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Estimation of the probability content in a specified interval using fiducial approach

Ngan Hoang-Nguyen-Thuy and K. Krishnamoorthy

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA, USA

ABSTRACT

Statistical methods for constructing confidence intervals for the probability content in a specified interval are proposed. Exact and approximate solutions based on the fiducial approach are described when the measurements on the variable of interest can be modelled by a location-scale (or log-location-scale) distribution. Methods are described for the normal, Weibull, two-parameter exponential and two-parameter Rayleigh distributions. For each case, the solutions are evaluated for their merits. Three examples, where it is desired to estimate the percentages of engineering products meet the specification limits, are provided to illustrate the methods.

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1. Introduction

Mechanical parts are manufactured to meet some tolerance specification limits so that they can be used for their intended purpose. For example, if a shaft is designed to have a 'sliding fit' in a hole, the shaft should be little smaller than the hole. Specifically, if a shaft with a nominal diameter of 10 mm is to have a sliding fit within a hole, the shaft might be specified with a tolerance range from 9.964 to 10 mm, and the hole might be specified with a tolerance range from 10.04 to 10.076 mm. Both the shaft and hole sizes will usually form normal distributions.[1] In electrical components production, an electrical specification might call for a resistor with a nominal value of 100 ohms, but will also state a tolerance such as $\pm 1\%$. Thus, in many applications, one needs to assess the percentage of parts that meet the specifications. For example, the acceptance sampling plan, an important statistical method, is commonly used in quality control. In particular, the plan is used to accept/reject a shipment of a product based on some quality characteristics of the parts in a sample from the shipment. Such methods are also used in different stages of production by a manufacturer. If an acceptance sampling plan is based on a continuous variable type data, and it is designed to accept/reject the shipment or a production process on the basis of the percentage of parts satisfy the tolerance specifications, then a confidence interval (CI) or hypothesis test for the true percentage of parts that meet specification is required to implement the acceptance sampling plan.

CONTACT K. Krishnamoorthy krishna@louisiana.edu

A tolerance interval maybe used to assess the percentage of parts that meet the tolerance specification (NIST[2]). Recall that a p-content – γ coverage tolerance interval or (p, γ) tolerance interval based on a sample is constructed so that it would include at least a proportion p of the sampled population with confidence γ. If a specification interval (*LSL*, *USL*), where the *LSL* and *USL* denote the lower and upper specification limits, respectively, includes a (p, γ) tolerance interval then it can be concluded that at least $100p\%$ of the parts meet specifications with confidence γ. However, it should be noted that if the tolerance interval overlaps with the specification interval, then no conclusion can be made as to the percentage of parts that meet specifications. Assuming normality for the continuous variable data, Owen [23] has proposed simultaneous confidence intervals (CIs) for the lower and upper percentiles of a normal distribution. Krishnamoorthy and Mathew [18] (Section 2.3.3) have used these simultaneous CIs to test if a specified percentage of parts are within the specification limits. If the null hypothesis of the test is rejected then it can be concluded that at least a specified percentage of parts are within the specification limits. However, the specification interval (*LSL*, *USL*) may include at least a specified percentage of parts even if the null hypothesis is not rejected. See Example 1 in the sequel. Recently, Young *et al.* [26] have investigated the problem of determining sample size so that a (p, γ) lower tolerance limit is greater than *LSL*. For such sample size, it is expected that the lower tolerance limit is greater than *LSL*, as a result, at least proportion p of the measurements meet the lower specification. These authors have also considered the problem of determining sample size so that a two-sided tolerance interval would include (*LSL*, *USL*).

In some applications, an engineering part is required to meet only the *LSL* and the problem is to estimate this probability based on the inspection of a sample of parts. In lifetime data analysis, the probability $P(X > t)$, where t is a specified time period, is referred to as the survival probability. Commonly a lower bound on this probability is required to quantify the effect of a treatment or to judge the minimum percentage of parts that meet the lower specification limit. No closed-form expression is available for the lower bound even in the normal case, and it can be computed only numerically. Specifically, if $T(X; p, \gamma)$ denotes the (p, γ) lower tolerance limit, then the value of p for which $T(X; p, \gamma) = t$ is a $100\gamma\%$ lower confidence limit for $P(X > t)$. However, to the best of our knowledge, no confidence interval is available for the probability content in a specified interval.

In this article, we provide some simple solutions based on the fiducial approach to the aforementioned problems. The concepts of fiducial distribution and fiducial inference were introduced by Fisher [8,9]. Even though there are some severe criticisms concerning the interpretation of fiducial distribution (Zabell [27]) and not a popular statistical method, Efron [7] has noted in Section 8 of his paper that 'maybe Fishers biggest blunder will become a big hit in the 21st century!'. The fiducial approach was resurfaced in the name of *generalized variable* approach introduced by Tsui and Weerahandi [24] and Weerahandi [25]. Hannig [11] has noted that the generalized variable approach is a special case of the fiducial approach, and all the results obtained using the generalized variable approach can be obtained using the fiducial approach. The fiducial approach is a useful tool to find solutions to many complex problems with satisfactory frequentist properties. See Clopper and Pearson [3], Garwood [10] and Chapman [1] for some classical results. For other problems where fiducial inference led to exact CIs, see Dawid and Stone [5], the articles by Krishnamoorthy and Mathew [16,17]. Applications of the fiducial approach to estimate the process capability indices (PCIs) can be found in Mathew *et al.* [22] and the recent article by Edirisinghe *et al.* [6].

The rest of the article is organized as follows. In the following section, we provide fiducial quantities for the parameters of a location-scale distribution. We show that the fiducial confidence interval for the survival probability $P(X > t)$ is exact when X has a location-scale distribution. In Section 3, we provide fiducial CIs for the probability contents in a specified finite interval for the normal, Weibull, two-parameter exponential and Rayleigh distributions. For each case, the accuracy of the fiducial CI is evaluated by Monte Carlo simulation studies. In Section 4, the methods are illustrated using three examples with real data. Some concluding remarks are given in Section 5.

2. Location-scale family of distributions

A family of probability distributions is called a location-scale family if their probability density function (pdf) can be expressed in the form

$$f(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0, \tag{1}$$

where μ is the location parameter and σ is the scale parameter. For example, the normal, Laplace, logistic and two-parameter exponential distributions are all location-scale distributions. The lognormal and Weibull distributions are log-location-scale distributions as the log-transformed samples from these distributions follow location-scale distributions.

Let $F(x|\mu,\sigma)$ denote the cumulative distribution function (cdf) and let $F^{-1}(p|\mu,\sigma)$ denote the inverse distribution function. Let $\widehat{\mu}$ and $\widehat{\sigma}$ be equivariant estimators of μ and σ, respectively, based on a sample of size n. Then $\frac{\widehat{\mu}-\mu}{\sigma}$, $\frac{\widehat{\sigma}}{\sigma}$ and $\frac{\widehat{\mu}-\mu}{\widehat{\sigma}}$ are all pivotal quantities (see Lawless [20], Theorem E2). That is, their distributions remain the same for all parameter values. As a consequence, the quantities

$$\frac{\widehat{\mu}-\mu}{\sigma} \stackrel{d}{=} \widehat{\mu}^* \quad \text{and} \quad \frac{\widehat{\sigma}}{\sigma} \stackrel{d}{=} \widehat{\sigma}^*, \tag{2}$$

where $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are equivariant estimators based on a sample of size n from the distribution $f(x|0,1)$, and the notation '$X \stackrel{d}{=} Y$' means X and Y are identically distributed.

Using the approach by Dawid and Stone [5], fiducial distributions of the location and scale parameters can be obtained as follows. Let $(\widehat{\mu}_0, \widehat{\sigma}_0)$ be an observed value of $(\widehat{\mu}, \widehat{\sigma})$. Solving the 'equations' in Equation (2) for μ and σ, and then replacing $(\widehat{\mu}, \widehat{\sigma})$ by $(\widehat{\mu}_0, \widehat{\sigma}_0)$, we obtain the fiducial quantities (FQs) for μ and σ as

$$Q_\mu = \widehat{\mu}_0 - \frac{\widehat{\mu}^*}{\widehat{\sigma}^*}\widehat{\sigma}_0 \quad \text{and} \quad Q_\sigma = \frac{\widehat{\sigma}_0}{\widehat{\sigma}^*}, \tag{3}$$

respectively. For a fixed $(\widehat{\mu}_0, \widehat{\sigma}_0)$, the distribution of the above FQs are called fiducial distributions.

2.1. Fiducial distribution of $F_X(t|\mu,\sigma)$

For a given t, let $P_t = P(X \le t|\mu,\sigma) = F_X(t|\mu,\sigma)$ and consider testing

$$H_0 : P_t = p_0 \quad \text{vs.} \quad H_a : P_t > p_0, \tag{4}$$

where p_0 is a specified value in $(0, 1)$. It can be readily verified that testing above hypotheses is equivalent to testing

$$H_0 : \mu + F^{-1}(p_0|0,1)\sigma = t \quad \text{vs.} \quad H_a : \mu + F^{-1}(p_0|0,1)\sigma < t. \tag{5}$$

Furthermore,

$$\frac{\mu + F^{-1}(p_0|0,1)\sigma - \widehat{\mu}}{\widehat{\sigma}} \overset{d}{=} \frac{F^{-1}(p_0|0,1) - \widehat{\mu}^*}{\widehat{\sigma}^*}, \tag{6}$$

where $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are equivariant estimators based on a sample of size n from the distribution $f(x|0,1)$. In view of Equation (6) and the probability integral transform, under $H_0 : \mu + F^{-1}(p_0|0,1)\sigma = t$, the p-value

$$P\left(\frac{F^{-1}(p_0|0,1) - \widehat{\mu}^*}{\widehat{\sigma}^*} < \frac{t - \widehat{\mu}}{\widehat{\sigma}}\right)$$

has the uniform $(0, 1)$ distribution. For a given level α and an observed value $(\widehat{\mu}_0, \widehat{\sigma}_0)$ of $(\widehat{\mu}, \widehat{\sigma})$, the test that rejects H_0 whenever the p-value

$$P\left(\frac{F^{-1}(p_0|0,1) - \widehat{\mu}^*}{\widehat{\sigma}^*} < \frac{t - \widehat{\mu}_0}{\widehat{\sigma}_0}\right) < \alpha \iff P\left(F^*\left(\widehat{\sigma}^*\left(\frac{t - \widehat{\mu}_0}{\widehat{\sigma}_0}\right) + \widehat{\mu}^*\right) > p_0\right) < \alpha, \tag{7}$$

where F^* is the cumulative distribution function of the location-scale distribution with $\mu = 0$ and $\sigma = 1$, is an exact level α test. For a given $(\widehat{\mu}_0, \widehat{\sigma}_0)$, the 'probable values' of $P_t = p_0$ are determined by the distribution of

$$Q_{P_t} = F^*\left(\widehat{\sigma}^*\left(\frac{t - \widehat{\mu}_0}{\widehat{\sigma}_0}\right) + \widehat{\mu}^*\right), \tag{8}$$

which is the fiducial distribution for P_t. The above quantity Q_{P_t} is called the fiducial quantity for $P_t = P(X \le t|\mu,\sigma)$.

Remark 2.1: It should be noted that a FQ for $F(t|\mu,\sigma)$ can be obtained in a straightforward manner by replacing (μ,σ) with their FQs (Q_μ, Q_σ). However, in the preceding paragraph, we deduced the FQ for $F(t|\mu,\sigma)$ from hypothesis test in order to show that the fiducial inferential results for $F(t|\mu,\sigma)$ are exact. To obtain the FQ for $F(t|\mu,\sigma)$ by substitution, let $F(x|0,1) = F^*(x)$. Recall that for a location-scale distribution,

$$F(t|\mu,\sigma) = F\left(\frac{t-\mu}{\sigma}\middle|0,1\right) = F^*\left(\frac{t-\mu}{\sigma}\right).$$

Replacing (μ,σ) by (Q_μ, Q_σ), we find the FQ in Equation (8).

2.2. Fiducial CIs for probability

2.2.1. Confidence Intervals for $F_X(t|\mu,\sigma)$

For a given $(\widehat{\mu}_0, \widehat{\sigma}_0)$, the fiducial distribution of $F_X(t|\mu,\sigma)$ is the distribution of

$$Q_{P_t} = F\left(\widehat{\sigma}^*\left(\frac{t-\widehat{\mu}_0}{\widehat{\sigma}_0}\right)+\widehat{\mu}^*\right). \tag{9}$$

A $1-2\alpha$ fiducial CI for P_t is formed by the lower and upper 100α percentiles of Q_{P_t}. Specifically, if $Q_{P_{t;q}}$ denotes the $100q$ percentile of Q_{P_t}, then $(Q_{P_t;\alpha},\ Q_{P_t;1-\alpha})$ is a $1-2\alpha$ CI for P_t. For a given $(\widehat{\mu}_0, \widehat{\sigma}_0)$, the distribution of the above fiducial quantity does not depend on any parameter, its percentiles can be estimated by Monte Carlo simulation. As the fiducial quantity is obtained by inverting an exact test, the fiducial CI for $P(X \le t|\mu,\sigma)$ is exact.

2.2.2. Confidence intervals for $P(L \le X \le U)$

Writing $P(L \le X \le U) = P(X \le U) - P(X \le L)$ and replacing the parameters with their FQs, we can obtain a fiducial quantity for $P_{LU} = P(L \le X \le U)$ as

$$Q_{P_{LU}} = Q_{P_U} - Q_{P_L} = F^*\left(\widehat{\sigma}^*\left(\frac{U-\widehat{\mu}_0}{\widehat{\sigma}_0}\right)+\widehat{\mu}^*\right) - F^*\left(\widehat{\sigma}^*\left(\frac{L-\widehat{\mu}_0}{\widehat{\sigma}_0}\right)+\widehat{\mu}^*\right). \tag{10}$$

For a given $(\widehat{\mu}_0, \widehat{\sigma}_0)$, the distribution of the above fiducial quantity does not depend on any parameter, and so its percentiles can be estimated by Monte Carlo simulation. Percentiles of the above fiducial quantity form a CI for the probability P_{LU}. However, such fiducial CIs for P_{LU} are not necessarily exact with respect to the joint distribution of $\widehat{\mu}$ and $\widehat{\sigma}$.

In the following sections, we shall illustrate the fiducial approach for some location-scale families of distributions.

3. Some location-scale distributions

In this section, we shall describe FQs for the probabilities under the normal, Weibull, exponential and Rayleigh distributions. Specifically, we point out efficient equivariant estimators and the methods to calculate them. For each case, we explain the methods of computing CIs for $P(X < t)$ and for $P(L \le X \le U)$, where X follows a location-scale distribution. Note that if (L_t, U_t) is a CI for $P(X < t)$, then $(1-U_t, 1-L_t)$ is a CI for $P(X > t)$.

3.1. Normal distribution

For the normal case, we shall use the usual equivariant estimators, the sample mean $\widehat{\mu} = \bar{X}$ and the variance $\widehat{\sigma}^2 = S^2$. Note that $\widehat{\mu}^*$ and $\widehat{\sigma}^{*^2}$ are the sample mean and variance based on a sample of size n from a standard normal distribution, and so $\widehat{\mu}^* \sim N(0, 1/n)$ independently of $\widehat{\sigma}^* \sim \sqrt{\chi^2_{n-1}/(n-1)}$. Let us denote the observed value $(\widehat{\mu}_0, \widehat{\sigma}_0)$ by $(\bar{x}, s)$. In these notations, the FQ for $P_t = P(X < t)$ is expressed as

$$Q_{P_t} = \Phi\left(\frac{W(t-\bar{x})}{s} + \frac{Z}{\sqrt{n}}\right), \tag{11}$$

where $Z \sim N(0,1)$ independently of $W \sim \sqrt{\chi^2_{n-1}/(n-1)}$ and $\Phi(x)$ is the cdf of the standard normal distribution. For a given $(t, \bar{x}, s)$, the percentiles of Q_{P_t} can be estimated by Monte Carlo simulation, and appropriate percentiles form a CI for $P(X < t)$. As noted earlier, the above CI is exact except for the simulation errors. R code for computing confidence limits for $P(X < t)$ is given in the appendix.

Alternatively, one can find a closed-form approximate CI for $P(X < t)$ using the modified normal approximation by Krishnamoorthy [14]. Let W_α denote the 100α percentile of W. To find a CI for $P(X < t)$, it is enough to find percentiles of $\frac{W(t-\bar{x})}{s} + \frac{Z}{\sqrt{n}}$. The lower 100α percentile is approximated as

$$L_t = \begin{cases} W_{.5}\left(\frac{t-\bar{x}}{s}\right) - \sqrt{\left(\frac{t-\bar{x}}{s}\right)^2 (W_{.5} - W_\alpha)^2 + z_\alpha^2/n} & \text{if } \frac{t-\bar{x}}{s} > 0, \\ W_{.5}\left(\frac{t-\bar{x}}{s}\right) - \sqrt{\left(\frac{t-\bar{x}}{s}\right)^2 (W_{.5} - W_{1-\alpha})^2 + z_\alpha^2/n} & \text{if } \frac{t-\bar{x}}{s} \le 0. \end{cases} \tag{12}$$

The $100(1-\alpha)$ percentile can be approximated as

$$U_t = \begin{cases} W_{.5}\left(\frac{t-\bar{x}}{s}\right) + \sqrt{\left(\frac{t-\bar{x}}{s}\right)^2 (W_{.5} - W_{1-\alpha})^2 + z_\alpha^2/n} & \text{if } \frac{t-\bar{x}}{s} > 0, \\ W_{.5}\left(\frac{t-\bar{x}}{s}\right) + \sqrt{\left(\frac{t-\bar{x}}{s}\right)^2 (W_{.5} - W_\alpha)^2 + z_\alpha^2/n} & \text{if } \frac{t-\bar{x}}{s} \le 0. \end{cases} \tag{13}$$

The approximate $100(1-2\alpha)$ CI for $P(X < t)$ is given by $(\Phi(L_t), \Phi(U_t))$. This approximate CI is as good as the one based on Monte Carlo simulation. For more details on this approximation, see Hoang-Nguyen-Thuy [12].

3.1.1. A Fiducial CI for $P_{LU} = P(L \le X \le U)$

Using (10), the fiducial quantity for P_{LU} can be expressed as

$$Q_{P_{LU}} = \Phi\left(\frac{W(U-\bar{x})}{s} + \frac{Z}{\sqrt{n}}\right) - \Phi\left(\frac{W(L-\bar{x})}{s} + \frac{Z}{\sqrt{n}}\right), \tag{14}$$

where W and Z are as defined in Equation (11). For a given $(L, U, \bar{x}, s)$, Monte Carlo simulation can be used to estimate the percentiles of $Q_{P_{LU}}$, and appropriate percentiles form a CI for P_{LU}.

3.1.2. On selecting tail error probabilities

The fiducial distribution is essentially the posterior distribution with no prior distributions of the parameters, and so a highest probability density (HPD) region can be used as a

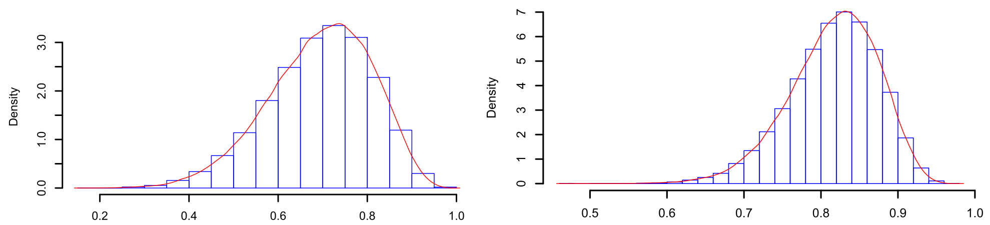

Figure 1. Histograms of the normal fiducial quantity (14); Left: $n = 10, \bar{x} = 0.070, s = 1.256, (L, U) = (-1, 2)$; Right: $n = 30, \bar{x} = -0.062, s = 0.932, (L, U) = (-1, 2)$.

fiducial CI for P_{LU}. One can find such HPD region using the R function as in the appendix. However, our Monte Carlo simulation studies (not reported here) indicated that such HPD intervals have poor coverage probabilities. For some parameter values, the coverage probability could be as low as .85 when the nominal level is .95. Alternatively, we can choose the tail error probabilities, so that the coverage probabilities are close to the nominal level for all parameter values as follows. Let α_l and α_r denote the left and right-tail error probabilities so that $\alpha_l + \alpha_r = \alpha$. Then $100\alpha_l$ lower percentile and $100\alpha_r$ upper percentile of $Q_{P_{LU}}$ form a $1 - \alpha$ fiducial CI for P_{LU}. In order to choose the values of α_l and α_r, we plotted the histogram of the fiducial quantity $Q_{P_{LU}}$ in (14) for some values of $(n, \bar{x}, s)$ as shown in Figure 1. The plots in Figure 1, and other plots (not reported here) for various values of $(n, \bar{x}, s)$ clearly indicated that the fiducial distribution for P_{LU} is left-skewed. So by choosing $\alpha_l > \alpha_r$, one could obtain a shorter CI for P_{LU}. On the basis of extensive numerical studies, for values $\alpha_l = 0.68\alpha$ and $\alpha_r = 0.32\alpha$, the $1 - \alpha$ fiducial CI for P_{LU} has satisfactory coverage probabilities for most cases. For example, to find a 95% fiducial CI for P_{LU}, we choose the left endpoint as the lower $100\alpha_l = 100 \times 0.68 \times 0.05 = 3.4$ th percentile of $Q_{P_{LU}}$ and the right endpoint as the upper $100\alpha_r = 100 \times 0.32 \times 0.05 = 1.6$ th percentile of $Q_{P_{LU}}$.

3.1.3. Coverage studies

To judge the accuracy of the fiducial CIs (a) formed by lower and upper $100\alpha/2$ percentiles of FQ in Equation (14) and (b) CI formed by the lower $100{\times}0.68 \times \alpha$ and upper $100{\times}0.32 \times \alpha$ percentiles of FQ in Equation (14), we estimated the coverage probabilities of these two CIs with the nominal level 0.95. The coverage probabilities along with the expected widths of the CIs are reported in Table 1. The estimated coverage probabilities and expected widths clearly indicate that the CI (b) is better than (a) for all sample sizes and L and U. For small sample sizes, the CI (a) is little liberal having coverage probabilities less than the nominal level 0.95. Even for large samples of size 30 or more, the CI (b) has the edge over the CI (a) in terms of coverage probabilities.

3.1.4. Lognormal distribution

If Y follows a lognormal distribution with parameters μ and σ^2, then $X = \ln(Y)$ has a normal distribution with mean μ and σ^2. So $P(L \le Y \le U) = P(\ln(L) \le X \le \ln(U))$, and the above method for the normal case can be applied to log-transformed sample from a lognormal population to find a CI for the probability $P(\ln(L) \le X \le \ln(U))$.

Table 1. Coverage probabilities (CP), left-tail error probabilities (LE), right-tail error probabilities (RE) and average lengths (AL) of 95% fiducial CIs for $P(L < X < U), X \sim N(0, 1)$.

	$n = 10$				$n = 20$			
	(a)		(b)		(a)		(b)	
$L(U)$	[LE]CP[RE]	(AL)	[LE]CP[RE]	(AL)	[LE]CP[RE]	(AL)	[LE]CP[RE]	(AL)
−0.3(0.3)	[0.006]0.936[0.058]	0.208	[0.009]0.950[0.041]	0.213	[0.010]0.943[0.048]	0.146	[0.014]0.954[0.032]	0.149
−0.6(0.6)	[0.007]0.935[0.058]	0.354	[0.011]0.949[0.040]	0.359	[0.010]0.943[0.047]	0.252	[0.015]0.954[0.031]	0.256
−1.0(1.0)	[0.008]0.935[0.057]	0.424	[0.012]0.948[0.039]	0.421	[0.010]0.942[0.047]	0.303	[0.016]0.953[0.031]	0.303
−2.0(2.0)	[0.011]0.936[0.053]	0.278	[0.016]0.947[0.037]	0.259	[0.013]0.941[0.046]	0.171	[0.018]0.953[0.030]	0.162
−2.3(2.3)	[0.011]0.937[0.052]	0.221	[0.017]0.947[0.036]	0.203	[0.014]0.941[0.046]	0.124	[0.018]0.952[0.029]	0.116
−2.5(2.5)	[0.012]0.937[0.051]	0.188	[0.018]0.946[0.036]	0.170	[0.014]0.941[0.045]	0.098	[0.019]0.952[0.029]	0.090
−0.3(1.0)	[0.009]0.932[0.059]	0.362	[0.012]0.949[0.039]	0.366	[0.012]0.937[0.051]	0.254	[0.017]0.947[0.036]	0.257
−1.0(0.3)	[0.009]0.926[0.065]	0.359	[0.012]0.942[0.046]	0.363	[0.010]0.944[0.046]	0.258	[0.015]0.955[0.031]	0.262
−1.5(−0.5)	[0.009]0.941[0.050]	0.254	[0.012]0.952[0.036]	0.260	[0.011]0.947[0.042]	0.187	[0.016]0.957[0.028]	0.190
0.5(1.5)	[0.010]0.942[0.047]	0.256	[0.014]0.954[0.032]	0.262	[0.014]0.948[0.038]	0.186	[0.019]0.955[0.025]	0.189
0.0(1.0)	[0.009]0.930[0.060]	0.295	[0.013]0.948[0.039]	0.301	[0.012]0.936[0.052]	0.205	[0.017]0.948[0.036]	0.209
0.3(2.3)	[0.016]0.950[0.034]	0.383	[0.022]0.956[0.022]	0.390	[0.021]0.950[0.028]	0.285	[0.028]0.952[0.019]	0.291
−0.5(2.5)	[0.020]0.942[0.038]	0.425	[0.026]0.948[0.026]	0.421	[0.020]0.945[0.036]	0.305	[0.027]0.949[0.023]	0.303
	$n = 30$				$n = 50$			
	(a)		(b)		(a)		(b)	
$L(U)$	[LE]CP[RE]	(AL)	[LE]CP[RE]	(AL)	[LE]CP[RE]	(AL)	[LE]CP[RE]	(AL)
−0.3(0.3)	[0.014]0.942[0.045]	0.116	[0.019]0.951[0.030]	0.118	[0.018]0.946[0.036]	0.090	[0.024]0.952[0.024]	0.092
−0.6(0.6)	[0.014]0.941[0.045]	0.202	[0.020]0.951[0.030]	0.204	[0.018]0.946[0.036]	0.157	[0.024]0.952[0.024]	0.159
−1.0(1.0)	[0.015]0.940[0.045]	0.244	[0.020]0.951[0.029]	0.244	[0.019]0.945[0.036]	0.190	[0.024]0.952[0.024]	0.190
−2.0(2.0)	[0.016]0.940[0.044]	0.130	[0.022]0.950[0.029]	0.124	[0.020]0.945[0.035]	0.093	[0.024]0.952[0.024]	0.090
−2.3(2.3)	[0.017]0.940[0.043]	0.090	[0.022]0.949[0.029]	0.085	[0.020]0.946[0.035]	0.061	[0.025]0.951[0.024]	0.059
−2.5(2.5)	[0.017]0.940[0.043]	0.068	[0.022]0.949[0.028]	0.064	[0.020]0.946[0.034]	0.045	[0.025]0.951[0.024]	0.042
−0.3(1.0)	[0.014]0.939[0.047]	0.205	[0.019]0.950[0.031]	0.209	[0.014]0.946[0.040]	0.162	[0.020]0.952[0.027]	0.164
−1.0(0.3)	[0.013]0.945[0.042]	0.208	[0.017]0.956[0.026]	0.211	[0.020]0.944[0.036]	0.160	[0.025]0.952[0.023]	0.161
−1.5(−0.5)	[0.015]0.947[0.038]	0.155	[0.021]0.952[0.027]	0.157	[0.019]0.953[0.028]	0.122	[0.024]0.959[0.017]	0.125
0.5(1.5)	[0.017]0.947[0.036]	0.154	[0.022]0.955[0.022]	0.157	[0.013]0.952[0.035]	0.122	[0.020]0.953[0.023]	0.121
0.0(1.0)	[0.014]0.938[0.048]	0.166	[0.020]0.950[0.030]	0.170	[0.014]0.944[0.042]	0.131	[0.019]0.952[0.029]	0.131
0.3(2.3)	[0.022]0.950[0.028]	0.239	[0.029]0.953[0.018]	0.243	[0.017]0.954[0.029]	0.191	[0.024]0.952[0.018]	0.191
−0.5(2.5)	[0.024]0.943[0.033]	0.253	[0.029]0.952[0.019]	0.255	[0.019]0.948[0.033]	0.202	[0.025]0.952[0.021]	0.202

(a) CI formed by lower and upper 2.5th percentiles of Equation (14).
(b) CI formed by the lower 3.4th and upper 1.6th percentile of Equation (14).

3.2. Weibull distribution

Let $X_1, \ldots, X_n$ be a sample from a two-parameter Weibull(c, b) distribution with the probability density function

$$f(x|b,c) = \frac{c}{b}\left(\frac{x}{b}\right)^{c-1} \exp\left\{-\left[\frac{x}{b}\right]^c\right\}, \quad x > 0, \; b > 0, \; c > 0.$$

Let $Y_i = \ln(X_i), i = 1, \ldots, n$. The maximum likelihood estimate (MLE) of c is the solution of the equation

$$\frac{1}{\widehat{c}} - \left(\sum_{i=1}^{n} X_i^{\widehat{c}} Y_i\right)\left(\sum_{i=1}^{n} X_i^{\widehat{c}}\right)^{-1} + \frac{1}{n}\sum_{i=1}^{n} Y_i = 0, \tag{15}$$

and the MLE of b is given by $\widehat{b} = (\frac{1}{n}\sum_{i=1}^{n} X_i^{\widehat{c}})^{1/\widehat{c}}$. See Cohen [4] or Krishnamoorthy *et al.* [15] The estimator $\widehat{c} = \frac{\pi}{\sqrt{6}S_y}$, where $S_y^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/(n-1)$ can be used as an initial value to find the MLE of c using the Newton–Raphson iterative scheme. The R code given in the appendix can be used to compute the MLEs for Weibull parameters.

3.2.1. Fiducial quantity for probabilities

The FQs for c and b are given by

$$Q_c = \frac{\widehat{c}_0}{\widehat{c}^*} \quad \text{and} \quad Q_b = \left(\frac{1}{\widehat{b}^*}\right)^{\frac{\widehat{c}^*}{\widehat{c}_0}} \widehat{b}_0, \tag{16}$$

respectively. See Krishnamoorthy *et al.* [15] Substituting these FQs for the parameters in the cdf of a Weibull(c,b) distribution $F(x|c,b) = 1 - \exp[-(\frac{x}{b})^c]$, we find the FQ for $P_t = P(X < t|c,b)$ as

$$Q_{P_t} = 1 - \exp\left(-\widehat{b}^* \left(\frac{t}{\widehat{b}_0}\right)^{\widehat{c}_0/\widehat{c}^*}\right). \tag{17}$$

A fiducial quantity for $P_{LU} = P(L \leq X \leq U)$ can be expressed as

$$Q_{P_{LU}} = \exp\left(-\widehat{b}^* \left(\frac{L}{\widehat{b}_0}\right)^{\widehat{c}_0/\widehat{c}^*}\right) - \exp\left(-\widehat{b}^* \left(\frac{U}{\widehat{b}_0}\right)^{\widehat{c}_0/\widehat{c}^*}\right). \tag{18}$$

3.2.2. Coverage studies

In order to determine the tail error probabilities α_l and α_r so that $100\alpha_l$ lower percentile and $100\alpha_r$ upper percentile of $Q_{P_{LU}}$ in Equation (18) form a $1 - (\alpha_l + \alpha_r) = 1 - \alpha$ fiducial CI for P_{LU}, we constructed histograms of the fiducial distribution for various values of $(n, \widehat{c}_0, \widehat{b}_0)$ and presented only two histograms in Figure 2. All our numerical studies indicated that the fiducial distributions for the Weibull case are nearly symmetric, and so we recommend to use $\alpha_l = \alpha_r = \alpha/2$. That is, we form the fiducial CI using the lower and upper $100\alpha/2$ percentiles of $Q_{P_{LU}}$ in Equation (18). The estimated coverage probabilities of such fiducial CIs are given in Table 2. Examination of table values indicates that the CI could be liberal for small sample sizes. For sample sizes of 20 or more, the coverage probabilities are close to the nominal level. Furthermore, we note that the precision of the CI increasing with increasing samples size. For $n \geq 20$, our fiducial CI for $P(L \leq X \leq U)$ are safe to use for a practical purpose.

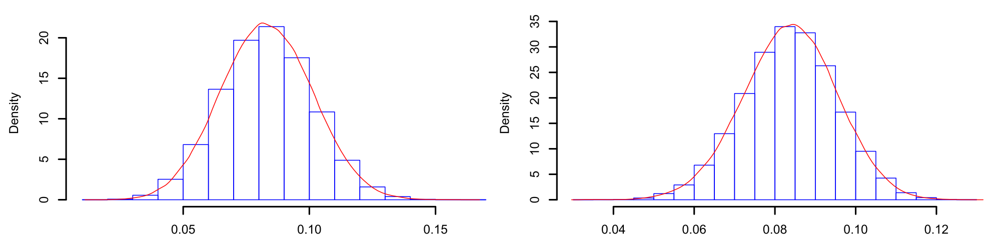

Figure 2. Histograms of the Weibull fiducial quantity (14); Left: $n = 10, \widehat{c}_0 = 0.8688, \widehat{b}_0 = 0.7208$, $(L, U) = (0.2, 0.3)$; Right: $n = 30, \widehat{c}_0 = 1.0685, \widehat{b}_0 = 0.8841$, $(L, U) = (0.2, 0.3)$.

Table 2. Coverage probabilities (CP) and average lengths (AL) of 95% fiducial CIs for $P(L < X < U)$, X ∼Weibull(c, b).

	Weibull(1, 1)				Weibull(1.5, 5)			
$L(U)$	$n = 10$ CP[AL]	$n = 20$ CP[AL]	$n = 30$ CP[AL]	$n = 40$ CP[AL]	$n = 10$ CP[AL]	$n = 20$ CP[AL]	$n = 30$ CP[AL]	$n = 40$ CP[AL]
0.2(0.3)	0.950[0.074]	0.951[0.054]	0.950[0.044]	0.951[0.038]	0.951[0.029]	0.947[0.020]	0.943[0.016]	0.950[0.013]
0.2(0.5)	0.938[0.181]	0.945[0.126]	0.945[0.101]	0.950[0.089]	0.945[0.083]	0.953[0.059]	0.952[0.048]	0.948[0.040]
0.2(0.7)	0.929[0.257]	0.936[0.177]	0.940[0.141]	0.950[0.123]	0.949[0.125]	0.948[0.091]	0.948[0.076]	0.946[0.065]
0.3(0.9)	0.922[0.267]	0.932[0.182]	0.940[0.146]	0.940[0.125]	0.949[0.136]	0.948[0.104]	0.946[0.086]	0.948[0.076]
0.4(1.0)	0.925[0.252]	0.934[0.173]	0.943[0.139]	0.947[0.121]	0.951[0.128]	0.953[0.099]	0.948[0.083]	0.953[0.074]
0.5(1.0)	0.927[0.209]	0.935[0.144]	0.944[0.115]	0.951[0.101]	0.955[0.105]	0.953[0.082]	0.946[0.068]	0.950[0.060]
0.6(1.2)	0.930[0.226]	0.939[0.158]	0.948[0.128]	0.941[0.109]	0.955[0.117]	0.948[0.090]	0.948[0.076]	0.950[0.067]
0.7(1.3)	0.930[0.213]	0.940[0.149]	0.949[0.122]	0.948[0.105]	0.954[0.113]	0.950[0.087]	0.948[0.073]	0.953[0.065]
1.0(1.5)	0.930[0.153]	0.939[0.107]	0.945[0.088]	0.948[0.076]	0.954[0.085]	0.950[0.065]	0.951[0.055]	0.952[0.049]
1.5(2.5)	0.924[0.178]	0.934[0.131]	0.939[0.108]	0.947[0.095]	0.953[0.137]	0.954[0.098]	0.947[0.081]	0.953[0.071]
2.0(3.0)	0.941[0.130]	0.942[0.099]	0.946[0.083]	0.947[0.073]	0.932[0.127]	0.941[0.087]	0.949[0.071]	0.949[.060]

3.3. Some other continuous distributions

3.3.1. Exponential distribution

Let $X_1, \ldots, X_n$ be a sample from a two-parameter exponential distribution with the pdf

$$f(x|\mu, \sigma) = \frac{1}{\sigma} \exp(-(x-\mu)/\sigma), \quad x > \mu, \ \sigma > 0. \tag{19}$$

The MLEs of μ and σ are given by $\widehat{\mu} = X_{(1)}$ and $\widehat{\sigma} = \bar{X} - X_{(1)}$, where $X_{(1)}$ is the smallest of the X_i's. The MLEs are equivariant and independent with $2n(\widehat{\mu} - \mu)/\sigma \sim \chi^2_2$ and $2n\widehat{\sigma}/\sigma \sim \chi^2_{2n-2}$. Notice that $\widehat{\mu}^* \sim \chi^2_2/(2n)$ independently of $\widehat{\sigma}^* \sim \chi^2_{2n-2}/(2n)$. Using these quantities in Equation (9), we can write FQ for $P_t = P(X \le t)$ as

$$Q_{P_t} = 1 - \exp\left(-\frac{1}{2n}\left[\chi^2_{2n-2}\left(\frac{t - \widehat{\mu}_0}{\widehat{\sigma}_0}\right) + \chi^2_2\right]\right), \tag{20}$$

where the χ^2 random variables are independent. The fiducial quantity for $P_{LU} = P(L \le X \le U)$ can be expressed as

$$Q_{P_{LU}} = \exp\left(-\frac{1}{2n}\left[\chi^2_{2n-2}\left(\frac{L - \widehat{\mu}_0}{\widehat{\sigma}_0}\right) + \chi^2_2\right]\right) - \exp\left(-\frac{1}{2n}\left[\chi^2_{2n-2}\left(\frac{U - \widehat{\mu}_0}{\widehat{\sigma}_0}\right) + \chi^2_2\right]\right). \tag{21}$$

3.3.2. Rayleigh distribution

The pdf of a two-parameter Rayleigh distribution with location parameter μ and the scale parameter σ is given by

$$f(x|\mu, \sigma) = \frac{(x-\mu)}{\sigma^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x > \mu, \ \sigma > 0. \tag{22}$$

The cdf is given by

$$F(x|\mu, \sigma) = 1 - e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x > \mu, \ \sigma > 0, \tag{23}$$

with the standard form $F^*(x) = F(x|0,1) = 1 - e^{-x^2/2}$. We shall denote the above distribution by Rayleigh(μ, σ).

Let $X_1, \dots, X_n$ be a sample from a two-parameter Rayleigh distribution. Let $\bar{X}$ denote the sample mean and define the sample variance as $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Krishnamoorthy *et al.* [19] have shown that the MLE $\widehat{\mu}$ of μ can be obtained as the root of the equation

$$h(\mu|X) = \frac{2n^2(\bar{X} - \mu)}{\sum_{i=1}^{n}(X_i - \mu)^2} - \sum_{i=1}^{n}(X_i - \mu)^{-1} = 0 \tag{24}$$

using the interval $(X_{(1)} - 12\widetilde{\sigma}/\sqrt{n},\ X_{(1)})$ as a root bracketing interval. Here, $\widetilde{\sigma}^2 = 2S^2/(4-\pi)$ is the moment estimate. The MLE $\widehat{\sigma} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}(X_i - \widehat{\mu})^2}$, where $\widehat{\mu}$ is the MLE of μ.

It follows from Equation (8) and (23) that the FQ for $P_t = P(X \le t|\mu,\sigma)$ can be expressed as

$$Q_{P_t} = 1 - e^{-\frac{1}{2}\left[\widehat{\sigma}^*\left(\frac{t-\widehat{\mu}_0}{\widehat{\sigma}_0}\right)+\widehat{\mu}^*\right]^2}, \tag{25}$$

where $\widehat{\mu}_0$ and $\widehat{\sigma}_0$ are the observed MLEs based on a sample of size n from the Rayleigh(μ, σ) distribution and $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are the MLEs based on a sample of size n from the Rayleigh$(0, 1)$ distribution. A FQ for $P_{LU} = P(X \le U|\mu,\sigma) - P(X \le L|\mu,\sigma)$ can be expressed as

$$Q_{P_{LU}} = e^{-\frac{1}{2}\left[\widehat{\sigma}^*\left(\frac{L-\widehat{\mu}_0}{\widehat{\sigma}_0}\right)+\widehat{\mu}^*\right]^2} - e^{-\frac{1}{2}\left[\widehat{\sigma}^*\left(\frac{U-\widehat{\mu}_0}{\widehat{\sigma}_0}\right)+\widehat{\mu}^*\right]^2}. \tag{26}$$

For a given $(\widehat{\mu}_0, \widehat{\sigma}_0)$, let $Q_{P_{LU};\alpha}$ denote the 100α percentile of $Q_{P_{LU}}$. For both the exponential and Rayleigh distributions, on the basis of shape of the fiducial distributions (simulation results are not reported here), we recommend to use the lower and upper $100\alpha/2$ percentiles of $Q_{P_{LU}}$ to form a $1-\alpha$ CI for the probability P_{LU}.

3.3.3. Coverage studies

The estimated coverage probabilities and average lengths of 95% CIs of probability contents for the exponential and Rayleigh distributions are given in Tables 3 and 4, respectively. We observe from Table 3 that the fiducial CIs for the exponential case could be slightly conservative for some sample size and parameter values. In general, we see that the coverage probabilities are slightly larger than the nominal level of 0.95. For the case of Rayleigh distribution, we observe from Table 4 that the CIs are satisfactory even for samples of size 10. The coverage probabilities, in the worst case, go as low as 0.93 when the nominal level is 0.95. On an overall basis, we see that CIs for both distributions are satisfactory and they can be recommended for applications.

4. Examples

Example 4.1: This example along with data is adapted from Liu *et al.* [21]. A manufacturing company of universal serial bus (USB) wants to produce micro-USB instead of mini USB, because the shape of modern digital devices is designed with thinner and lighter components. Accordingly, a particular model of micro-USB receptacle interface is considered

Table 3. Coverage probabilities (CP) and average lengths (AL) of 95% fiducial CIs for $P(L < X < U), X \sim$ Exponential(μ, σ).

	Exponential(0, 1)					Exponential(1, 2)			
	$n = 10$	$n = 20$	$n = 30$	$n = 40$		$n = 10$	$n = 20$	$n = 30$	$n = 40$
$L(U)$	CP[AL]	CP[AL]	CP[AL]	CP[AL]	$L(U)$	CP[AL]	CP[AL]	CP[AL]	CP[AL]
0.2(0.4)	0.951[0.168]	0.946[0.101]	0.955[0.081]	0.956[0.070]	1.2(1.4)	0.949[0.126]	0.946[0.072]	0.955[0.058]	0.956[0.049]
0.3(0.5)	0.956[0.128]	0.947[0.078]	0.956[0.063]	0.956[0.054]	1.3(1.5)	0.950[0.110]	0.946[0.064]	0.955[0.051]	0.956[0.044]
0.4(0.7)	0.960[0.131]	0.955[0.079]	0.959[0.063]	0.957[0.054]	1.4(1.7)	0.951[0.134]	0.946[0.080]	0.956[0.065]	0.956[0.055]
0.4(1.5)	0.941[0.233]	0.954[0.123]	0.962[0.089]	0.965[0.069]	1.4(2.5)	0.956[0.308]	0.948[0.186]	0.957[0.150]	0.956[0.128]
0.5(2.5)	0.944[0.246]	0.956[0.139]	0.967[0.101]	0.963[0.081]	1.5(2.5)	0.957[0.258]	0.950[0.156]	0.958[0.126]	0.956[0.107]
0.6(.75)	0.960[0.048]	0.957[0.028]	0.963[0.021]	0.964[0.018]	1.6(1.7)	0.952[0.039]	0.947[0.023]	0.955[0.019]	0.955[0.016]
0.7(2.0)	0.947[0.183]	0.958[0.105]	0.969[0.077]	0.965[0.062]	1.7(3.0)	0.959[0.222]	0.956[0.129]	0.961[0.102]	0.962[0.085]
1.0(2.5)	0.965[0.187]	0.969[0.132]	0.960[0.110]	0.949[0.096]	2.0(3.5)	0.951[0.174]	0.960[0.092]	0.966[0.065]	0.969[0.053]
2.0(3.0)	0.955[0.110]	0.946[0.088]	0.952[0.077]	0.951[0.069]	3.0(4.0)	0.948[0.078]	0.959[0.045]	0.970[0.033]	0.966[0.027]
2.0(4.0)	0.953[0.184]	0.945[0.146]	0.952[0.128]	0.951[0.114]	3.0(5.0)	0.965[0.139]	0.963[0.089]	0.972[0.070]	0.964[0.061]
2.0(5.0)	0.952[0.232]	0.944[0.181]	0.952[0.157]	0.951[0.140]	3.0(6.0)	0.968[0.189]	0.961[0.132]	0.965[0.109]	0.954[0.096]

Table 4. Coverage probabilities (CP) and average lengths (AL) of 95% fiducial CIs for $P(L < X < U), X \sim$ Rayleigh(μ, σ).

	Rayleigh(0, 1)					Rayleigh(1, 3)			
	$n = 10$	$n = 20$	$n = 30$	$n = 40$		$n = 10$	$n = 20$	$n = 30$	$n = 40$
$L(U)$	CP[AL]	CP[AL]	CP[AL]	CP[AL]	$L(U)$	CP[AL]	CP[AL]	CP[AL]	CP[AL]
0.2(0.4)	0.956[0.138]	0.950[0.084]	0.949[0.064]	0.948[0.054]	1.2(1.4)	0.956[0.056]	0.948[0.036]	0.956[0.028]	0.945[0.023]
0.3(0.5)	0.944[0.126]	0.945[0.077]	0.944[0.059]	0.942[0.050]	1.3(1.5)	0.957[0.054]	0.950[0.035]	0.956[0.027]	0.946[0.022]
0.4(0.7)	0.930[0.171]	0.937[0.109]	0.942[0.086]	0.939[0.073]	1.4(1.7)	0.960[0.078]	0.954[0.049]	0.957[0.038]	0.945[0.031]
0.4(1.5)	0.944[0.422]	0.947[0.293]	0.953[0.238]	0.950[0.205]	1.4(2.5)	0.957[0.241]	0.954[0.149]	0.959[0.115]	0.943[0.095]
0.5(2.5)	0.940[0.379]	0.936[0.243]	0.938[0.186]	0.940[0.157]	1.5(2.5)	0.955[0.217]	0.953[0.134]	0.958[0.104]	0.944[0.085]
0.6(.75)	0.928[0.085]	0.940[0.056]	0.947[0.045]	0.943[0.039]	1.6(1.7)	0.961[0.025]	0.954[0.016]	0.957[0.012]	0.944[0.010]
0.7(2.0)	0.939[0.387]	0.938[0.251]	0.941[0.195]	0.945[0.165]	1.7(3.0)	0.942[0.251]	0.948[0.158]	0.947[0.123]	0.941[0.103]
1.0(3.0)	0.952[0.378]	0.955[0.264]	0.950[0.211]	0.952[0.181]	2.0(4.0)	0.928[0.332]	0.942[0.223]	0.944[0.179]	0.938[0.153]
2.0(3.0)	0.956[0.214]	0.952[0.169]	0.954[0.144]	0.948[0.128]	3.0(4.0)	0.944[0.182]	0.947[0.123]	0.953[0.100]	0.947[0.085]
2.0(4.0)	0.953[0.298]	0.952[0.225]	0.954[0.186]	0.948[0.164]	3.0(5.0)	0.952[0.313]	0.948[0.213]	0.955[0.172]	0.952[0.146]
2.0(5.0)	0.952[0.324]	0.952[0.236]	0.954[0.192]	0.948[0.168]	3.0(6.0)	0.951[0.375]	0.948[0.253]	0.953[0.202]	0.951[0.169]

Table 5. Front widths of a sample of 40 micro USB receptacle interface.

6.905	6.907	6.896	6.895	6.897	6.898	6.903	6.905	6.896	6.903
6.889	6.892	6.898	6.890	6.895	6.901	6.899	6.891	6.893	6.904
6.898	6.896	6.900	6.906	6.902	6.907	6.908	6.907	6.903	6.893
6.907	6.903	6.900	6.893	6.887	6.897	6.900	6.893	6.902	6.899

with the target width of the receptacle interface 6.9 mm, and the tolerance 0.02. That is, the upper and lower specification limits are USL $= 6.92$ and LSL $= 6.88$.

A sample of 107 micro USB receptacles is randomly drawn from the entire lot and the measurements are shown in Table 5 of Liu *et al.* [21]. The normal probability plot of the data by these authors clearly indicates that a normal model fits the data well. For simplicity and illustration purpose, a sample of 40 measurements are taken randomly from 107 measurements and are presented in Table 5.

To apply the test proposed in Krishnamoorthy and Mathew [18], we computed the mean as $\bar{x} = 6.8989$ and the standard deviation as $s = 0.005602$. To check if 95% of micro USB receptacles meet the specifications 6.90 ± .02, the hypotheses of interest are

$$H_0 : 6.88 \geq \mu - z_{.975}\sigma \quad \text{or} \quad \mu + z_{.975}\sigma \geq 6.92 \quad \text{vs.} \quad H_a : (\mu \pm z_{.975}\sigma) \subset (6.88, 6.92), \tag{27}$$

where z_p is the 100p percentile of the standard normal distribution. The null hypothesis will be rejected at the level 0.01, if the specification interval (6.88, 6.92) includes a 99% CI for $(\mu - z_{.975}\sigma,\ \mu + z_{.975}\sigma)$. Owen [23] has proposed a CI of the form $\bar{x} \pm ks$ for $(\mu - z_{.975}\sigma,\ \mu + z_{.975}\sigma)$. For $n = 40$, $p = 0.95$ (because $(1+p)/2 = 0.975$) and confidence level 0.99, the factor k from Table B4 of Krishnamoorthy and Mathew [18] is 2.517. Using the calculated mean and the factor, we found the 99% CI as (6.885, 6.913). Note that the specification interval (6.88, 6.92) includes the CI (6.885, 6.913), and so we conclude that at least 95% receptacles meet the specifications with confidence 99%.

In this type of applications, an important question could be the percentage of receptacles meet the specifications. However, the above testing method checks not only the minimum percentage of receptacles that are within specifications, but also tests if the percentages in both tails are equal, which is somewhat redundant. In our method, we need to simply find a 99% lower confidence limit for

$$P_{LU} = \Phi\left(\frac{6.92 - \mu}{\sigma}\right) - \Phi\left(\frac{6.88 - \mu}{\sigma}\right),$$

where μ and σ denote the mean and SD of the distribution of the widths of micro-USB receptacles. To find a 99% lower confidence limit for the above probability, we used the following R code:

```
#######################################################
N = 1000000; n = 40; df = n-1
xb = 6.8989; s = .005602
W = sqrt(rchisq(N,df)/df); Zn = rnorm(N)/sqrt(n)
PivN = pnorm(W*(6.92-xb)/s-Zn)-pnorm(W*(6.88-xb)/s-Zn)
print(quantile(PivN, .01),4)
    1%
0.9901
PivN = 1-pnorm(W*(6.88-xb)/s-Zn)
print(quantile(PivN, .05),4)
     5%
0.9964
#######################################################
```

From the above R code, we see that at least 99% of the receptacles meet the specification limits with confidence 99%.

For the sake of illustration, let us estimate a 95% lower confidence limit for $P(X > L) = P(X > 6.88)$. The traditional approach is to find the value of p for which the $(p, 0.95)$ lower tolerance limit $\bar{x} - \frac{1}{\sqrt{n}} t_{n-1;.95}(z_p\sqrt{n})s = 6.88$. That is, we need to determine the value of

p so that

$$t_{39;.95}(z_p\sqrt{40}) = \sqrt{40}\frac{(6.8989 - 6.88)}{0.005602}.$$

Using a numerical search method, we find the value of p as 0.9964. If we use the fiducial approach, then we need to find the 5th percentile of $1 - \Phi(W(6.88 - \bar{x})/s - Z/\sqrt{40})$. Using the simulation, as shown in the second part of the above R code, we find the 5th percentile as 0.9964.

Example 4.2: The data in the following Table 6 represent lifetime of air conditioning equipments. The lifetimes are operating hours for plane number 7909 with 13 Boeing 720 aircrafts, and they are as given in Table 6. The data were analyzed by Keating *et al.* [13] using a gamma model. In particular, these authors have noted that the data fit a gamma model, and they have used the data for illustrating a gamma model-based inference. The Weibull probability model (Minitab 14) in Figure 3 clearly indicates that the data fit a Weibull model quite well.

Table 6. Lifetime data on air conditioning equipments.

90	10	60	186	61	49	14	24	56	20	79	84
44	59	29	118	25	156	310	76	26	44	23	62
130	208	70	101	208							

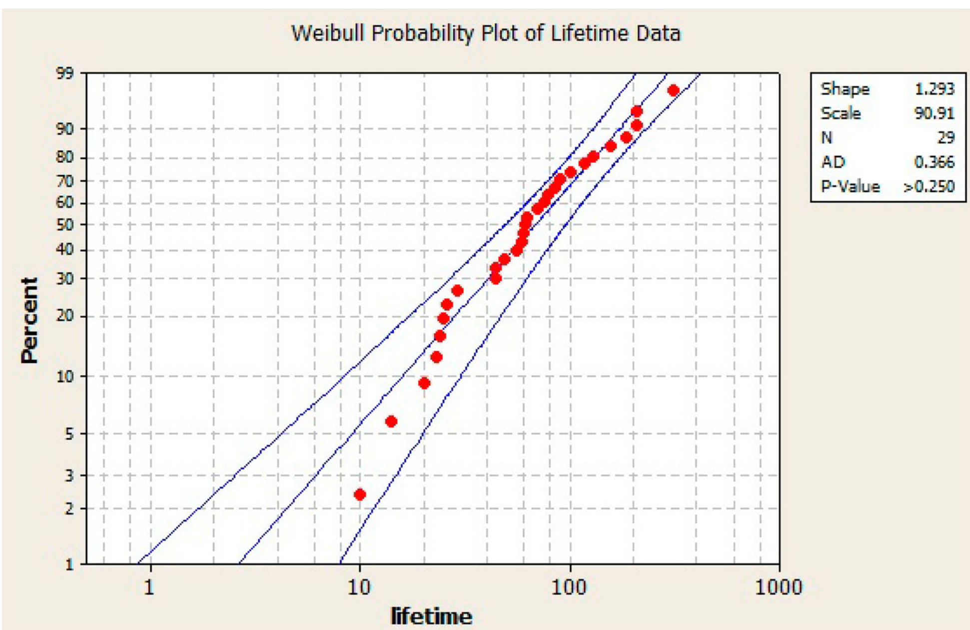

Figure 3. Weibull probability plot of lifetime data in Table 6.

The MLEs are $\widehat{c} = 1.293$ and $\widehat{b} = 90.91$. To find a 90% CI for $P(50 \le X \le 200)$, we estimated the lower 5th and the upper 5th percentiles of

$$Q_{P_{LU}} = \exp\left(-\widehat{b}^* \left(\frac{50}{90.91}\right)^{1.293/\widehat{c}^*}\right) - \exp\left(-\widehat{b}^* \left(\frac{200}{90.91}\right)^{1.293/\widehat{c}^*}\right) \quad (28)$$

as 0.43 and 0.66, respectively. That is, 43–66% of the air conditioning equipments survive between 50 and 200 operating hours with confidence 90%. We also computed 90% HPD credible interval using R function

hdi(density(Q, n = 10^5), credMass = .90, tole = 1e-10, allowSplit = TRUE)[1:2]

as (0.43, 0.66) , same as the one formed by the lower and upper 5th percentiles of $Q_{P_{LU}}$. In the above R function, Q is the vector of 10^5 replicates of the fiducial quantity $Q_{P_{LU}}$ in (28). Notice that 0.43 is the 95% lower confidence limit for the $P(50 \le X \le 200)$. This means that at least 43% of air conditioning equipments work for 50–200 operating hours with confidence 95%. These confidence limits were estimated using Monte Carlo simulation with 100,000 runs. See the R functions in the supplemental file.

Example 4.3: A manufacturing factory needs drills of different sizes in the production process. The factory purchases the 1.88-mm drills from a supplier. Lifetime data for the drills are collected during the production process and reported in Table 1 of Chen *et al.* [2]. The data are reproduced here in Table 7. Krishnamoorthy *et al.* [19] have shown that a Rayleigh distribution fits the data quite well.

The MLEs are $\widehat{\mu}_0 = 72.84$ and $\widehat{\sigma}_0 = 14.79$. Substituting these numbers in Equation (26), we find

$$Q_{P_{LU}} = e^{-\frac{1}{2}\left[\widehat{\sigma}^*\left(\frac{80-72.84}{14.79}\right)+\widehat{\mu}^*\right]^2} - e^{-\frac{1}{2}\left[\widehat{\sigma}^*\left(\frac{100-72.84}{14.79}\right)+\widehat{\mu}^*\right]^2}, \quad (29)$$

where $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are the MLEs based on a sample of size $n = 45$ generated from the Rayleigh(0, 1) distribution. Using Monte Carlo simulation with 100,000 runs, we estimated the lower and the upper 2.5th percentiles of $Q_{P_{LU}}$ as 0.58 and 0.76, respectively. Thus, The 95% CI for $P(80 \le X \le 100)$ is (0.58, 0.76). The HPD credible interval using the R function

hdi(density(Q, n = 10^5), credMass = 0.95, tole = 1e-10, allowSplit = TRUE)[1:2]

is computed as (0.59 0.77). In the above R code, Q is the vector of 10^5 replicates of $Q_{P_{LU}}$ defined in (29).

Table 7. Lifetime (in minutes) of a sample of 1.88 mm drills.

105	105	95	87	112	80	95	97	77	103	78	87	107	96	79
91	108	97	80	76	92	85	76	96	77	80	100	94	82	104
91	95	93	99	99	94	84	99	91	85	86	79	89	89	100

Suppose it is desired to find a 95% lower confidence limit for $P(X > 80)$. In this case, the FQ can be obtained from Equation (25) as

$$1 - Q_{P_t} = e^{-\frac{1}{2}\left[\widehat{\sigma}^*\left(\frac{80-72.84}{14.79}\right)+\widehat{\mu}^*\right]^2}.$$

Using simulation with 100,000 runs, we estimated the lower 5th percentile of $1 - Q_{P_t}$ as 0.812. That is, at least 81.2% of the drills last 80 or more minutes with confidence 0.95.

5. Concluding remarks

Tolerance intervals and simultaneous tests on lower and upper percentiles are used to assess the probability content in a specified interval. However, these methods are not specifically intended for estimating the probability content, and they do not always produce meaningful results. One-sided tolerance limits are used to find one-sided confidence limits for the tail-probabilities such as the survival probability. In this article, we proposed a simple exact solution to find not only one-sided confidence limits, but also two-sided CIs for tail probabilities such as the survival probability. We also provided approximate solutions to find CIs for the probability content in a specified interval. The proposed fiducial CIs are approximate for the probability content in a specified interval, and our simulation studies indicated that the CIs are satisfactory for practical purpose for moderate to large sample sizes. The proposed methods can be readily extended to the cases where the samples are type II censored (failure censored). More details of the results on the censored case can be found in Hoang-Nguyen-Thuy [12]. To help practitioners, we provided R functions in a supplemental file to compute CIs for tail probabilities and for the probability content in a specified interval.

Notes

1. https://www.wikizero.com/en/Engineering_tolerance.
2. https://www.itl.nist.gov/div898/handbook/prc/section2/prc263.htm.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix

The following R code for computing a 95% CI for $P(X < t)$, where $X \sim N(\mu, \sigma^2)$.

```
# N is the number of runs
Z = rnorm(N); W = sqrt(rchisq(N, n-1)/(n-1))
Q = W*((t- xbar)/s)+Z/sqrt(n)
perc = quantile(Q, c(.025, .975))
CI = pnorm(perc)
```

```
####################################################################
```

R code to find HPD region for $P(L \leq X \leq U)$, where $X \sim N(\mu, \sigma^2)$.

```
# package "HDInterval" is needed
# cl = confidence level; n = sample size; N = number of simulation
       runs
Z = rnorm(N); W = sqrt(rchisq(N, n-1)/(n-1))
Q = pnorm(W*((U- xbar)/s)+Z/sqrt(n)) - pnorm(W*((L- xbar)/s)
    +Z/sqrt(n))
hdi(density(Q, n = N), credMass = cl, tole=1e-10, allowSplit
    =TRUE)[1:2]
##################################################################
```

The following R code is to compute the MLEs for Weibull parameters.

```
# package "survival" is needed; x = vector of sample data
model = survreg(Surv(x, rep(1, length(x)))~1, dist="weibull")
c.hat = 1/unname(model$scale); b.hat = exp(unname(model$coef))
##################################################################
```