1.76)	.949 (1.41)	.950 (1.43)	.949 (1.41)
5	.951 (2.19)	.949 (2.22)	.950 (2.20)	.950 (1.76)	.950 (1.78)	.950 (1.76)
8	.949 (3.51)	.951 (3.55)	.949 (3.51)	.950 (2.81)	.949 (2.85)	.949 (2.82)
10	.950 (4.39)	.949 (4.44)	.950 (4.40)	.949 (3.52)	.948 (3.56)	.950 (3.52)

4 Confidence Intervals for the Mean

The mean of the Maxwell(μ, σ) distribution is given by $\mu + c\sigma$ with $c = 2/\sqrt{\pi}$; see Eq. (5). Let $\widehat{\mu}_e$ and $\widehat{\sigma}_e$ be equivariant estimators based on a sample of size n from a Maxwell(μ, σ) distribution. Let l_m and u_m denote the lower and upper 100α percentiles of the pivotal quantity in (12) with $c = 2/\sqrt{\pi}$. Then,

$$(\widehat{\mu}_e + l_m\widehat{\sigma}_e,\ \widehat{\mu}_e + u_m\widehat{\sigma}_e) \tag{14}$$

is a $100(1 - 2\alpha)\%$ CI for the mean.

Precision Studies

It can be readily verified that the MEs, MLEs and the modified MLEs are equivariant, and so the pivotal quantities on the basis of these estimators can be used to find exact CIs for the mean.

The CIs based on equivariant estimators are exact except for the simulation error in obtaining the percentage points. However, to judge the accuracy of our simulation studies, we have estimated the coverage probabilities and expected widths of different CIs using simulation consisting of 100,000 runs, and reported them in Table 1. Since the coverage probability and expected width do not depend on the location parameter, without loss of generality, we chose $\mu = 0$ in our simulation study. As expected, the coverage probabilities are practically same as the nominal level 0.95 for all the cases.

Table 2 Percentiles for computing 90%, 95% and 99% CIs for the mean based on the MLEs

n	5%	95%	2.5%	97.5%	.5%	99.5%
4	0.542	1.92	0.342	2.21	− 0.315	3.19
5	0.671	1.73	0.539	1.93	0.168	2.50
6	0.740	1.63	0.641	1.79	0.375	2.20
7	0.786	1.57	0.700	1.69	0.488	2.00
8	0.818	1.52	0.745	1.62	0.565	1.88
9	0.844	1.48	0.781	1.58	0.628	1.80
10	0.861	1.45	0.802	1.54	0.670	1.74
11	0.876	1.43	0.821	1.51	0.702	1.69
12	0.887	1.41	0.838	1.48	0.727	1.64
13	0.901	1.40	0.855	1.47	0.753	1.61
14	0.911	1.39	0.867	1.45	0.775	1.59
15	0.919	1.37	0.878	1.43	0.791	1.56
16	0.928	1.36	0.887	1.42	0.800	1.54
17	0.934	1.35	0.895	1.41	0.811	1.52
18	0.940	1.35	0.902	1.39	0.823	1.50
19	0.946	1.34	0.909	1.39	0.834	1.49
20	0.951	1.33	0.918	1.38	0.848	1.48
25	0.972	1.30	0.942	1.34	0.881	1.42
30	0.986	1.29	0.959	1.32	0.903	1.40
35	0.996	1.27	0.971	1.30	0.923	1.37
40	1.006	1.26	0.983	1.29	0.937	1.35
45	1.013	1.25	0.992	1.28	0.947	1.34
50	1.020	1.25	0.999	1.27	0.960	1.32
60	1.029	1.23	1.010	1.26	0.974	1.30
70	1.036	1.23	1.019	1.25	0.986	1.29
80	1.043	1.22	1.027	1.24	0.995	1.28
90	1.047	1.21	1.032	1.23	1.002	1.27
100	1.052	1.21	1.038	1.23	1.009	1.26

The expected widths in Table 1 indicate that these three CIs have similar precisions and only minute differences exist among the expected widths. The MLE-CIs are slightly better than those based on the MEs and they are very similar to MMLE confidence intervals.

The percentiles needed to find 90%, 95% and 99% MLE-CIs were computed for n ranging from 4 to 100, and reported in Table 2. To estimate the percentiles, we used simulation consisting of 100,000 runs. If percentiles are desired for values of n and confidence coefficients that are not reported, they can be obtained using the R function `perc.ci.mean()` given in "Appendix B".

5 One-Sided Tolerance Limits and Survival Probability

One-sided tolerance limits (TLs) are one-sided confidence limits for appropriate quantiles. Specifically, a $100\gamma\%$ upper confidence limit for the upper pth quantile is called (p, γ) upper TL and a $100\gamma\%$ lower confidence limit for the lower p quantiles is called (p, γ) lower TL. The pth quantile of a Maxwell(μ, σ) distribution is $\mu + c\sigma$ with

$$c = q_p(0, 1) = \sqrt{G^{-1}(p|3/2)}, \tag{15}$$

where $G^{-1}(p|a)$ denote the quantile function of a gamma$(a, 1)$ distribution. Hence, we can use the percentiles of $(c - \widehat{\mu}_e^*)/\sigma_e^*$ in (12) with $c = q_p(0, 1)$ to find confidence limits for $\mu + q_p(0, 1)\sigma$. Let $W_p = (q_p(0, 1) - \widehat{\mu}_e^*)/\sigma_e^*$ and let $W_{p,q}$ denote the $100q$ percentile of W_p. Then

$$\widehat{\mu}_e + W_{p,\gamma}\widehat{\sigma}_e \tag{16}$$

is a (p, γ) upper tolerance limit for the Maxwell distribution. In the above, $\widehat{\mu}_e$ and $\widehat{\sigma}_e$ are some equivariant estimators based on a sample of size n from a Maxwell(μ, σ) distribution. A (p, γ) lower tolerance limit can be obtained similarly as $\widehat{\mu}_e + W_{1-p,1-\gamma}\widehat{\sigma}_e$.
Precision Studies

Note that the tolerance limits based on any equivariant estimators are exact in the sense that the coverage probabilities are always equal to the nominal level. However, tolerance limits based on different estimators could be different, and so it is of interest to find the method that produces better tolerance limits. In the case of upper tolerance limits, equivariant estimators that produce upper TL with smaller expected values are preferable, and for the case of LTL the equivariant estimators that produce larger LTL are preferable. We evaluated expected values of upper TL based on MLEs, MEs and MMLEs and presented them in Table 3. Since the upper TLs are based on equivariant estimators, without loss of generality, we can take $\mu = 0$ for comparing the upper TLs with respect to expected values. The reported expected widths clearly indicate that the expected widths of upper TLs based on the MLEs and MMLEs are the same while the expected widths of the upper TLs based on the MEs are slightly larger than those of the other upper TLs. Recall that smaller upper TLs are better.

We computed the factors for computing (p, γ) one-sided tolerance limits for $p =$ .80, .90, .95, .99, $\gamma = .95$ and values of n ranging from 4 to 100, and reported them in Table 4. We used 100,000 simulation runs to estimate the percentiles of W_p and W_{1-p}. Factors for values of (n, p, γ) that are not reported in Table 4, can be obtained using the R function `fac.os.TLs()` in "Appendix B".
Survival Probability
For a lifetime random variable, the survival probability is defined as

$$\tau = P(X > t|\mu, \sigma) = 1 - G\left((t - \mu)^2/\sigma^2|3/2\right),$$

Table 3 Expected values of (.90, .95) upper tolerance limits

	$n=5$			$n=10$			$n=15$			$n=20$			$n=30$		
σ	MLE	ME	MMLE	MLE	ME	MMLE	MLE	ME	MMLE	MLE	ME	MMLE	MLE	ME	MMLE
.5	1.43	1.43	1.43	1.17	1.18	1.17	1.10	1.10	1.10	1.06	1.06	1.06	1.02	1.02	1.02
1	2.85	2.86	2.85	2.35	2.36	2.35	2.19	2.20	2.19	2.12	2.13	2.12	2.04	2.05	2.04
2	5.71	5.73	5.71	4.70	4.72	4.70	4.38	4.40	4.38	4.23	4.25	4.24	4.08	4.09	4.08
4	11.42	11.47	11.43	9.40	9.44	9.40	8.77	8.81	8.77	8.47	8.50	8.47	8.15	8.18	8.15
5	14.27	14.33	14.28	11.74	11.79	11.74	10.96	11.01	10.96	10.59	10.63	10.59	10.20	10.23	10.20
8	22.83	22.93	22.84	18.79	18.87	18.79	17.54	17.62	17.54	16.95	17.01	16.95	16.30	16.36	16.31
10	28.49	28.61	28.50	23.48	23.58	23.49	21.92	22.03	21.92	21.18	21.27	21.19	20.38	20.45	20.38

Table 4 (p, .95) one-sided tolerance limits based on the MLEs

	Lower tolerance factor p				Upper tolerance factor			
n	.80	.90	.95	.99	.80	.90	.95	.99
4	− 0.418	− 0.908	− 1.311	− 1.855	3.05	3.84	4.52	5.85
5	− 0.119	− 0.481	− 0.772	− 1.230	2.64	3.24	3.77	4.84
6	0.040	− 0.268	− 0.524	− 0.895	2.42	2.94	3.40	4.31
7	0.156	− 0.147	− 0.359	− 0.691	2.27	2.76	3.16	4.00
8	0.228	− 0.041	− 0.252	− 0.579	2.19	2.63	3.02	3.77
9	0.282	0.020	− 0.179	− 0.482	2.12	2.53	2.90	3.62
10	0.322	0.069	− 0.116	− 0.404	2.06	2.46	2.81	3.49
11	0.353	0.109	− 0.071	− 0.346	2.02	2.40	2.75	3.41
12	0.378	0.141	− 0.034	− 0.303	1.99	2.36	2.69	3.33
13	0.399	0.169	− 0.005	− 0.266	1.96	2.33	2.64	3.27
14	0.418	0.192	0.022	− 0.228	1.93	2.29	2.60	3.21
15	0.431	0.212	0.048	− 0.208	1.92	2.26	2.57	3.17
16	0.445	0.226	0.064	− 0.181	1.89	2.24	2.54	3.13
17	0.457	0.241	0.082	− 0.160	1.88	2.21	2.51	3.09
18	0.469	0.255	0.100	− 0.142	1.87	2.20	2.49	3.05
19	0.479	0.267	0.109	− 0.125	1.85	2.18	2.47	3.03
20	0.485	0.278	0.124	− 0.111	1.84	2.17	2.45	3.01
25	0.518	0.319	0.171	− 0.054	1.79	2.11	2.38	2.91
30	0.541	0.344	0.199	− 0.017	1.76	2.07	2.33	2.85
35	0.557	0.365	0.222	0.011	1.74	2.04	2.30	2.80
40	0.569	0.379	0.239	0.029	1.72	2.02	2.27	2.77
45	0.579	0.392	0.254	0.046	1.71	2.00	2.25	2.74
50	0.587	0.402	0.265	0.058	1.70	1.99	2.23	2.71
60	0.599	0.418	0.282	0.080	1.68	1.96	2.20	2.68
70	0.609	0.429	0.296	0.096	1.67	1.94	2.18	2.65
80	0.618	0.438	0.306	0.107	1.66	1.93	2.17	2.63
90	0.625	0.445	0.314	0.117	1.65	1.92	2.16	2.61
100	0.628	0.451	0.321	0.124	1.64	1.91	2.15	2.60

where t is a specified mission time. To assess the survival time, a lower confidence limit for τ is needed, which can be deduced from a lower tolerance limit (see Section 1.1.3 of Krishnamoorthy and Mathew [11]) as follows. Let $L(p; \widehat{\mu}, \widehat{\sigma})$ be a (p, γ) lower tolerance limit for a Maxwell distribution. Then, by definition of the lower tolerance limit, we have

$$P_{\widehat{\mu},\widehat{\sigma}} \left\{ P_X \left(X \geq L(p; \widehat{\mu}, \widehat{\sigma}) \middle| \widehat{\mu}, \widehat{\sigma} \right) \geq p \right\} = \gamma. \tag{17}$$

For a given $(\widehat{\mu}, \widehat{\sigma})$, let p be determined so that $L(p; \widehat{\mu}, \widehat{\sigma}) = t$. Then (17) implies that $P_X(X > t) \geq p$ with probability γ, and so p is the $100\gamma\%$ lower confidence limit for $\tau = P(X > t)$. To find the value of p such that $L(p; \widehat{\mu}, \widehat{\sigma}) = t$ for a Maxwell distribution, we need to equate the (p, γ) lower tolerance limit to t or equivalently,

$$W_{1-p,1-\gamma} = \frac{t - \widehat{\mu}}{\widehat{\sigma}},$$

where $W_{1-p} = [\sqrt{G^{-1}((1-p)|3/2)} - \widehat{\mu}^*]/\widehat{\sigma}^*$ and $W_{1-p;\alpha}$ is the 100α percentile of W_{1-p}. Thus, for a given $(\widehat{\mu}, \widehat{\sigma}, \gamma)$, we need to determine p so that

$$100(1-\gamma) \text{ percentile of } W_{1-p} = \frac{\sqrt{G^{-1}((1-p)|3/2)} - \widehat{\mu}^*}{\widehat{\sigma}^*} = \frac{t - \widehat{\mu}}{\widehat{\sigma}},$$

where $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are equivariant estimators based on a sample of size n from a standard Maxwell distribution. The value of p that satisfies the above equation can be found using Monte Carlo simulation and a root finding method with root bracketing interval $[.001, \bar{F}(t)]$, where $\bar{F}(t) = 1 - G\left((t-\widehat{\mu})^2/\widehat{\sigma}^2|3/2\right)$. The following Algorithm 1 can be used to find the root. The lower confidence limit based on this approach is exact except for simulation error.

Algorithm 1

1. For a given sample of size n and a value of t, compute the MLEs $\widehat{\mu}$ and $\widehat{\sigma}$ and compute the estimate $p_0 = 1 - G((t-\widehat{\mu})/\sigma^2|3/2)$ of $\tau = P(X > t)$ and the value of $t_0 = (t - \widehat{\mu})/\widehat{\sigma}$.
2. Generate, say, 100,000 samples, each of size n, from the standard Maxwell distribution.
3. Calculate the MLEs $\widehat{\mu}_i^*$ and $\widehat{\sigma}_i^*$ based on the ith sample generated in the preceding step, $i = 1, \ldots, 100{,}000$
4. Denote the $100(1-\gamma)$ percentile of $\frac{\sqrt{G^{-1}((1-p)|3/2)} - \widehat{\mu}^*}{\widehat{\sigma}^*}$ by Q_p and set $f(p) = Q_p - t_0$. Note that for a given p, Q_p can be estimated using the simulated estimates $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ in Step 3.
5. Using the value of p_0 in Step 1 and $p_1 = .001$, say, as the root bracketing values, the solution to the equation $f(p) = 0$ can be found using a bisection method. The root of the equation is a $100\gamma\%$ lower confidence limit for $\tau = P(X > t)$.

Note that to compute $f(p)$ defined in the above algorithm at various values of p, we need to carry out the simulation in Step 3 only once. The bisection scheme converges in a fewer steps with the bracketing interval in Step 5.

6 Prediction Intervals for the Mean of a Future Sample

We shall now see an exact method of finding a prediction interval (PI) for the mean of a future sample of size m based on an available sample of size n. Let $(\widehat{\mu}_e, \widehat{\sigma}_e)$ be equivariant estimator of (μ, σ) based on the current sample of size n from a Maxwell(μ, σ) distribution, and let $\bar{Y}$ denote the mean of a future sample of size

m from the same Maxwell distribution. Let $\widehat{\mu}_e^*$ and $\widehat{\sigma}_e^*$ are equivariant estimators based on a sample of size n from the standard Maxwell distribution and $\bar{Y}^*$ is the mean of an independent sample of size m from the standard Maxwell distribution. Then $(\bar{Y} - \widehat{\mu}_e)/\widehat{\sigma}_e \stackrel{d}{=} (\bar{Y}^* - \widehat{\mu}_e^*)/\widehat{\sigma}_e^*$. If h_α denotes 100α percentile of $(\bar{Y}^* - \widehat{\mu}_e^*)/\widehat{\sigma}_e^*$, then

$$(\widehat{\mu}_e + h_\alpha \widehat{\sigma}_e,\ \widehat{\mu}_e + h_{1-\alpha}\widehat{\sigma}_e) \tag{18}$$

is a $100(1-2\alpha)\%$ PI for a future sample mean $\bar{Y}$. Note that the percentile h_α can be estimated by Monte Carlo simulation since the distribution of $(\bar{Y}^* - \widehat{\mu}_e^*)/\widehat{\mu}_e^*$ does not depend on any parameter.

We could use any of the three estimators MLEs, MEs and MMLEs in (18) to find a PI for $\bar{Y}$. Our simulation studies (see Table 5) indicate that the PIs based on the MEs and MMLEs are very similar (in terms of precision) and the ones based on the MLEs are slightly narrower than others. The differences among the PIs seem to be negligible when sample sizes are 30 or more.

In the following Table 6, we provide critical values needed to find 95% PIs based on the MLEs. We have provided critical values for only a few values n and m. Critical values for any n, m and confidence level can be computed the R code given in "Appendix B".

7 Tolerance Intervals

We shall now find factors for constructing equal-tailed and two-sided TIs using the general approach given in Hoang-Nguyen-Thuy and Krishnamoorthy [7]. Let $Q_p = \mu + q_p(0,1)\sigma$, where $q_p(0,1)$ is defined in (15), denote the $100p$ percentile of the Maxwell(μ, σ) distribution. Note that the interval $\left(Q_{\frac{1-p}{2}},\ Q_{\frac{1+p}{2}}\right)$ includes $100p$ percentage of the distribution. An interval (L_e, U_e) that includes the above interval with confidence γ is called (p, γ) equal-tailed TI. Note that the interval (L_e, U_e) not only includes at least $100p$ percent of the population, but also no more than $100(1-p)/2$ percent of the population is less than L_e and no more than $100(1-p)/2$ percent of the population is greater than U_e. A (p, γ) two-sided TI includes at least $100p$ percent of the population and it does not have to include $Q_{\frac{1-p}{2}}$ or $Q_{\frac{1+p}{2}}$. Formally, the two-sided TI $(L(\boldsymbol{X}), U(\boldsymbol{X}))$ is determined so that

$$P_{\boldsymbol{X}}\left(F(U(\boldsymbol{X})|\mu,\sigma) - F(L(\boldsymbol{X})|\mu,\sigma) \geq p\right) = \gamma,$$

where the CDF $F(x|\mu,\sigma)$ is defined in (3).

7.1 Equal-Tailed TIs

Let $q_p^* = [q_p(0,1) - \widehat{\mu}_e^*]/\widehat{\sigma}_e^*$, where $q_p(0,1)$ denote the $100p$ percentile of the standard Maxwell distribution. Furthermore, let $q_{p,\alpha}^*$ denote the 100α percentile of

Table 5 Expected widths of 95% prediction intervals for the mean of a future sample of size m

	$n = 10, m = 3$			$n = 10, m = 7$			$n = 15, m = 5$			$n = 15, m = 15$			$n = 30, m = 20$		
σ	MLE	ME	MMLE	MLE	ME	MMLE	MLE	ME	MMLE	MLE	ME	MMLE	MLE	ME	MMLE
.5	0.66	0.69	0.69	0.50	0.52	0.52	0.43	0.44	0.44	0.33	0.34	0.34	0.28	0.28	0.28
1	1.32	1.38	1.37	1.00	1.04	1.03	0.86	0.88	0.88	0.67	0.69	0.69	0.55	0.56	0.56
2	2.64	2.75	2.74	1.99	2.08	2.07	1.72	1.77	1.76	1.34	1.37	1.37	1.10	1.12	1.11
4	5.27	5.51	5.48	3.99	4.16	4.14	3.43	3.54	3.51	2.67	2.75	2.74	2.21	2.24	2.22
5	6.58	6.87	6.84	4.98	5.19	5.17	4.29	4.42	4.39	3.34	3.44	3.42	2.76	2.80	2.78
8	10.54	11.01	10.95	7.97	8.32	8.27	6.87	7.08	7.02	5.35	5.50	5.48	4.41	4.48	4.44
10	13.18	13.77	13.70	9.97	10.40	10.34	8.59	8.85	8.78	6.69	6.87	6.85	5.52	5.60	5.55

Table 6 Lower and upper percentiles for computing 95% PIs for the mean of a future sample of size m based on the background sample of size n using the MLEs

$n = 10$			$n = 15$			$n = 30$			$n = 40$		
m	2.5%	97.5%	m	2.5%	97.5%	m	2.5%	97.5%	m	2.5%	97.5%
1	.113	2.48	1	.197	2.36	1	.267	2.26	1	.283	2.23
3	.456	1.97	5	.629	1.73	5	.685	1.64	5	.696	1.62
5	.564	1.82	8	.703	1.64	10	.789	1.51	10	.804	1.49
7	.620	1.75	10	.730	1.60	15	.834	1.46	15	.849	1.44
9	.654	1.71	15	.773	1.55	20	.859	1.43	20	.875	1.41
11	.677	1.69	20	.796	1.53	25	.876	1.41	30	.905	1.37
15	.707	1.65	25	.810	1.51	35	.897	1.39	40	.922	1.35

q_p^*. Then, a (p, γ) lower tolerance limit is given by

$$LTL = \widehat{\mu}_e + q^*_{1-p,1-\gamma}\widehat{\sigma}_e$$

and a (p, γ) upper tolerance limit is given by

$$UTL = \widehat{\mu}_e + q^*_{p,\gamma}\widehat{\sigma}_e.$$

That is, at least $100p$ percent of the population is greater than the LTL with confidence γ. Similarly, the UTL can be interpreted as at least $100p$ percent of the population is less than the UTL with confidence γ. Noting that LTL is a $100\gamma\%$ lower confidence limit for lower quantile $\mu + q_{1-p}(0, 1)\sigma$ and UTL is a $100\gamma\%$ upper confidence limit for $\mu + q_p(0, 1)\sigma$, we find

$$\begin{aligned} &P\left(\widehat{\mu}_e + q^*_{1-p,1-\gamma}\widehat{\sigma}_e \le \mu + q_{1-p}(0,1)\sigma\right) = \gamma \quad \text{and} \\ &P\left(\widehat{\mu}_e + q^*_{p,\gamma}\widehat{\sigma}_e \ge \mu + q_p(0,1)\sigma\right) = \gamma. \end{aligned} \tag{19}$$

It follows from the above first probability statement that

$$P\left(\widehat{\mu}_e + q^*_{\frac{1-p}{2},\frac{1-\gamma}{2}}\widehat{\sigma}_e \le \mu + q_{\frac{1-p}{2}}(0,1)\sigma\right) = 1 - \frac{1-\gamma}{2}$$

and from the second probability that

$$P\left(\widehat{\mu}_e + q^*_{\frac{1+p}{2},\frac{1+\gamma}{2}}\widehat{\sigma}_e \ge \mu + q_{\frac{1+p}{2}}(0,1)\sigma\right) = 1 - \frac{1-\gamma}{2}.$$

Thus, using Bonferroni inequality and noting that $1 - (1-\gamma)/2 - (1-\gamma)/2 = \gamma$, we find

$$P\left(\widehat{\mu}_e + q^*_{\frac{1-p}{2},\frac{1-\gamma}{2}}\widehat{\sigma}_e \le \mu + q_{\frac{1-p}{2}}(0,1)\sigma \ \text{ and } \ \mu + q_{\frac{1+p}{2}}(0,1)\sigma \le \widehat{\mu}_e + q^*_{\frac{1+p}{2},\frac{1+\gamma}{2}}\widehat{\sigma}_e\right) \ge \gamma. \tag{20}$$

That is, the interval

$$\left(\widehat{\mu}_e + q_{\frac{1-p}{2},\frac{1-\gamma}{2}}\widehat{\sigma}_e,\ \widehat{\mu}_e + q_{\frac{1+p}{2},\frac{1+\gamma}{2}}\widehat{\sigma}_e\right) \tag{21}$$

would include at least $100p$ percent of the population with confidence at least γ.

The coverage probability of the TI in (21) does not depend on any parameter, and so the coverage probability in (20) can be expressed as

$$\begin{aligned} H_{\widehat{\mu}_e^*,\widehat{\sigma}_e^*}(\gamma|0,1) = P\Bigg(&\widehat{\mu}_e^* + q^*_{\frac{1-p}{2};\frac{1-\gamma}{2}}\widehat{\sigma}_e^* \le q_{\frac{1-p}{2}}(0,1) \text{ and } q_{\frac{1+p}{2}}(0,1) \\ \le\ &\widehat{\mu}_e^* + q^*_{\frac{1+p}{2};\frac{1+\gamma}{2}}\widehat{\sigma}_e^*\Bigg) \ge \gamma. \end{aligned} \tag{22}$$

Note that, for any given confidence level γ', $H_{\widehat{\mu}_e^*,\widehat{\sigma}_e^*}(\gamma'|0,1)$ is the actual coverage probability of the TI in (21) with γ replaced by γ'. In view of (22), we can choose γ' so that $H_{\widehat{\mu}_e^*,\widehat{\sigma}_e^*}(\gamma'|0,1) = \gamma$. A Monte Carlo estimate of the coverage probability $H_{\widehat{\mu}_e^*,\widehat{\sigma}_e^*}(\gamma'|0,1)$ along with a root bracketing interval $[\gamma - .4, \gamma]$, say, can be used to find the root of the equation $\widehat{H}_{\widehat{\mu}_e^*,\widehat{\sigma}_e^*}(\gamma'|0,1) - \gamma = 0$. Let γ_e' denote the root of the equation. Then the interval

$$\left(\widehat{\mu}_e + q^*_{\frac{1-p}{2};\frac{1-\gamma_e'}{2}}\widehat{\sigma}_e,\ \widehat{\mu}_e + q^*_{\frac{1+p}{2};\frac{1+\gamma_e'}{2}}\widehat{\sigma}_e\right) \tag{23}$$

is an exact equal-tailed TI.

7.2 Two-Sided Tolerance Intervals

Notice that the TI in (21) would include at least $100p\%$ of the population with probability at least γ. So the probability content in the interval (21) is

$$F\left(\widehat{\mu}_e + q^*_{\frac{1+p}{2},\frac{1+\gamma}{2}}\widehat{\sigma}_e\right) - F\left(\widehat{\mu}_e + q^*_{\frac{1-p}{2},\frac{1-\gamma}{2}}\widehat{\sigma}_e\right) \ge p$$

with probability at least γ. That is,

$$P\left(F\left(\widehat{\mu}_e + q^*_{\frac{1+p}{2},\frac{1+\gamma}{2}}\widehat{\sigma}_e\right) - F\left(\widehat{\mu}_e + q^*_{\frac{1-p}{2},\frac{1-\gamma}{2}}\widehat{\sigma}_e\right) \ge p\right) \ge \gamma. \tag{24}$$

Since the above probability does not depend on any parameter, we can express the probability as

$$\begin{aligned} G_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1) = P\Bigg\{&F_X\left(\widehat{\mu}^* + q^*_{\frac{1+p}{2};\frac{1+\gamma}{2}}\widehat{\sigma}^*\middle|0,1\right) \\ &-F_X\left(\widehat{\mu}^* + q^*_{\frac{1-p}{2};\frac{1-\gamma}{2}}\widehat{\sigma}^*\middle|0,1\right) \ge p\Bigg\} \ge \gamma, \end{aligned} \tag{25}$$

where $F_X(x|0,1)$ denotes the CDF of the standard Maxwell distribution. Let $\gamma_t' \le \gamma$ be such that

$$G_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma_t'|0,1) = P\left\{F_X\left(\widehat{\mu}^* + q^*_{\frac{1+p}{2};\frac{1+\gamma_t'}{2}}\widehat{\sigma}^*\middle|0,1\right) - F_X\left(\widehat{\mu}^* + q^*_{\frac{1-p}{2};\frac{1-\gamma_t'}{2}}\widehat{\sigma}^*\middle|0,1\right) \ge p\right\} = \gamma. \tag{26}$$

Then

$$\left(\widehat{\mu} + q^*_{\frac{1-p}{2};\frac{1-\gamma_t'}{2}}\widehat{\sigma},\ \widehat{\mu} + q^*_{\frac{1+p}{2};\frac{1+\gamma_t'}{2}}\widehat{\sigma}\right) \tag{27}$$

is an exact two-sided TI. The root of the Eq. (26) can be found along the lines for finding equal-tailed TI described in the preceding paragraph. For more details and algorithm to compute the equal-tailed and two-side TI factors, see Hoang-Nguyen-Thuy and Krishnamoorthy [7].

We compared tolerance intervals based on the MLEs, MEs, and MMLEs in terms of expected widths. The comparison results are very similar to those in the preceding sections, and so we do not report the comparison results here. That is, TIs based on the MLEs are a little shorter than those based on the MEs and MMLEs. So we computed $(p, .95)$ factors based on the MLEs to compute two-sided as well as equal-tailed TIs and reported them in Table 7 for $p = .80$, .90, .95 and .99, and for various values of sample size ranging from 5 to 100. Factors for any values of (n, p, γ) can be computed using the R function `TFs.Maxwell()` in "Appendix B".

8 Examples

Example 1 The data in Table 8 are from Chen et al. [3], and they represent the lifetimes of 1.88-mm drills from a supplier. These data were collected during the production process of drills in a factory. Krishnamoorthy et al. [11] and Hoang-Nguyen-Thuy and Krishnamoorthy [8] modeled the data using a two-parameter Rayleigh distribution and constructed CIs, PIs and TIs. These data also fit a two-parameter Maxwell distribution quite well; see the Q-Q plot in Figure 1.

We computed various point estimates, 95% CIs for the mean lifetime of drills, (.90, .95) lower tolerance limits (LTLs), 95% PIs for the mean $\bar{X}_{15}$ of a future sample of size five, (.90, .95) tolerance intervals and 95% lower confidence limit for $P(X > 76)$ and presented them in Table 9. All these intervals were computed using the R functions in "Appendix B". As noted earlier, comparison of various intervals indicates that all statistical intervals based on different equivariant estimates are practically similar and only minute differences exist among them. The results are also in agreement with our earlier comparison studies. In particular, we note that interval estimates based on the MLEs are slightly narrower than corresponding other intervals.

By fitting a two-parameter Rayleigh distribution for the data in Table 8, Krishnamoorthy et al. [9] have obtained the following results: The 95% CI for the mean

Table 7 Lower (upper) factors for computing $(p, .95)$ TIs based on the MLEs

	Two-sided				Equal-tailed			
	p							
n	.80	.90	.95	.99	.80	.90	.95	.99
	$L_t(U_t)$	$L_t(U_t)$	$L_t(U_t)$	$L_t(U_t)$	$L_e(U_e)$	$L_e(U_e)$	$L_e(U_e)$	$L_e(U_e)$
[.3em] 5	− .478(3.24)	− .830(3.86)	− 1.10(4.41)	− 1.52(5.50)	− .774(3.64)	− 1.08(4.19)	− 1.31(4.72)	− 1.62(5.72)
6	− .236(2.89)	− .538(3.41)	− .759(3.89)	− 1.11(4.81)	− .482(3.24)	− .754(3.73)	− .933(4.17)	− 1.25(5.06)
7	− .093(2.70)	− .356(3.17)	− .552(3.58)	− .876(4.44)	− .313(2.99)	− .543(3.44)	− .711(3.82)	− .993(4.62)
8	.006(2.55)	− .243(2.99)	− .431(3.38)	− .723(4.15)	− .194(2.84)	− .403(3.23)	− .578(3.61)	− .821(4.33)
9	.074(2.45)	− .157(2.87)	− .332(3.24)	− .608(3.97)	− .103(2.71)	− .308(3.10)	− .458(3.44)	− .698(4.13)
10	.120(2.39)	− .092(2.77)	− .261(3.12)	− .522(3.83)	− .036(2.62)	− .236(2.99)	− .382(3.32)	− .620(3.98)
11	.167(2.33)	− .051(2.71)	− .203(3.04)	− .449(3.69)	.006(2.55)	− .171(2.91)	− .319(3.21)	− .542(3.84)
12	.196(2.28)	− .005(2.65)	− .160(2.97)	− .396(3.59)	.047(2.49)	− .123(2.83)	− .279(3.15)	− .479(3.73)
13	.222(2.24)	.026(2.59)	− .123(2.91)	− .361(3.53)	.081(2.44)	− .093(2.78)	− .231(3.08)	− .430(3.66)
14	.245(2.21)	.055(2.56)	− .095(2.86)	− .320(3.46)	.113(2.41)	− .059(2.73)	− .195(3.01)	− .393(3.60)
15	.264(2.19)	.077(2.51)	− .068(2.82)	− .290(3.40)	.133(2.37)	− .034(2.69)	− .159(2.97)	− .361(3.54)
16	.278(2.16)	.097(2.48)	− .043(2.77)	− .260(3.36)	.160(2.34)	− .007(2.65)	− .133(2.93)	− .326(3.49)
17	.296(2.14)	.116(2.46)	− .023(2.75)	− .235(3.31)	.176(2.32)	.012(2.62)	− .112(2.89)	− .304(3.44)
18	.305(2.12)	.127(2.44)	− .004(2.72)	− .217(3.28)	.194(2.29)	.031(2.59)	− .091(2.87)	− .286(3.40)

Table 7 continued

	Two-sided				Equal-tailed			
	p							
n	.80	.90	.95	.99	.80	.90	.95	.99
19	.318(2.11)	.142(2.42)	.010(2.69)	− .195(3.25)	.200(2.27)	.049(2.57)	− .072(2.83)	− .266(3.36)
20	.330(2.09)	.156(2.40)	.028(2.67)	− .179(3.22)	.217(2.26)	.059(2.55)	− .054(2.81)	− .243(3.33)
25	.363(2.04)	.199(2.33)	.077(2.59)	− .117(3.11)	.270(2.18)	.118(2.46)	.006(2.71)	− .170(3.21)
30	.389(2.00)	.231(2.28)	.109(2.54)	− .076(3.03)	.304(2.13)	.158(2.40)	.043(2.64)	− .129(3.13)
35	.405(1.98)	.252(2.25)	.135(2.49)	− .043(2.98)	.327(2.09)	.184(2.36)	.072(2.60)	− .095(3.07)
40	.418(1.96)	.268(2.22)	.155(2.46)	− .020(2.93)	.347(2.07)	.204(2.33)	.097(2.56)	− .068(3.02)
45	.429(1.94)	.282(2.20)	.169(2.44)	− .001(2.90)	.361(2.04)	.220(2.30)	.113(2.53)	− .049(2.99)
50	.437(1.93)	.292(2.19)	.182(2.42)	.010(2.88)	.373(2.03)	.233(2.28)	.129(2.51)	− .032(2.96)
60	.450(1.91)	.308(2.16)	.199(2.39)	.033(2.84)	.391(2.00)	.255(2.25)	.152(2.47)	− .006(2.91)
70	.459(1.90)	.319(2.15)	.212(2.37)	.047(2.81)	.405(1.98)	.271(2.22)	.168(2.44)	.011(2.87)
80	.466(1.88)	.328(2.13)	.222(2.35)	.060(2.79)	.417(1.96)	.283(2.21)	.180(2.42)	.026(2.85)
90	.472(1.88)	.334(2.12)	.231(2.33)	.070(2.77)	.424(1.95)	.293(2.19)	.193(2.40)	.038(2.83)
100	.476(1.87)	.341(2.11)	.238(2.32)	.079(2.75)	.433(1.94)	.301(2.18)	.200(2.39)	.049(2.81)

The factors L_t and U_t are found so that $(\widehat{\mu} + L_t\widehat{\sigma},\ \widehat{\mu} + U_t\widehat{\sigma})$ is a $(p, 1-\alpha)$ two-sided TI. The factors L_e and U_e are found so that $(\widehat{\mu} + L_e\widehat{\sigma},\ \widehat{\mu} + U_e\widehat{\sigma})$ is a $(p, 1-\alpha)$ equal-tailed TI

Table 8 Lifetime (in min) of a sample of 1.88-mm drills

105	105	95	87	112	80	95	97	77	103	78	87	107	96	79
91	108	97	80	76	92	85	76	96	77	80	100	94	82	104
91	95	93	99	99	94	84	99	91	85	86	79	89	89	100

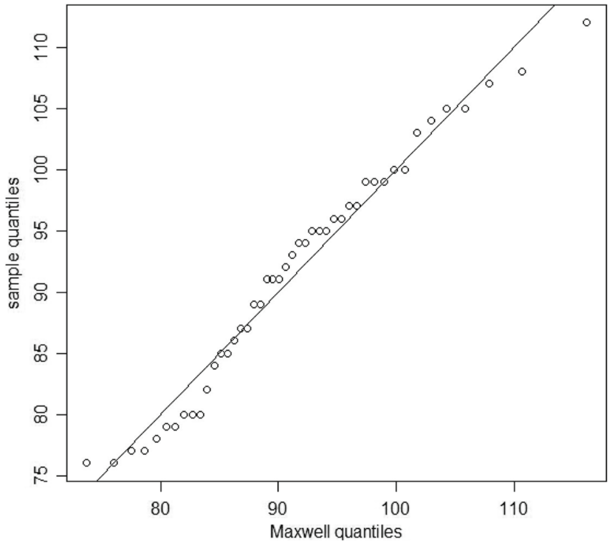

Fig. 1 Maxwell Q-Q plot for drills' life data

on the basis of the MLEs is (88.64, 94.52) and the one based on the MEs is (88.68, 94.58). Note that both CIs are in agreement with the corresponding ones in Table 9 which are based on a Maxwell model. The (.90, .95) lower tolerance limit based on the MLEs and two-parameter Rayleigh model is 77.0 and the one based on the MEs is 76.8. These two tolerance limits are also in agreement with the corresponding ones in Table 9.

Example 2 For this example, we shall use the flood data given in Dumonceaux and Antile [5]. The data are reproduced here in Table 10, which are maximum flood levels (in millions of cubic feet per second per 4-year period cycle) of Susquehenna River at Harrisburg, PA, for the period of 1890-1969. Application of the K-S test by Dey et al. [4] indicated that the data fit a Maxwell distribution. Our Q-Q plot in Figure 2 also indicates that the data fit a Maxwell distribution satisfactorily.

We computed various statistical intervals using the R functions given in "Appendix B", and presented them in Table 11. We once again observe that the comparison results are in agreement with our earlier comparison of various statistical intervals. In particular, the MLE-CIs are slightly narrower than the ME-intervals for estimation of mean, prediction and tolerance intervals. The MLE-intervals and the MMLE-intervals are practically the same.

Table 9 Statistical intervals for the drill data in Table 8

	Estimates of (μ, σ)	95% CI for mean		(.90,.95) LTL	
		Percentiles	CI	Factor	LTL
MLEs	(68.726, 20.124)	(.992, 1.280)	(88.69, 94.48)	.392	76.61
MEs	(68.385, 20.417)	(.993, 1.281)	(88.66, 94.54)	.391	76.37
MMLEs	(68.743, 20.255)	(.984 1.273)	(88.67, 94.53)	.384	76.52

	95% PI for $\overline{X}_{15}$		(.90, .95) TI		95% LCL for $P(X > 76)$
	Percentiles	PI	Factor	TI	
MLEs	(.854, 1.434)	(85.91, 97.59)	(.282, 2.20)	(74.40, 113.00)	.913
MEs	(.859, 1.432)	(85.92, 97.62)	(.281, 2.20)	(74.12, 113.30)	.907
MMLEs	(.844, 1.426)	(85.84, 97.63)	(.274, 2.19)	(74.29, 113.10)	.911

Table 10 Flood levels (10^6 cubic feet per second) data

.654	.613	.315	.449	.297	.402	.379	.423	.379	.3235
.269	.740	.418	.412	.494	.416	.338	.392	.484	.265

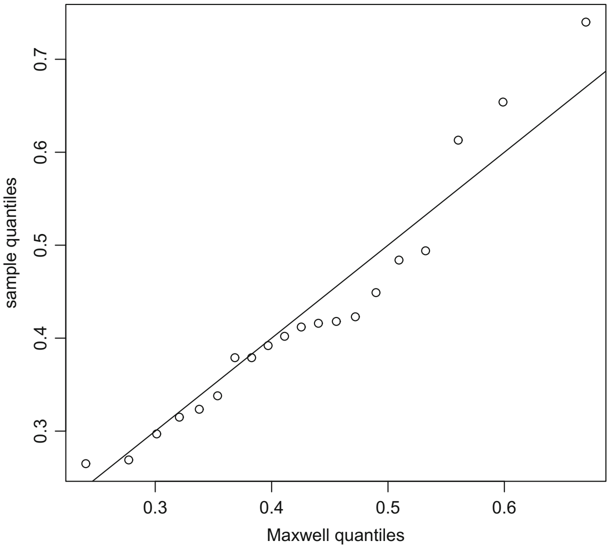

Fig. 2 Maxwell Q-Q plot for maximum flood levels data

9 Concluding Remarks

In this article, we have compared various exact statistical intervals that can be constructed using equivariant estimators. In particular, we compared the interval estimates based on equivariant MEs, MLEs and MMLEs which can be readily obtained. Our extensive comparison studies indicated that only little differences exist among these intervals in terms of precision. In general, the interval estimates based on the MLEs are slightly better than intervals based on other estimates for small sample sizes. For moderate to large samples, interval estimates based on all three equivariant estimates are very similar. It should be noted that even though the MEs and MMLEs are in closed-form, computer programs are necessary to find CIs, PIs and tolerance intervals based on them. Furthermore, method of moments and modified maximum likelihood method could produce inaccurate estimates of μ that are greater than $x_{(1)}$. Since computer programs are necessary to construct statistical intervals based on equivariant estimators, one may prefer to use the interval estimates based on natural MLEs. To help practitioners, we provide R functions in the "Appendix B", which can be used to find CIs, PIs, one-sided tolerance limits, confidence bound on survival probability and tolerance intervals and equal-tailed TIs.

Table 11 Statistical intervals for the flood level data in Table 10

	Estimates of (μ, σ)	95% CI for mean		(.90,.95) UTL	
		Percentiles	CI	Factor	UTL
MLEs	(.1634, .2343)	(.916, 1.38)	(.378, .487)	4.04	1.11
MEs	(.1263, .2631)	(.923, 1.37)	(.369, .487)	3.98	1.17
MMLEs	(.1585, .2410)	(.904, 1.36)	(.376, .486)	3.98	1.12

	95% PI for $\overline{X}_5$		(.90, .95) TI		95% LCL for $P(X > .450)$
	Percentiles	PI	Factor	TI	
MLEs	(.659, 1.68)	(.318, .557)	(.158, 2.40)	(.200, .726)	.264
MEs	(.672, 1.67)	(.303, .566)	(.175, 2.37)	(.172, .750)	.259
MMLEs	(.645, 1.66)	(.314, .559)	(.151, 2.37)	(.195, .730)	.263

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A

We here derive an interval based on a sample of size n that would include the location parameter μ with a specified probability. The derivation below is similar to the one by Hoang-Nguyen-Thuy and Krishnamoorthy (2020) for Rayleigh distributions. Let $X_{(1)}$ denote the smallest order statistic for a sample of size n from a Maxwell(μ, σ) distribution. Note that the distribution of $X_{(1)}$ is given by

$$P(X_{(1)} \le x) = 1 - (1 - F(x|\mu, \sigma))^n \,,$$

where $F(x|\mu, \sigma)$ is the CDF in (3). For a given $P \in (0, 1)$, let us determine the value of t so that $P\left(X_{(1)} - t \le \mu \le X_{(1)}\right) = P(\mu \le X_{(1)} \le \mu + t) = P$. Let $G(x|\Gamma(3/2))$ denote the gamma distribution function with the shape parameter 3/2 and the scale parameter 1. For a given P, we need to determine t so that

$$\begin{aligned} P(\mu \le X_{(1)} \le \mu + t) &= 1 - \left(1 - G(t^2/\sigma^2|\Gamma(3/2))\right)^n \\ &= P. \end{aligned}$$

Solving the above equation for t, we obtain

$$t = \sigma\sqrt{G^{-1}\left(1 - (1 - P)^{1/n}|3/2\right)},$$

where $G^{-1}(q|a)$ is the quantile function of gamma$(a, 1)$. In practice, σ is unknown, and so using the moment estimate $\widehat{\sigma}_M$, we estimate t by

$$\widehat{t} = \widehat{\sigma}_M\sqrt{G^{-1}\left(1 - (1 - P)^{1/n}|3/2\right)}.$$

Note that we estimated the value of t by replacing σ by $\widehat{\sigma}_M$. By choosing $P = .999$, the interval $\left(X_{(1)} - \widehat{t},\ X_{(1)}\right)$ is expected include with high probability.

Appendix B

```
# This function generates n pseudo random numbers
    from a Maxwell(mu, sigma)
# distribution

rmaxwell = function(n, mu, sigma){
z = rgamma(n, 3/2)
x = mu + sqrt(z)*sigma
return(x)
}
# Usage:
x = rmaxwell(n = 20, mu = 2, sigma = 3)

# This function computes the MLEs for a given sample x
MLEs = function(x){
n = length(x); Pf = 1-(1-.999)^(1/n)
qp = sqrt(qgamma(Pf, 3/2))
sigsq.x = (n-1)*var(x)/n
xb = mean(x); s = sqrt(sigsq.x); xmin = min(x)
sigh = s*sqrt(2*pi/(3*pi-8))
tconst = sigh*qp
fn = function(v){
gt = n*(xb-v)-.6666667*(sigsq.x+(xb-v)^2)*sum(1/(x-v))
}
muh = uniroot(fn, c(xmin-tconst, xmin))[[1]]
sigh = sqrt(2*(sum((x-muh)^2))/3/n)
return(c(muh,sigh))
}
# This function computes the percentiles
     to compute 100cl
# for  the mean; sample size is n; nr
     = number of simulation runs

perc.ci.mean = function(nr, n, cl){
al = (1-cl)/2; cs = 2/sqrt(pi)
x = matrix(rmaxwell(nr*n,0,1),nr,n)
ml = apply(x, 1, function(x) MLEs(x))
muh = ml[1,]; sigh = ml[2,]
piv = (cs-muh)/sigh
crt = quantile(piv, c(al,1-al))
print(crt,3)
}
# Usage:
> perc.ci.mean(10^5, n = 20, cl = .95)
2.5
```

```
0.915 1.380

# This function computes the percentiles to
    construct 100cl
# for  the mean of a future sample of size m;
    nr = number of simulation runs

PI.fac = function(nr, n, m, cl){
x = matrix(rmaxwell(nr*n,0,1),nr,n)
ml = apply(x, 1, function(x) MLEs(x))
muh = ml[1,]; sigh = ml[2,]
y = matrix(rmaxwell(nr*m,0,1),nr,m)
yb = apply(y, 1, function(x)  mean(x))
piv = (yb-muh)/sigh
crt = unname(quantile(piv,c(.025,.975)))
print(crt,3)
}
PI.fac(10^5, n = 45, m = 5, cl = .95)

# This function computes (p, cl) one-sided
    tolerance factors
fac.os.TLs = function(nr, n,  p, cl){
x = matrix(rmaxwell(nr*n,0,1),nr,n)
ml = apply(x, 1, function(x) MLEs(x))
muh = ml[1,]; sigh = ml[2,]
UppP = sqrt(qgamma(p,3/2))
piv = (UppP-muh)/sigh
crtU = quantile(piv, cl)
LowP = sqrt(qgamma(1-p,3/2))
piv = (LowP-muh)/sigh
crtL = quantile(piv, 1-cl)
print(c(crtL, crtU),3)
}
fac.os.TLs(10^5, n = 10, p = .9, cl = .95)

# Lower confidence limit for P(X > t)

CI.Surv.Prob = function(nr, x, t,  cl){ # x = sample
n = length(x); al = 1-cl
mls = MLEs(x);  muh0 = mls[1]; sigh0 = mls[2]
xm = matrix(rmaxwell(nr*n,0,1), nr, n)
mles = apply(xm, 1, function(x) MLEs(x))
muh = mles[1,]; sigh = mles[2,]
t0 = (t-muh0)/sigh0;  p0 = 1-pgamma(t0^2,1.5)
#
fn = function(y){
```

```
Qp = quantile((sqrt(qgamma(1-y,1.5))-muh)/sigh, al)
return(Qp-t0)
}
Low = uniroot(fn, c(.001, p0))[[1]]
print(Low)
}

# Two-sided or equal-tailed tolerance factors
TFs.Maxwell = function(nr,n,p,gam,tails){
ql = sqrt(qgamma((1-p)/2,1.5))
qu = sqrt(qgamma((1+p)/2,1.5))
x = matrix(rmaxwell(nr*n,0,1), nr, n)
mles = apply(x, 1, function(x) MLEs(x))
muh = mles[1,]; sigh = mles[2,]
# one-sided factor
TF = function(nr, n, muh, sigh, p, gam){
wu = (qu-muh)/sigh; wl = (ql-muh)/sigh
UTF = quantile(wu, gam); LTF = quantile(wl, 1-gam)
return(c(LTF,UTF))
}
# CDF of Maxwell
pmaxwell = function(x, mu, sig){
x = pmax(mu,x)
z = (x-mu)^2/sig^2
return(pgamma(z,1.5))
}
fn = function(gamt){
fac = TF(nr,n, muh, sigh, (1+p)/2, (1+gamt)/2)
Low = muh + fac[1]*sigh; Upp = muh + fac[2]*sigh
if(tails == "E-T"){
cont = Low <= ql & qu <= Upp}
else{
cont = pmaxwell(Upp,0,1)-pmaxwell(Low,0,1)}
covr = mean(cont >= p)
return(covr-gam)
}
# bisection method
xl = .5; xr = gam; k = 1
repeat{
fl = fn(xl); fr = fn(xr)
xm = (xl+xr)/2
fm = fn(xm)
if(abs(fm) < 1.0e-5 | k > 50){break}
if(fl*fm > 0){xl = xm}
else{xr = xm}
k = k + 1
```

```
}
fac = unname(TF(nr,n, muh, sigh, (1+p)/2, (1+xm)/2))
print(fac,3)
}
> TFs.Maxwell(10^5, n = 20, p = .9, gam = .95,
    tails = "T-S")
[1] 0.157 2.401
```

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