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ORIGINAL PAPER

# A method for computing tolerance intervals for a location-scale family of distributions

Ngan Hoang-Nguyen-Thuy[1] · K. Krishnamoorthy[1]

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**Abstract**

The problems of computing two-sided tolerance intervals (TIs) and equal-tailed TIs for a location-scale family of distributions are considered. The TIs are constructed using one-sided tolerance limits with the Bonferroni adjustments and then adjusting the confidence levels so that the coverage probabilities of the TIs are equal to the specified nominal confidence level. The methods are simple, exact and can be used to find TIs for all location-scale families of distributions including log-location-scale families. The computational methods are illustrated for the normal, Weibull, two-parameter Rayleigh and two-parameter exponential distributions. The computational method is applicable to find TIs based on a type II censored sample. Factors for computing two-sided TIs and equal-tailed TIs are tabulated and R functions to find tolerance factors are provided in a supplementary file. The methods are illustrated using a few practical examples.

**Keywords** Asymmetric location-scale · Bisection method · Bonferroni · Content · Coverage level · Equivariant estimators · type II censored

## 1 Introduction

In many practical applications, such as medical, environmental and engineering, it is desired to find an interval estimate based on a sample that would capture at least a proportion $p$ of the sampled population with confidence $\gamma$. Such a statistical interval is referred to as the tolerance interval (TI). A tolerance interval based on a random sample is constructed so that it would include at least a proportion $p$ of the sampled population with confidence level $\gamma$. This type of interval estimate is referred to as a $p$ content—$\gamma$

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✉ K. Krishnamoorthy
krishna@louisiana.edu

[1] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, USA

coverage tolerance interval (TI) or simply $(p, \gamma)$ TI. Another type of TI $(L_e, U_e)$ is constructed so that at most a proportion $(1-p)/2$ of the population is less than the lower endpoint $L_e$ and at most a proportion $(1-p)/2$ of the population is greater than the upper endpoint $U_e$ (Owen 1964). This type of TI controls the percentages in both tails and is referred to as the $(p, \gamma)$ "equal-tailed" TI. A $(p, \gamma)$ one-sided lower tolerance limit (TL) is constructed so that at least a proportion $p$ of the population falls above the limit with confidence $\gamma$ while a $(p, \gamma)$ one-sided upper TL is constructed so that at least a proportion $p$ of the population falls below the limit with confidence $\gamma$. For earlier work and numerous applications of TIs, see the book by Guttman (1970), and the book by Krishnamoorthy and Mathew (2009). Tolerance intervals for many commonly used distributions can be computed using the R package "tolerance" by Young (2010).

To define TIs formally, let $Q_p$, $.5 < p < 1$, denote the 100$p$th percentile of a population. The $(p, \gamma)$ one-sided upper TL is a $\gamma$ level upper confidence limit for $Q_p$, and the $(p, \gamma)$ one-sided lower TL is a $\gamma$ level lower confidence limit for $Q_{1-p}$. The $(p, \gamma)$ equal-tailed TI $(L_e(\boldsymbol{X}), U_e(\mathbf{X}))$ based on a sample $\boldsymbol{X}$ is constructed so that

$$P_{\boldsymbol{X}}\left(L_e(\boldsymbol{X}) \le Q_{\frac{1-p}{2}} \text{ and } Q_{\frac{1+p}{2}} \le U_e(\boldsymbol{X})\right) = \gamma. \tag{1}$$

That is, the random interval $[L_e(\boldsymbol{X}), U_e(\boldsymbol{X})]$ includes the interval $\left(Q_{\frac{1-p}{2}}, Q_{\frac{1+p}{2}}\right)$ with probability $\gamma$. Since the percentiles are a function of parameters, calculation of one-sided TLs or equal-tailed TIs simplifies to construction of confidence limits for some parametric functions. However, the problem of computing a two-sided TI $(L_t(\boldsymbol{X}), U_t(\boldsymbol{X}))$ can't be simplified to the problem of estimating some population quantiles as it is defined as

$$P_{\boldsymbol{X}}\left\{P_X\left(L(\boldsymbol{X}) \le X \le U(\boldsymbol{X}) | \boldsymbol{X}\right) \ge p\right\} = P_{\boldsymbol{X}}\left\{F_X\left(U(\boldsymbol{X})\right) - F_X\left(L(\boldsymbol{X})\right) \ge p\right\} = \gamma, \tag{2}$$

where $F_X(x)$ denotes the cumulative distribution function (cdf) associated with the sampled population. Following the approach of Wald and Wolfowitz (1946) for the normal case, Krishnamoorthy and Xie (2011) have provided a general approach for constructing TIs for a symmetric location-scale family of distributions based on a censored or uncensored sample. However, their approach is not applicable for asymmetric location-scale families such as the family of two-parameter exponential distributions and log-location-scale families such as the family of Weibull distributions. Yuan et al. (2016) have proposed an exact numerical/simulation approach which essentially involves calculation of coverage probabilities of TIs over a grid of (lower, upper) factors and then choose a pair for which the coverage probability is close to the specified nominal confidence level and satisfy another constraint.

Construction of a tolerance interval for a location-scale family of distributions involves computing a factor that depends on the sample size, content level $p$ and the confidence level $\gamma$. Since the one-sided tolerance limits (TLs) are one-sided confidence limits for appropriate population percentiles, pivot-based methods are commonly used to find the necessary factors. In general, it is easier to find factors for one-sided TLs than those for two-sided TIs. An equal-tailed TI or a two-sided TI can be easily deduced from the one-sided tolerance limits by adjusting the content level and using the Bonferroni adjustment to the confidence level. But such two-sided TIs could be overly conservative yielding intervals that are unnecessarily wide. However, by adjusting the confidence level so as to satisfy the probability requirement in (1) or in (2), an equal-tailed TI or a two-sided TI can be obtained. As the coverage probability of a TI based on equivariant estimators for a location-scale distribution does not depend on any parameter, the TIs formed by the one-sided tolerance limits with the adjusted confidence level are exact. They are exact in the sense that the coverage probabilities are equal to the nominal confidence level for all parameter values. As shown in the sequel, such modified TLs are easier to obtain numerically or by simulation. Furthermore, this approach can be easily extended to the type II censored case.

The rest of the article is organized as follows. In the following section, we outline the methods for computing factors to construct one-sided tolerance limits and calculation of coverage probabilities of a two-sided TI. In Sect. 3, we describe an existing method and our new method to find two-sided and equal-tailed TIs for a location-scale family of distributions. In Sect. 4, we illustrate the methods for the normal, Weibull, two-parameter Rayleigh and two-parameter exponential distributions. Tolerance factors, based on the adjusted confidence levels, for computing two-sided TIs and equal-tailed TIs are reported, and R functions to compute factors are provided in a supplemental file. As noted earlier, the tolerance intervals based on the adjusted one-sided tolerance factors are exact. In Sect. 5, we briefly outline our approach to handle type II censored samples. The methods are illustrated using some practical examples in Sect. 6, and some concluding remarks are given in Sect. 7.

## 2 One-sided TIs and calculation of coverage probabilities

As our method of constructing a two-sided TI or an equal-tailed TI is based on one-sided tolerance limits, we shall now describe the construction of one-sided tolerance limits based on equivariant estimators for a location-scale family of distributions.

### 2.1 One-sided tolerance intervals

Consider a location-scale family with the probability density function (pdf) of the form

$$f(x|\mu,\sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \ \sigma > 0, \tag{3}$$

where $\mu$ is the location parameter and $\sigma$ is the scale parameter. Let $\widehat{\mu}$ and $\widehat{\sigma}$ be equivariant estimators of $\mu$ and $\sigma$, respectively, based on a sample of size $n$. Then $(\widehat{\mu}-\mu)/\sigma$, $\widehat{\sigma}/\sigma$ and $(\widehat{\mu}-\mu)/\widehat{\sigma}$ are all pivotal quantities. In other words, the distributions of these quantities do not depend on any parameter. Furthermore, for a location-scale distribution, the maximum likelihood estimates are equivariant estimators. See Theorem E2 of Lawless (2003).

For $.5 < p < 1$, let $Q_p(\mu, \sigma)$ denote the $100p$ percentile of the distribution $f(x|\mu, \sigma)$. Let $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ be equivariant estimators based on a sample of size $n$ from the distribution $f(x|0, 1)$. The percentile $Q_p(\mu, \sigma)$ of a location-scale distribution is location-scale equivariant, and so

$$\frac{Q_p(\mu, \sigma) - \widehat{\mu}}{\widehat{\sigma}} \sim \frac{Q_p(0, 1) - \widehat{\mu}^*}{\widehat{\sigma}^*} = q_p^*, \text{ say},$$

where "$X \sim Y$" means that $X$ and $Y$ are identically distributed. So the percentiles of $q_p^*$ can be used to set confidence bound on $Q_p(\mu, \sigma)$. Specifically, if $q_{p,\gamma}^*$ is the $100\gamma$ percentile of $q_p^*$, then

$$\widehat{\mu} + q_{p;\gamma}^* \widehat{\sigma}, \tag{4}$$

is a $100\gamma\%$ upper confidence limit for $Q_p(\mu, \sigma)$ or $(p, \gamma)$ one-sided upper tolerance limit. The percentiles of $q_p^*$ can be calculated numerically or estimated by Monte Carlo simulation. In similar notations, the one-sided lower TL can be expressed as

$$\widehat{\mu} + q_{1-p;1-\gamma}^* \widehat{\sigma}, \tag{5}$$

where $q_{1-p;1-\gamma}^*$ is the $100(1-\gamma)$ percentile of $q_{1-p}^* = [Q_{1-p}(0, 1) - \widehat{\mu}^*]/\widehat{\sigma}^*$.

For a symmetric location-scale family, the percentile $q_{1-p;1-\gamma}^* = -q_{p,\gamma}^*$, and so it is reasonable to seek a two-sided TI of the form $\widehat{\mu} \pm k\widehat{\sigma}$, where $k$ is the factor that is determined on the basis of the sample size, content level $p$ and the coverage level $\gamma$. For asymmetric location-scale family, $q_{1-p;1-\gamma}^* \neq -q_{p;\gamma}^*$, and in such cases one may consider a $(p, \gamma)$ two-sided TI of the form $(\widehat{\mu} + k_l\widehat{\sigma},\ \widehat{\mu} + k_u\widehat{\sigma})$, where $k_l < k_u$.

## 2.2 Calculation of coverage probabilities of tolerance intervals

*Equal-Tailed Tolerance Interval*

Consider an equal-tailed TI of the form $(\widehat{\mu} + E_l\widehat{\sigma},\ \widehat{\mu} + E_u\widehat{\sigma})$, where $\widehat{\mu}$ and $\widehat{\sigma}$ are equivariant estimators and the factors $E_l$ and $E_u$ are determined so that $E_l < E_u$. The coverage probability of this equal-tailed TI is given by

$$H_{\widehat{\mu},\widehat{\sigma}}(\gamma|\mu, \sigma) = P_{\widehat{\mu},\widehat{\sigma}}\left(\widehat{\mu} + E_l\widehat{\sigma} \leq Q_{\frac{1-p}{2}}(\mu, \sigma) \text{ and } Q_{\frac{1+p}{2}}(\mu, \sigma) \leq \widehat{\mu} + E_u\widehat{\sigma}\right). \tag{6}$$

Since $\widehat{\mu}$ and $\widehat{\sigma}$ are equivariant estimators, the probability $H_{\widehat{\mu},\widehat{\sigma}}(\gamma|\mu, \sigma)$ does not depend on the parameter values, as a result, $H_{\widehat{\mu},\widehat{\sigma}}(\gamma|\mu, \sigma) = H_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0, 1)$ for all $\mu$ and $\sigma > 0$, where $(\widehat{\mu}^*, \widehat{\sigma}^*)$ are the equivariant estimators based on a sample of size $n$ from the distribution with the pdf $f(x|0, 1)$. Thus, the coverage probability of the TI is given by

$$H_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0, 1) = P\left(\widehat{\mu}^* + E_l\widehat{\sigma}^* \leq Q_{\frac{1-p}{2}}(0, 1) \text{ and } Q_{\frac{1+p}{2}}(0, 1) \leq \widehat{\mu}^* + E_u\widehat{\sigma}^*\right). \tag{7}$$

For an exact TI, the above probability should be equal to $\gamma$.

In most cases, the coverage probability $H_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1)$ can be estimated only by Monte Carlo simulation, which can be carried out as follows. Generate $N$ samples, each of size $n$, from $f(x|0,1)$. Let $\widehat{\mu}_i^*$ and $\widehat{\sigma}_i^*$ denote the equivariant estimators based on the $i$th sample, $i=1,\ldots,N$. Then

$$\widehat{H}_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1) = \frac{1}{N}\sum_{i=1}^{N} I\left[\widehat{\mu}_i^* + E_l\widehat{\sigma}_i^* \leq Q_{\frac{1-p}{2}}(0,1) \text{ and } Q_{\frac{1+p}{2}}(0,1) \leq \widehat{\mu}_i^* + E_u\widehat{\sigma}_i^*\right], \tag{8}$$

where $I[x]$ is the indicator function, is an estimate of $H_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1)$.

*Two-Sided Tolerance Interval*

Consider a two-sided TI of the form $(\widehat{\mu}+T_l\widehat{\sigma},\ \widehat{\mu}+T_u\widehat{\sigma})$, where $(T_l, T_u)$ are the (lower, upper) factors. Using the equivariance arguments as in the preceding section and the definition (2) of the two-sided TI, the coverage probability can be calculated using the expression

$$G_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1) = P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left\{F_X\left(\widehat{\mu}^* + T_u\widehat{\sigma}^*|0,1\right) - F_X\left(\widehat{\mu}^* + T_l\widehat{\sigma}^*|0,1\right) \geq p\right\}, \tag{9}$$

where $F_X(x|0,1)$ denotes the cdf of $f(x|0,1)$ and $(\widehat{\mu}^*, \widehat{\sigma}^*)$ are the equivariant estimators based on a sample of size $n$ from the distribution $f(x|0,1)$. A Monte Carlo estimate of $G_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1)$ is given by

$$\widehat{G}_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1) = \frac{1}{N}\sum_{i=1}^{N} I\left[F_X(\widehat{\mu}_i^* + T_u\widehat{\sigma}_i^*|0,1) - F_X(\widehat{\mu}_i^* + T_l\widehat{\sigma}_i^*|0,1) \geq p\right], \tag{10}$$

where $N$, $\widehat{\mu}_i^*$ and $\widehat{\sigma}_i^*$ are as defined in (8).

## 3 Methods for computing tolerance intervals

### 3.1 The method by Yuan et al. (2016)

We shall first describe the exact numerical/simulation method by Yuan et al. (2016) for constructing a two-sided TI of the form $(\widehat{\mu}+T_l\widehat{\sigma},\ \widehat{\mu}+T_u\widehat{\sigma})$. An estimate of the coverage probability of such tolerance interval is given by

$$CP(T_l, T_u) = \frac{1}{N}\sum_{i=1}^{N} I\left[F_X(\widehat{\mu}_i^* + T_u\widehat{\sigma}_i^*|0,1) - F_X(\widehat{\mu}_i^* + T_l\widehat{\sigma}_i^*|0,1) \geq p\right], \tag{11}$$

where $N$ is the number of simulation runs. Furthermore, define

$$CP_U(T_u) = \frac{1}{N}\sum_{i=1}^{N} I\left[F_X(\widehat{\mu}^* + T_u\widehat{\sigma}^*|0,1) \geq p/2\right] \quad \text{and}$$
$$CP_L(T_l) = \frac{1}{N}\sum_{i=1}^{N} I\left[F_X(\widehat{\mu}^* + T_l\widehat{\sigma}^*|0,1) \leq 1 - p/2\right].$$

To find the factors, first one needs to compute the coverage probabilities over a grid of $T_l$ and $T_u$ values, and then collect factors $(T_l, T_u)$ for which $CP(T_l, T_u) = \gamma$. The intersection point of the curves $CP(T_l, T_u) = \gamma$ and $CP_u(T_u) = CP_l(T_l)$ gives the required factors. The intersection point of these two curves can be obtained using linear interpolation.

It should be noted that one needs to specify ranges of $T_l$ and $T_u$ to use the above searching method. In general, specifying ranges is not an easy task, and one has to determine the ranges case by case depending on the sample size, content level $p$ and the confidence level $\gamma$. Instead of searching a pair of factors over a rectangular region, we take the factors of the form

$$T_l(p,\gamma') = q^*_{\frac{1-p}{2},\frac{1-\gamma'}{2}} \quad \text{and} \quad T_u(p,\gamma') = q^*_{\frac{1+p}{2},\frac{1+\gamma'}{2}}$$

where $q^*_{p,\gamma}$ is as defined in (4), and search for the value of $\gamma'$ for which the coverage probability of the TI is specified nominal level $\gamma$. In other words, we calculate the factors for TIs as confidence-adjusted one-sided factors as shown in the sequel. Furthermore, the search for $\gamma'$ is restricted to the interval $[\gamma - \delta, \gamma]$ for some small $\delta > 0$.

### 3.2 Tolerance intervals based on confidence-adjusted one-sided factors

We shall now find factors for constructing two-sided or equal-tailed TIs on the basis of confidence-adjusted one-sided tolerance factors.

*Factors for Equal-tailed Tolerance Intervals*

It follows from the definition of one-sided tolerance limits that

$$P\left(\widehat{\mu} + q^*_{\frac{1-p}{2};1-\gamma}\widehat{\sigma} \leq Q_{\frac{1-p}{2}}(\mu,\sigma)\right) = \gamma \quad \text{and}$$
$$P\left(Q_{\frac{1+p}{2}}(\mu,\sigma) \leq \widehat{\mu} + q^*_{\frac{1+p}{2};\gamma}\widehat{\sigma}\right) = \gamma.$$

Applying the Bonferroni inequality and using (7), we see that the coverage probability of the interval

$$\left(\widehat{\mu} + q^*_{\frac{1-p}{2};\frac{1-\gamma}{2}}\widehat{\sigma},\ \widehat{\mu} + q^*_{\frac{1+p}{2};\frac{1+\gamma}{2}}\widehat{\sigma}\right) \tag{12}$$

is

$$H_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1) = P\left(\widehat{\mu}^* + q^*_{\frac{1-p}{2};\frac{1-\gamma}{2}}\widehat{\sigma}^* \le Q_{\frac{1-p}{2}}(0,1) \text{ and } Q_{\frac{1+p}{2}}(0,1) \le \widehat{\mu}^* + q^*_{\frac{1+p}{2};\frac{1+\gamma}{2}}\widehat{\sigma}^*\right) \ge \gamma. \quad (13)$$

Let $\gamma'_e \le \gamma$ be the adjusted confidence level so that $H_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma'_e|0,1) = \gamma$. Then the interval

$$\left(\widehat{\mu} + q^*_{\frac{1-p}{2};\frac{1-\gamma'_e}{2}}\widehat{\sigma},\ \widehat{\mu} + q^*_{\frac{1+p}{2};\frac{1+\gamma'_e}{2}}\widehat{\sigma}\right) \quad (14)$$

is an exact equal-tailed TI.

*Factors for Two-Sided Tolerance Intervals*

Notice that the TI in (12) also satisfies the probability inequality

$$G_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma|0,1) = P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left\{F_X\left(\widehat{\mu}^* + q^*_{\frac{1+p}{2};\frac{1+\gamma}{2}}\widehat{\sigma}^*\middle|0,1\right) - F_X\left(\widehat{\mu}^* + q^*_{\frac{1-p}{2};\frac{1-\gamma}{2}}\widehat{\sigma}^*\middle|0,1\right) \ge p\right\} \ge \gamma, \quad (15)$$

where $F_X(x|0,1)$ denotes the cdf of $f(x|0,1)$ and $(\widehat{\mu}^*, \widehat{\sigma}^*)$ are the equivariant estimators based on a sample of size $n$ from the distribution $f(x|0,1)$. Let $\gamma'_t \le \gamma$ be such that

$$G_{\widehat{\mu}^*,\widehat{\sigma}^*}(\gamma'_t|0,1) = P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left\{F_X\left(\widehat{\mu}^* + q^*_{\frac{1+p}{2};\frac{1+\gamma'_t}{2}}\widehat{\sigma}^*\middle|0,1\right) - F_X\left(\widehat{\mu}^* + q^*_{\frac{1-p}{2};\frac{1-\gamma'_t}{2}}\widehat{\sigma}^*\middle|0,1\right) \ge p\right\} = \gamma. \quad (16)$$

Then

$$\left(\widehat{\mu} + q^*_{\frac{1-p}{2};\frac{1-\gamma'_t}{2}}\widehat{\sigma},\ \widehat{\mu} + q^*_{\frac{1+p}{2};\frac{1+\gamma'_t}{2}}\widehat{\sigma}\right) \quad (17)$$

is an exact two-sided TI.

*Computational Algorithm*

To provide some computational details on finding the confidence level $\gamma'_t$ that satisfies (16), we first note that exact numerical methods for computing one-sided factors are available only for a few distributions. For most of the distributions, the factors for computing one-sided tolerance limits can be obtained only by Monte Carlo simulation. Keeping this fact in mind, we shall provide the following Algorithm 1 to compute the adjusted-confidence level $\gamma'_t$ that satisfies (16).

**Algorithm 1**

For a given $n$, content level $p$ and coverage level $\gamma$,

1. Generate $N$ samples, each of size $n$, from the location-scale distribution with $\mu = 0$ and $\sigma = 1$.
2. Compute the equivariant estimators $\widehat{\mu}_i^*$ and $\widehat{\sigma}_i^*$ based on the $i$th sample, $i = 1, \ldots, N$.
3. Set $U_i = \frac{Q_{\frac{1+p}{2}}(0,1) - \widehat{\mu}_i^*}{\widehat{\sigma}_i^*}$ and $L_i = \frac{Q_{\frac{1-p}{2}}(0,1) - \widehat{\mu}_i^*}{\widehat{\sigma}_i^*}$ $i = 1, \ldots, N$.
4. Set $U_p^* = (U_1, \ldots, U_N)$ and $L_p^* = (L_1, \ldots, L_N)$.
5. Let $q^*_{\frac{1+p}{2};\frac{1+\xi}{2}}$ be the $100(1+\xi)/2$ percentile of $U_p^*$ and let $q^*_{\frac{1-p}{2};\frac{1-\xi}{2}}$ be the $100(1-\xi)/2$ percentile of $L_p^*$.
6. Set the function $f(\xi) = \widehat{G}_{\widehat{\mu}^*,\widehat{\sigma}^*}(\xi|0, 1) - \gamma$, where $\widehat{G}$ (see Eqn (10)) is given by

$$\widehat{G}_{\widehat{\mu}^*,\widehat{\sigma}^*}(\xi|0, 1) = \frac{1}{N}\sum_{i=1}^{N} I\left[F_X(\widehat{\mu}_i^* + q^*_{\frac{1+p}{2};\frac{1+\xi}{2}}\widehat{\sigma}_i^*|0, 1) - F_X(\widehat{\mu}_i^* + q^*_{\frac{1-p}{2};\frac{1-\xi}{2}}\widehat{\sigma}_i^*|0, 1) \geq p\right].$$

7. Solve the equation $f(\xi) = 0$ using a bisection method with $[\gamma - .4, \gamma]$, say, as a root bracketing interval.
8. The root of the equation is the adjusted coverage level $\gamma_t'$ that satisfies (16).
9. The percentiles $q^*_{\frac{1-p}{2};\frac{1-\gamma_t'}{2}}$ and $q^*_{\frac{1+p}{2};\frac{1+\gamma_t'}{2}}$ are the $(p, \gamma)$ two-sided tolerance factors.

An adjusted-confidence level $\gamma_e'$ to compute an equal-tailed TI can be obtained using Algorithm 1 by replacing $\widehat{G}_{\widehat{\mu}^*,\widehat{\sigma}^*}(\xi|0, 1)$ in step 6 with $\widehat{H}_{\widehat{\mu}^*,\widehat{\sigma}^*}(\xi|0, 1)$ defined in (8).

## 4 Computation of tolerance intervals

### 4.1 Normal

It should be noted that calculations of factors for computing one-sided as well as two-sided TIs for a normal distribution are well-known and widely available in the literature; see Wald and Wolfowitz (1946) or Section 2.3 of Krishnamoorthy and Mathew (2009). We here use the normal case just to illustrate our approach and show that the modified one-sided tolerance factors and the available exact factors for computing two-sided TIs are the same.

*Two-Sided Tolerance Intervals*

Let $(\bar{X}, S^2)$ denote the (mean, variance) based on a sample of size $n$ from a $N(\mu, \sigma^2)$ distribution. Note that $\bar{X}$ and $S^2$ are equivariant estimators of $\mu$ and $\sigma^2$, respectively. The $(p, \gamma)$ factor for one-sided tolerance limit is given by $k_{p,\gamma} = t_{n-1;\gamma}(z_p\sqrt{n})/\sqrt{n}$, where $t_{m;\alpha}(\delta)$ denotes the $\alpha$ quantile of the noncentral $t$ distribution with degrees of freedom (df) $m$ and the noncentrality parameter $\delta$ and $z_p$ denotes the $100p$ percentile of the standard normal distribution. For writing convenience, let

$$\kappa_{p,\gamma_t'} = k_{\frac{1+p}{2},\frac{1+\gamma_t'}{2}} = \frac{1}{\sqrt{n}} t_{n-1;\frac{1+\gamma_t'}{2}}\left(z_{\frac{1+p}{2}}\sqrt{n}\right). \quad (18)$$

Let $(\bar{X}^*, S^{*2})$ denote the (mean, variance) based on a sample of size $n$ from a $N(0, 1)$ distribution. The value of $\gamma_t'$ is determined so that

$$\begin{aligned} G(\gamma_t'|0, 1) &= P_{\bar{X}^*, S^*}\left\{\Phi\left(\bar{X}^* + \kappa_{p,\gamma_t'} S^*\right) - \Phi\left(\bar{X}^* - \kappa_{p,\gamma_t'} S^*\right) \geq p\right\} \\ &= P_{Z_n, U}\left\{\Phi\left(Z_n + \kappa_{p,\gamma_t'} U\right) - \Phi\left(Z_n - \kappa_{p,\gamma_t'} U\right) \geq p\right\} \\ &= \gamma, \end{aligned} \tag{19}$$

where $Z_n \sim N(0, 1/n)$ independently of $U^2 \sim \chi^2_{n-1}/(n-1)$ and $\Phi(x)$ is the standard normal cumulative distribution function. That is, $\gamma_t'$ is the root of the equation $G(\gamma_t'|0, 1) - \gamma = 0$, which can be obtained using a bisection method with $[\gamma - .4, \gamma]$ as the root bracketing interval. The probability $G(\gamma_t'|0, 1)$ can be evaluated numerically; see Section 2.3 of Krishnamoorthy and Mathew (2009).

*Equal-Tailed Tolerance Intervals*

To find an equal-tailed tolerance factor, we need to determine coverage level $\gamma_e'$ so that

$$H(\gamma_e'|0, 1) = P_{Z_n, U}\left(Z_n - \kappa_{p,\gamma_e'} U < -z_{\frac{1+p}{2}} \text{ and } z_{\frac{1+p}{2}} < Z_n + \kappa_{p,\gamma_e'} U\right) = \gamma, \tag{20}$$

where $Z_n$ and $U$ are as defined in (19) and $\kappa_{p,\gamma_e'}$ is as defined in (18). Thus, $\gamma_e'$ is the root of the equation $H(\gamma_e'|0, 1) - \gamma = 0$, which can be obtained using a bisection method with $[\gamma - .4, \gamma]$ as the root bracketing interval. Following the approach in Owen (1964), the probability $H(\gamma_e'|0, 1)$ can be evaluated numerically.

The values of $\gamma_t'$ that satisfy (19) are calculated for some values of $(n, p, \gamma)$ and they are reported in Table 1. The values of $\gamma_e'$ that satisfy (20) are also reported in Table 1 for the same values of $(n, p, \gamma)$.

Using the *StatCalc* that accompanies the book by Krishnamoorthy (2016) or the R-package “tolerance” by Young (2010), it can be readily verified that the exact two-sided factors on the basis of Wald and Wolfowitz’s (1946) and equal-tailed factors based on Owen’s (1964) approach and the ones in the above Table 1 are the same for all cases.

As an example, suppose it is desired to construct a (.90, .95) TI based on a sample of size $n = 15$. In this case, the value of $\gamma_t' = .8756$ (see Table 1),

$$\frac{1}{\sqrt{n}} t_{n-1;(1+\gamma_t')/2}\left(z_{\frac{1+p}{2}}\sqrt{n}\right) = \frac{1}{\sqrt{15}} t_{14,.9378}(1.6449 \times \sqrt{15}) = 2.492,$$

and $\bar{X} \pm 2.492S$, where $(\bar{X}, S)$ is the (mean, SD) based on a sample of size 15, is a (.90, .95) two-sided TI. To compute (.90, .95) equal-tailed TI based on the same sample size 15, we found the adjusted confidence level $\gamma_e' = 0.9449$ (see Table 1), and the equal-tailed factor is

$$\frac{1}{\sqrt{n}} t_{n-1;(1+\gamma_e')/2}\left(z_{\frac{1+p}{2}}\sqrt{n}\right) = \frac{1}{\sqrt{15}} t_{14,.9725}(1.6449 \times \sqrt{15}) = 2.765,$$

**Table 1** Adjusted coverage levels $\gamma_t'$ and $\gamma_e'$ and factor $\kappa_{p,\gamma_t'}$ for constructing two-sided TIs and factor $\kappa_{p,\gamma_e'}$ for constructing equal-tailed TIs for normal distributions

| | Factors for two-sided TIs | | | | | | Factors for equal-tailed TIs | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(p, \gamma)$ | (.90,.95) | | (.95,.95) | | (.99,.95) | | (.90,.95) | | (.95,.95) | | (.99,.95) | |
| $n$ | $\gamma_t'$ | $\kappa_{p,\gamma_t'}$ | $\gamma_t'$ | $\kappa_{p,\gamma_t'}$ | $\gamma_t'$ | $\kappa_{p,\gamma_t'}$ | $\gamma_e'$ | $\kappa_{p,\gamma_e'}$ | $\gamma_e'$ | $\kappa_{p,\gamma_e'}$ | $\gamma_e'$ | $\kappa_{p,\gamma_e'}$ |
| 5 | .9068 | 4.290 | .9110 | 5.077 | .9143 | 6.598 | .9391 | 4.848 | .9362 | 5.582 | .9314 | 7.026 |
| 7 | .8987 | 3.390 | .9056 | 4.020 | .9118 | 5.241 | .9416 | 3.815 | .9389 | 4.407 | .9342 | 5.570 |
| 10 | .8883 | 2.856 | .8984 | 3.393 | .9079 | 4.437 | .9435 | 3.197 | .9410 | 3.705 | .9364 | 4.703 |
| 15 | .8756 | 2.492 | .8891 | 2.965 | .9024 | 3.885 | .9449 | 2.765 | .9426 | 3.216 | .9383 | 4.103 |
| 20 | .8665 | 2.319 | .8823 | 2.760 | .8981 | 3.621 | .9456 | 2.554 | .9435 | 2.978 | .9392 | 3.811 |
| 30 | .8542 | 2.145 | .8729 | 2.555 | .8919 | 3.355 | .9464 | 2.338 | .9444 | 2.734 | .9403 | 3.513 |

and $\bar{X} \pm 2.765S$ is a (.90, .95) equal-tailed TI.

### 4.2 Weibull

Let $X_1, \ldots, X_n$ be a sample from a two-parameter Weibull$(b, c)$ distribution with the probability density function

$$f(x|b, c) = \frac{c}{b}\left(\frac{x}{b}\right)^{c-1} \exp\left\{-\left[\frac{x}{b}\right]^c\right\}, \; x > 0, \; b > 0, \; c > 0.$$

Let $Y_i = \ln(X_i)$, $i = 1, \ldots, n$. The maximum likelihood estimate (MLE) of $c$ is the solution of the equation

$$1/\widehat{c} - \left[\sum_{i=1}^{n} X_i^{\widehat{c}} Y_i\right] \Big/ \sum_{i=1}^{n} X_i^{\widehat{c}} + \frac{1}{n}\sum_{i=1}^{n} Y_i = 0, \tag{21}$$

and the MLE of $b$ is given by $\widehat{b} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\widehat{c}}\right)^{1/\widehat{c}}$. See Cohen (1965) or Krishnamoorthy et al. (2009). The estimator $\widehat{c} = \frac{\pi}{\sqrt{6}S_y}$, where $S_y^2 = \sum_{i=1}^{n}(Y_i - \bar{Y})^2/(n-1)$ can be used as an initial value to find the MLE of $c$ using the Newton–Raphson iterative scheme.

*One-sided tolerance limits*

Recall that the $(p, \gamma)$ upper tolerance limit is a $\gamma$ level upper confidence limit for the $p$th quantile $q_p = b(-\ln(1-p))^{\frac{1}{c}}$ of the Weibull$(b, c)$ distribution. Let $\widehat{b}^*$ and $\widehat{c}^*$ denote the maximum likelihood estimates (MLEs) based on a sample of size $n$ from a Weibull$(1, 1)$ distribution. Let

$$w_p = \widehat{c}^* \left(-\ln(\widehat{b}^*) + \ln(-\ln(1-p))\right). \tag{22}$$

Let $w_{p;q}$ denote the $q$th quantile $w_p$. Then a $(p, \gamma)$ one-sided lower and upper tolerance limits are given by

$$\widehat{b}\exp\left(w_{1-p;1-\gamma}/\widehat{c}\right) \quad \text{and} \quad \widehat{b}\exp\left(w_{p;\gamma}/\widehat{c}\right), \tag{23}$$

respectively. For more details on one-sided tolerance limits for a Weibull distribution, see Section 7.5 of Krishnamoorthy and Mathew (2009).

*Two-sided tolerance intervals*

Following the lines of Sect. 3.2, a $(p, \gamma)$ two-sided tolerance interval can be expressed as

$$\begin{aligned}&\left(L_{p,\gamma_t'}(\widehat{b}, \widehat{c}), \; U_{p,\gamma_t'}(\widehat{b}, \widehat{c})\right)\\ &\quad = \left(\widehat{b}\exp\left(w_{\frac{1-p}{2};\frac{1-\gamma_t'}{2}}\Big/\widehat{c}\right), \; \widehat{b}\exp\left(w_{\frac{1+p}{2};\frac{1+\gamma_t'}{2}}\Big/\widehat{c}\right)\right)\end{aligned} \tag{24}$$

where $\gamma_t'$ is to be determined so that

$$P_{\widehat{b}^*,\widehat{c}^*}\left\{F\left(U_{p,\gamma_t'}(\widehat{b}^*,\widehat{c}^*)\Big|1,1\right)-F\left(L_{p,\gamma_t'}(\widehat{b}^*,\widehat{c}^*)\Big|1,1\right)\geq p\right\}=\gamma, \quad (25)$$

where $F(x|1,1)=1-e^{-x}$ is the cdf of the Weibull(1,1) distribution.

*Equal-tailed tolerance interval*

For an equal-tailed TI of the form (24), the adjusted confidence level $\gamma_e'$ is determined so that

$$P_{\widehat{b}^*,\widehat{c}^*}\left(L_{p,\gamma_e'}(\widehat{b}^*,\widehat{c}^*)\leq\left[-\ln\left(\frac{1+p}{2}\right)\right] \text{ and } \left[-\ln\left(\frac{1-p}{2}\right)\right] \leq U_{p,\gamma_e'}(\widehat{b}^*,\widehat{c}^*)\right)=\gamma, \quad (26)$$

where $\widehat{b}^*$ and $\widehat{c}^*$ are the MLEs based on a sample of size $n$ from a Weibull(1, 1) distribution.

*Calculation of Tolerance Factors*

The adjusted confidence levels $\gamma_t'$ and $\gamma_e'$ for computing two-sided and equal-tailed TIs for Weibull distributions were estimated using Algorithm 1 with 100,000 simulation runs for values of $n$ ranging from 5 to 100, and $(p,\gamma)=(.90,.95)$, $(.95,.95)$ and $(.99,.95)$. The estimated values of $\gamma_t'$ and $\gamma_e'$ along with the factors are reported in Table 2. To illustrate the calculation of factors, let us suppose that it is desired to find $(p,\gamma)=(.90,.95)$ tolerance interval for a Weibull distribution based on a sample of size $n=15$. From Table 2, we find $\gamma_t'=.876$. Note that $(1+p)/2=.95$, $(1-p)/2=.05$, $(1+\gamma_t')/2=.938$ and $(1-\gamma_t')/2=.062$. The lower tolerance factor is 6.2 percentile of $w_{.05}=\widehat{c}^*\left(-\ln(\widehat{b}^*)+\ln(-\ln(.95))\right)$ [see Eq. (22)], which is estimated using simulation with 100,000 runs as $-4.72$. The upper tolerance factor is 93.8 percentile of $w_{.95}=\widehat{c}^*\left(-\ln(\widehat{b}^*)+\ln(-\ln(.05))\right)$, and is estimated as 1.82. The (.90, .95) TI is $\left(\widehat{b}\exp(-4.72/\widehat{c}),\ \widehat{b}\exp(1.82/\widehat{c})\right)$. In Table 2, we present lower $(L_t)$ and upper $(U_t)$ factors for constructing $(p,\gamma)$ TIs for a Weibull distribution.

In Table 2, we also present lower $(L_e)$ and upper $(U_e)$ factors for constructing $(p,\gamma)$ equal-tailed TIs for a Weibull distribution. As an example, when $n=10$ and $(p,\gamma)=(.95,.95)$, the lower and upper factors are $-7.53$ and 2.79, respectively. The (.95,.95) equal-tailed TI is expressed as $\left(\widehat{b}\exp(-7.53/\widehat{c}),\ \widehat{b}\exp(2.79/\widehat{c})\right)$.

### 4.3 Rayleigh distribution

The pdf of a two-parameter Rayleigh distribution with location parameter $\mu$ and the scale parameter $\sigma$ is given by

$$f(x|\mu,\sigma)=\frac{(x-\mu)}{\sigma^2}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x>\mu,\ \ \sigma>0.$$

We shall denote the above distribution by Rayleigh$(\mu,\sigma)$. Recently, Krishnamoorthy et al. (2019) have proposed a pivotal-based method for constructing one-sided

**Table 2** Factors $L_t(U_t)$ for constructing two-sided TIs and factors $L_e(U_e)$ for constructing equal-tailed TIs for Weibull distributions

| | Two-sided factors | | | | | | Equal-tailed factors | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.90,.95) | | (.95,.95) | | (.99,.95) | | (.90,.95) | | (.95,.95) | | (.99,.95) | |
| $n$ | $\gamma_t'$ | $L_t(U_t)$ | $\gamma_t'$ | $L_t(U_t)$ | $\gamma_t'$ | $L_t(U_t)$ | $\gamma_e'$ | $L_e(U_e)$ | $\gamma_e'$ | $L_e(U_e)$ | $\gamma_t'$ | $L_e(U_e)$ |
| $(p,\gamma)$ | | | | | | | | | | | | |
| 5 | .907 | − 8.88(3.67) | .911 | − 11.0(4.31) | .914 | − 15.7(5.29) | .939 | − 10.1(4.21) | .936 | − 12.0(4.72) | .931 | − 16.98(5.71) |
| 6 | .903 | − 7.54(3.06) | .908 | − 9.35(3.60) | .913 | − 13.5(4.49) | .941 | − 8.50(3.53) | .938 | − 10.3(3.98) | .933 | − 14.49(4.77) |
| 7 | .899 | − 6.79(2.72) | .906 | − 8.38(3.16) | .912 | − 12.2(3.98) | .942 | − 7.64(3.10) | .939 | − 9.18(3.51) | .934 | − 12.95(4.23) |
| 8 | .895 | − 6.24(2.45) | .903 | − 7.76(2.89) | .910 | − 11.2(3.62) | .942 | − 6.98(2.79) | .940 | − 8.51(3.21) | .935 | − 11.86(3.86) |
| 9 | .892 | − 5.83(2.30) | .901 | − 7.26(2.69) | .909 | − 10.5(3.39) | .943 | − 6.59(2.60) | .940 | − 7.97(2.98) | .936 | − 11.20(3.63) |
| 10 | .888 | − 5.54(2.18) | .898 | − 6.87(2.54) | .908 | − 9.97(3.20) | .944 | − 6.22(2.46) | .941 | − 7.53(2.79) | .936 | − 10.55(3.40) |
| 11 | .885 | − 5.30(2.06) | .896 | − 6.58(2.43) | .907 | − 9.57(3.07) | .944 | − 5.93(2.34) | .941 | − 7.19(2.66) | .937 | − 10.14(3.25) |
| 12 | .883 | − 5.14(1.99) | .894 | − 6.36(2.33) | .906 | − 9.21(2.94) | .944 | − 5.70(2.25) | .942 | − 6.96(2.58) | .937 | − 9.77(3.13) |
| 13 | .880 | − 4.95(1.92) | .893 | − 6.16(2.26) | .904 | − 8.95(2.84) | .945 | − 5.51(2.16) | .942 | − 6.73(2.49) | .938 | − 9.45(3.02) |
| 14 | .878 | − 4.83(1.86) | .891 | − 6.01(2.19) | .903 | − 8.71(2.77) | .945 | − 5.40(2.11) | .942 | − 6.56(2.41) | .938 | − 9.21(2.94) |
| 15 | .876 | − 4.72(1.82) | .889 | − 5.88(2.14) | .902 | − 8.52(2.70) | .945 | − 5.26(2.04) | .943 | − 6.40(2.34) | .938 | − 9.02(2.87) |
| 16 | .874 | − 4.63(1.78) | .888 | − 5.77(2.09) | .902 | − 8.33(2.65) | .945 | − 5.15(1.99) | .943 | − 6.25(2.29) | .939 | − 8.80(2.81) |
| 17 | .872 | − 4.55(1.75) | .886 | − 5.66(2.05) | .901 | − 8.18(2.58) | .945 | − 5.03(1.94) | .943 | − 6.13(2.24) | .939 | − 8.64(2.75) |
| 18 | .870 | − 4.47(1.71) | .885 | − 5.59(2.02) | .900 | − 8.05(2.55) | .945 | − 4.94(1.91) | .943 | − 6.04(2.20) | .939 | − 8.51(2.71) |
| 19 | .868 | − 4.40(1.68) | .884 | − 5.49(1.98) | .899 | − 7.93(2.51) | .946 | − 4.87(1.88) | .943 | − 5.92(2.16) | .939 | − 8.38(2.65) |
| 20 | .867 | − 4.35(1.66) | .882 | − 5.40(1.96) | .898 | − 7.81(2.47) | .946 | − 4.80(1.84) | .944 | − 5.82(2.13) | .939 | − 8.26(2.62) |

**Table 2** continued

| | Two-sided factors | | | | | | Equal-tailed factors | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.90,.95) | | (.95,.95) | | (.99,.95) | | (.90,.95) | | (.95,.95) | | (.99,.95) | |
| $n$ | $\gamma_t'$ | $L_t(U_t)$ | $\gamma_t'$ | $L_t(U_t)$ | $\gamma_t'$ | $L_t(U_t)$ | $\gamma_e'$ | $L_e(U_e)$ | $\gamma_e'$ | $L_e(U_e)$ | $\gamma_t'$ | $L_e(U_e)$ |
| 25 | .860 | − 4.12(1.57) | .877 | − 5.13(1.85) | .895 | − 7.43(2.34) | .946 | − 4.54(1.73) | .944 | − 5.52(2.00) | .940 | − 7.79(2.46) |
| 30 | .854 | − 3.97(1.50) | .873 | − 4.94(1.77) | .892 | − 7.14(2.25) | .946 | − 4.34(1.65) | .944 | − 5.29(1.91) | .940 | − 7.51(2.37) |
| 35 | .850 | − 3.87(1.46) | .869 | − 4.81(1.72) | .890 | − 6.95(2.18) | .947 | − 4.22(1.60) | .945 | − 5.13(1.85) | .941 | − 7.28(2.30) |
| 40 | .846 | − 3.78(1.43) | .867 | − 4.71(1.68) | .888 | − 6.81(2.14) | .947 | − 4.11(1.56) | .945 | − 5.02(1.81) | .941 | − 7.11(2.24) |
| 45 | .843 | − 3.72(1.40) | .864 | − 4.62(1.65) | .886 | − 6.68(2.10) | .947 | − 4.02(1.52) | .945 | − 4.91(1.76) | .941 | − 6.98(2.20) |
| 50 | .841 | − 3.66(1.38) | .862 | − 4.55(1.62) | .884 | − 6.58(2.07) | .947 | − 3.95(1.49) | .945 | − 4.84(1.74) | .941 | − 6.88(2.16) |
| 60 | .836 | − 3.59(1.34) | .858 | − 4.46(1.59) | .882 | − 6.45(2.02) | .947 | − 3.85(1.45) | .945 | − 4.70(1.69) | .942 | − 6.71(2.11) |
| 70 | .833 | − 3.53(1.32) | .856 | − 4.38(1.56) | .880 | − 6.32(1.98) | .947 | − 3.77(1.42) | .946 | − 4.63(1.65) | .942 | − 6.57(2.07) |
| 80 | .830 | − 3.48(1.30) | .853 | − 4.32(1.54) | .878 | − 6.24(1.96) | .948 | − 3.70(1.39) | .946 | − 4.55(1.62) | .942 | − 6.48(2.03) |
| 90 | .828 | − 3.45(1.29) | .852 | − 4.27(1.52) | .877 | − 6.17(1.94) | .948 | − 3.65(1.37) | .946 | − 4.48(1.60) | .942 | − 6.40(2.01) |
| 100 | .826 | − 3.41(1.27) | .850 | − 4.24(1.50) | .876 | − 6.13(1.92) | .948 | − 3.61(1.36) | .946 | − 4.43(1.59) | .942 | − 6.33(1.99) |

tolerance limits. To describe their approach, let $X_1, \ldots, X_n$ be a sample from a two-parameter Rayleigh distribution. Let $\bar{X}$ denote the sample mean and define the sample variance as $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. The moment estimates derived in Dey et al. (2014) are $\widetilde{\sigma}^2 = 2S^2/(4-\pi)$ and $\widetilde{\mu} = \bar{X} - \sqrt{\frac{\pi}{2}}\,\widetilde{\sigma}$. The MLE of $\mu$ is the root of the equation

$$h(\mu|X) = \frac{2n^2(\bar{X}-\mu)}{\sum_{i=1}^n (X_i-\mu)^2} - \sum_{i=1}^n (X_i-\mu)^{-1} = 0. \tag{27}$$

Krishnamoorthy et al. (2019) have shown that the MLE of $\mu$ lies in the interval $\left(X_{(1)} - 12\widetilde{\sigma}/\sqrt{n},\ X_{(1)}\right)$, where $\widetilde{\sigma}$ is the moment estimate, and the MLE of $\mu$ can be obtained as the root of the equation $h(\mu|X) = 0$ using the interval $(X_{(1)} - 12\widetilde{\sigma}/\sqrt{n},\ X_{(1)})$ as a root bracketing interval (e.g., R function uniroot()). Let $\widehat{\mu}$ denote the MLE of $\mu$ thus obtained. Then the MLE of $\widehat{\sigma} = \sqrt{\frac{1}{2n}\sum_{i=1}^n (X_i - \widehat{\mu})^2}$.

*One-sided tolerance limits*

The $p$th quantile of the Rayleigh$(a, b)$ distribution can be expressed as

$$Q_p(\mu, \sigma) = \mu + \sigma\sqrt{-2\ln(1-p)}. \tag{28}$$

Noting that a $(p, \gamma)$ one-sided upper tolerance limit is a $100\gamma\%$ upper confidence limit for $Q_p(\mu, \sigma)$, the $(p, \gamma)$ upper tolerance limit is expressed as $\widehat{\mu} + q^*_{p,\gamma}\widehat{\sigma}$, where $q^*_{p,\gamma}$ is the $100\gamma$ percentile of $q^*_p = (Q_p(0, 1) - \widehat{\mu}^*)/\widehat{\sigma}^*$ and $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are MLEs based on a sample of size $n$ from a Rayleigh(0,1) distribution. Similarly, a $(p, \gamma)$ lower tolerance limit can be expressed as $\widehat{\mu} + q^*_{1-p,1-\gamma}\widehat{\sigma}$, where $q^*_{1-p,1-\gamma}$ is the $100(1-\gamma)$ percentile of $\left(Q_{1-p}(0, 1) - \widehat{\mu}^*\right)/\widehat{\sigma}^*$, and $Q_{1-p}(0, 1) = \sqrt{-2\ln p}$.

*Two-sided tolerance intervals*

A $(p, \gamma)$ two-sided tolerance interval can be expressed as

$$\left(L_{p,\gamma'_t}(\widehat{\mu}, \widehat{\sigma}), U_{p,\gamma'_t}(\widehat{\mu}, \widehat{\sigma})\right) = \left(\widehat{\mu} + q^*_{\frac{1-p}{2};\frac{1-\gamma'_t}{2}}\widehat{\sigma},\ \widehat{\mu} + q^*_{\frac{1+p}{2};\frac{1+\gamma'_t}{2}}\widehat{\sigma}\right),$$

where $\gamma'_t$ is to be determined so that

$$P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left\{F\left(U_{p,\gamma'_t}(\widehat{\mu}^*, \widehat{\sigma}^*)\Big|0, 1\right) - F\left(L_{p,\gamma'_t}(\widehat{\mu}^*, \widehat{\sigma}^*)\Big|0, 1\right) \geq p\right\} = \gamma. \tag{29}$$

In the above, $F(x|0, 1) = 1 - e^{-x^2/2},\ x > 0$, is the cdf of the Rayleigh(0,1) distribution, and $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are the MLEs based on a sample of size $n$ from the Rayleigh(0, 1) distribution.

*Equal-tailed tolerance intervals*

A $(p, \gamma)$ equal-tailed TI can be expressed as $\left(L_{p,\gamma'_e}(\widehat{\mu}, \widehat{\sigma}), U_{p,\gamma'_e}(\widehat{\mu}, \widehat{\sigma})\right)$, where $\gamma'_e$ is to be determined so that

$$P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left(L_{p,\gamma_e'}(\widehat{\mu}^*,\widehat{\sigma}^*) \le \sqrt{-2\ln((1+p)/2)} \text{ and } \sqrt{-2\ln((1-p)/2)} \le U_{p,\gamma_e'}(\widehat{\mu}^*,\widehat{\sigma}^*)\right) = \gamma, \tag{30}$$

where $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ are the MLEs based on a sample of size $n$ from the Rayleigh(0,1) distribution.

*Calculation of Tolerance Factors*

To construct a TI for a Rayleigh distribution, the factors based on the adjusted-confidence levels were estimated using Algorithm 1 with 100,000 simulation runs. The adjusted-confidence levels along with factors are reported in Table 3 for some values of $(p, \gamma)$ and $n$ ranging from 5 to 100. As an example, suppose it is desired to find a (.90, .95) TI based on a sample of size 15. The required factors from Table 3 are $(L_t, U_t) = (-.101, 3.25)$, and the two-sided TI is $(\widehat{\mu} - .101\widehat{\sigma},\ \widehat{\mu} + 3.25\widehat{\sigma})$, where $\widehat{\mu}$ and $\widehat{\sigma}$ are the MLEs based on a sample of size $n$. The factors for constructing (.90, .95) equal-tailed TI are $(L_e, U_e) = (-.241, 3.50)$, and the equal-tailed TI is $(\widehat{\mu} - .241\widehat{\sigma},\ \widehat{\mu} + 3.50\widehat{\sigma})$.

### 4.4 Exponential distribution

Let $X_1, \ldots, X_n$ be a sample from a two-parameter exponential distribution with the pdf

$$f(x|\mu,\sigma) = \frac{1}{\sigma}\exp(-(x-\mu)/\sigma), \quad x > \mu,\ \sigma > 0. \tag{31}$$

The MLEs of $\mu$ and $\sigma$ are given by $\widehat{\mu} = X_{(1)}$ and $\widehat{\sigma} = \bar{X} - X_{(1)}$, where $X_{(1)}$ is the smallest of the $X_i$'s. The MLEs are equivariant and independent with $2n(\widehat{\mu}-\mu)/\sigma \sim \chi^2_2$ and $2n\widehat{\sigma}/\sigma \sim \chi^2_{2n-2}$.

*One-sided tolerance limits*

Note that the $p$ quantile of a two-parameter exponential distribution is given by $q_p = \mu - \sigma\ln(1-p)$. A $\gamma$ level upper confidence limit (one-sided upper tolerance limit) is $\widehat{\mu}+E_{p;\gamma}\widehat{\sigma}$, where $E_{p;\gamma}$ is the $\gamma$ quantile of $E_p = [-2n\ln(1-p)-\chi^2_2]/\chi^2_{2n-2}$. Similarly, a $(p,\gamma)$ one-sided lower tolerance limit can be expressed as $\widehat{\mu}+E_{1-p;1-\gamma}\widehat{\sigma}$. The percentiles of $E_p$ can be estimated in a straightforward manner using Monte Carlo simulation. Krishnamoorthy and Xia (2018) have provided an exact numerical method of computing the percentiles of $E_p$.

*Two-sided tolerance intervals*

A $(p,\gamma)$ two-sided tolerance interval can be expressed as

$$\left(L_{p,\gamma_t'}(\widehat{\mu},\widehat{\sigma}), U_{p,\gamma_t'}(\widehat{\mu},\widehat{\sigma})\right) = \left(\widehat{\mu} + E_{\frac{1-p}{2};\frac{1-\gamma_t'}{2}}\widehat{\sigma},\ \widehat{\mu} + E_{\frac{1+p}{2};\frac{1+\gamma_t'}{2}}\widehat{\sigma}\right), \tag{32}$$

where $\gamma_t'$ is to be determined so that

$$P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left\{F\left(L_{p,\gamma_t'}(\widehat{\mu}^*,\widehat{\sigma}^*)\middle|0,1\right) - F\left(U_{p,\gamma_t'}(\widehat{\mu}^*,\widehat{\sigma}^*)\middle|0,1\right) \ge p\right\} = \gamma. \tag{33}$$

**Table 3** Factors $L_t$ and $U_t$ for constructing two-sided TIs and factors $L_e$ and $U_e$ for constructing equal-tailed TIs for Rayleigh distributions

| $(p, \gamma)$ | Factors for two-sided TI | | | | | | | | | Factors for equal-sided TI | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | |
| $n$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ |
| 5 | .911 | −1.30 | 5.27 | .917 | −1.58 | 6.11 | .922 | −1.98 | 7.79 | .941 | −1.63 | 5.80 | .940 | −1.84 | 6.57 | .936 | −2.14 | 8.15 |
| 6 | .905 | −.910 | 4.60 | .913 | −1.15 | 5.30 | .921 | −1.50 | 6.74 | .943 | −1.19 | 5.11 | .940 | −1.37 | 5.77 | .938 | −1.65 | 7.08 |
| 7 | .901 | −.670 | 4.22 | .911 | −.895 | 4.87 | .921 | −1.20 | 6.15 | .943 | −.903 | 4.64 | .942 | −1.09 | 5.22 | .939 | −1.34 | 6.42 |
| 8 | .897 | −.517 | 3.97 | .908 | −.727 | 4.55 | .920 | −1.00 | 5.72 | .944 | −.735 | 4.33 | .942 | −.893 | 4.87 | .939 | −1.11 | 5.96 |
| 9 | .893 | −.412 | 3.79 | .904 | −.597 | 4.33 | .919 | −.867 | 5.42 | .945 | −.597 | 4.11 | .943 | −.766 | 4.63 | .940 | −.974 | 5.67 |
| 10 | .887 | −.321 | 3.63 | .904 | −.512 | 4.15 | .918 | −.755 | 5.20 | .945 | −.509 | 3.97 | .944 | −.656 | 4.46 | .941 | −.861 | 5.44 |
| 11 | .887 | −.264 | 3.53 | .900 | −.442 | 4.03 | .916 | −.675 | 5.03 | .946 | −.430 | 3.82 | .944 | −.585 | 4.30 | .941 | −.767 | 5.23 |
| 12 | .880 | −.205 | 3.43 | .896 | −.376 | 3.90 | .915 | −.607 | 4.87 | .946 | −.369 | 3.73 | .944 | −.511 | 4.16 | .942 | −.696 | 5.07 |
| 13 | .878 | −.169 | 3.36 | .895 | −.336 | 3.83 | .915 | −.548 | 4.75 | .946 | −.320 | 3.64 | .945 | −.464 | 4.07 | .942 | −.636 | 4.96 |
| 14 | .875 | −.134 | 3.30 | .891 | −.293 | 3.75 | .912 | −.503 | 4.65 | .947 | −.280 | 3.56 | .945 | −.410 | 3.98 | .943 | −.590 | 4.84 |
| 15 | .872 | −.101 | 3.25 | .890 | −.258 | 3.69 | .912 | −.469 | 4.57 | .947 | −.241 | 3.50 | .945 | −.374 | 3.93 | .943 | −.548 | 4.76 |
| 16 | .872 | −.077 | 3.21 | .890 | −.230 | 3.64 | .911 | −.434 | 4.49 | .947 | −.208 | 3.44 | .945 | −.344 | 3.84 | .943 | −.514 | 4.68 |
| 17 | .866 | −.053 | 3.17 | .888 | −.202 | 3.58 | .909 | −.408 | 4.43 | .947 | −.182 | 3.40 | .945 | −.314 | 3.79 | .943 | −.483 | 4.61 |
| 18 | .864 | −.034 | 3.13 | .883 | −.175 | 3.54 | .907 | −.375 | 4.37 | .947 | −.159 | 3.36 | .946 | −.283 | 3.75 | .943 | −.452 | 4.56 |
| 19 | .861 | −.015 | 3.09 | .882 | −.159 | 3.50 | .907 | −.355 | 4.33 | .947 | −.140 | 3.33 | .945 | −.265 | 3.72 | .943 | −.425 | 4.49 |
| 20 | .859 | −.001 | 3.07 | .879 | −.138 | 3.47 | .906 | −.334 | 4.28 | .947 | −.118 | 3.29 | .946 | −.244 | 3.67 | .944 | −.402 | 4.44 |

**Table 3** continued

| $(p,\gamma)$ | Factors for two-sided TI | | | | | | | | | Factors for equal-sided TI | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | |
| $n$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ |
| 25 | .849 | .058 | 2.96 | .874 | −.076 | 3.34 | .902 | −.256 | 4.11 | .947 | −.044 | 3.16 | .946 | −.164 | 3.53 | .944 | −.318 | 4.26 |
| 30 | .842 | .097 | 2.89 | .865 | −.029 | 3.25 | .897 | −.204 | 4.00 | .947 | .004 | 3.07 | .946 | −.111 | 3.43 | .944 | −.266 | 4.15 |
| 35 | .837 | .126 | 2.85 | .860 | .001 | 3.20 | .894 | −.168 | 3.91 | .948 | .038 | 3.01 | .947 | −.075 | 3.35 | .945 | −.223 | 4.05 |
| 40 | .832 | .143 | 2.80 | .858 | .024 | 3.15 | .891 | −.140 | 3.86 | .948 | .065 | 2.96 | .947 | −.046 | 3.30 | .945 | −.194 | 3.98 |
| 45 | .823 | .162 | 2.77 | .853 | .041 | 3.11 | .887 | −.119 | 3.81 | .948 | .087 | 2.92 | .947 | −.024 | 3.25 | .945 | −.169 | 3.93 |
| 50 | .825 | .172 | 2.75 | .848 | .056 | 3.08 | .887 | −.101 | 3.76 | .948 | .103 | 2.89 | .947 | −.005 | 3.22 | .946 | −.150 | 3.88 |
| 60 | .817 | .192 | 2.71 | .844 | .079 | 3.04 | .881 | −.073 | 3.70 | .949 | .129 | 2.84 | .947 | .021 | 3.16 | .946 | −.119 | 3.81 |
| 70 | .808 | .207 | 2.69 | .841 | .094 | 3.01 | .877 | −.054 | 3.65 | .949 | .147 | 2.80 | .948 | .043 | 3.12 | .946 | −.098 | 3.76 |
| 80 | .805 | .217 | 2.67 | .836 | .109 | 2.98 | .873 | −.039 | 3.62 | .949 | .164 | 2.78 | .947 | .060 | 3.09 | .946 | −.079 | 3.72 |
| 90 | .803 | .225 | 2.65 | .834 | .118 | 2.96 | .871 | −.028 | 3.60 | .949 | .175 | 2.75 | .947 | .072 | 3.06 | .946 | −.064 | 3.69 |
| 100 | .797 | .234 | 2.64 | .830 | .126 | 2.95 | .868 | −.018 | 3.57 | .949 | .185 | 2.74 | .947 | .083 | 3.04 | .946 | −.054 | 3.66 |

In the above equation, $(\widehat{\mu}^*, \widehat{\sigma}^*)$ are the MLEs based on a sample of size $n$ from the exponential(0, 1) distribution, and $F(x|0, 1) = 1 - e^{-x},\ x > 0$, is the cdf of the exponential(0,1) distribution.

*Equal-tailed tolerance intervals*

A $(p, \gamma)$ two-sided tolerance interval is given by $\left(L_{p,\gamma_e'}(\widehat{\mu}, \widehat{\sigma}), U_{p,\gamma_e'}(\widehat{\mu}, \widehat{\sigma})\right)$, where $\gamma_e'$ is to be determined so that

$$P_{\widehat{\mu}^*,\widehat{\sigma}^*}\left(L_{p,\gamma_e'}(\widehat{\mu}^*, \widehat{\sigma}^*) \le -\ln\left(\frac{1+p}{2}\right) \text{ and } -\ln\left(\frac{1-p}{2}\right) \le U_{p,\gamma_e'}(\widehat{\mu}^*, \widehat{\sigma}^*)\right) = \gamma. \tag{34}$$

*Calculation of Tolerance Factors*

We estimated the adjusted-coverage levels that satisfy (33) for some values of $(n, p, \gamma)$ using Algorithm 1, and presented them in Table 4. For example, when $(n, p, \gamma) = (20, .90, .95)$, we estimated the adjusted-confidence level that satisfies (33) using Algorithm 1 as $\gamma_t' = .859$. The upper tolerance factor is $(1+\gamma_t')/2 = .9295$ quantile of $E_{(1+p)/2} = [-2n\ln((1-p)/2) - \chi_2^2]/\chi_{2n-2}^2$, which is estimated as 4.53. The lower factor is $(1-\gamma_t')/2 = .0705$ quantile of $E_{(1-p)/2} = [-2n\ln((1+p)/2) - \chi_2^2]/\chi_{2n-2}^2$, which is estimated as $-.089$. All the factors in Table 4 were estimated using Algorithm 1 with 100,000 simulation runs.

We also estimated adjusted-confidence levels for computing equal-tailed TIs for an exponential distribution, and presented them in Table 4. Adjusted-confidence levels $\gamma_e'$ are given for some values of $(n, p)$ and $\gamma = .95$. Notice that the adjusted-confidence levels $\gamma_e'$ coincide with the nominal confidence levels $\gamma$ for sample of sizes around 15 or more. In other words, for $n$ is around 15 or more, the interval formed by the lower tolerance limit $\widehat{\mu} + E_{\frac{1-p}{2},\frac{1-\gamma}{2}}\widehat{\sigma}$ and the upper tolerance limit $\widehat{\mu} + E_{\frac{1+p}{2},\frac{1+\gamma}{2}}\widehat{\sigma}$ is a $(p, \gamma)$ equal-tailed TI.

## 5 Censored case

The pivotal quantities based on the MLEs and the computation of tolerance factors described earlier for the uncensored case are also valid if the samples are type II censored case. The pivotal quantities based on the MLEs are not valid when the samples are type I censored. However, we can use our computational method to find approximate tolerance intervals based on a type I censored sample. We shall briefly outline the methods for some distributions below.

*Weibull Distribution*

Let $X_{(1)} < \cdots < X_{(r)}$ be a set of failure times recorded from a sample of $n$ test items. Let $X_{i*} = X_{(i)}, i = 1, \ldots, r$ and $X_{i*} = X_{(r)}, i = r+1, \ldots, n$. Then the MLE $\widehat{c}$ of $c$ is the solution of the equation

$$\frac{1}{\widehat{c}} - \left[\sum_{i=1}^{n} X_{i*}^{\widehat{c}}\ln(X_{i*})\right] \Big/ \sum_{i=1}^{n} X_{i*}^{\widehat{c}} + \frac{1}{r}\sum_{i=1}^{r}\ln(X_{i*}) = 0, \tag{35}$$

**Table 4** Factors for constructing two-sided and equal-tailed TIs for an exponential distribution

| $(p, \gamma)$ | Factors for two-sided TI | | | | | | | | | Factors for equal-sided TI | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | |
| $n$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ |
| 4 | .920 | − 1.73 | 14.8 | .925 | − 1.89 | 19.0 | .930 | − 2.03 | 28.6 | .943 | − 2.06 | 16.9 | .942 | − 2.14 | 21.0 | .941 | − 2.21 | 30.5 |
| 5 | .916 | − 1.07 | 10.9 | .923 | − 1.19 | 14.0 | .930 | − 1.30 | 21.0 | .944 | − 1.29 | 12.4 | .943 | − 1.36 | 15.4 | .943 | − 1.42 | 22.4 |
| 6 | .911 | − .753 | 8.95 | .920 | − .849 | 11.5 | .929 | − .940 | 17.3 | .946 | − .937 | 10.3 | .945 | − .990 | 12.7 | .944 | − 1.03 | 18.4 |
| 7 | .907 | − .570 | 7.80 | .919 | − .655 | 10.0 | .928 | − .730 | 15.0 | .946 | − .720 | 8.94 | .946 | − .770 | 11.1 | .945 | − .808 | 16.0 |
| 8 | .903 | − .453 | 7.06 | .915 | − .524 | 9.02 | .927 | − .595 | 13.5 | .947 | − .582 | 8.07 | .946 | − .628 | 9.99 | .946 | − .664 | 14.4 |
| 9 | .898 | − .370 | 6.52 | .913 | − .438 | 8.34 | .926 | − .502 | 12.5 | .947 | − .486 | 7.45 | .946 | − .528 | 9.21 | .945 | − .560 | 13.2 |
| 10 | .895 | − .311 | 6.12 | .911 | − .373 | 7.82 | .925 | − .432 | 11.7 | .948 | − .416 | 6.99 | .947 | − .454 | 8.63 | .946 | − .486 | 12.4 |
| 11 | .891 | − .264 | 5.80 | .909 | − .324 | 7.43 | .925 | − .380 | 11.1 | .948 | − .361 | 6.63 | .947 | − .398 | 8.18 | .946 | − .427 | 11.7 |
| 12 | .887 | − .228 | 5.55 | .906 | − .284 | 7.10 | .923 | − .337 | 10.6 | .948 | − .317 | 6.33 | .947 | − .354 | 7.82 | .946 | − .381 | 11.2 |
| 13 | .883 | − .199 | 5.35 | .904 | − .253 | 6.83 | .922 | − .303 | 10.2 | .948 | − .283 | 6.10 | .947 | − .318 | 7.52 | .947 | − .346 | 10.8 |
| 14 | .879 | − .174 | 5.17 | .901 | − .227 | 6.61 | .921 | − .275 | 9.88 | .949 | − .254 | 5.90 | .948 | − .288 | 7.28 | .947 | − .314 | 10.4 |
| 15 | .875 | − .154 | 5.02 | .900 | − .206 | 6.43 | .921 | − .252 | 9.60 | .948 | − .228 | 5.71 | .948 | − .263 | 7.08 | .947 | − .289 | 10.2 |
| 16 | .872 | − .137 | 4.90 | .898 | − .187 | 6.26 | .920 | − .233 | 9.36 | .948 | − .208 | 5.57 | .948 | − .241 | 6.89 | .947 | − .267 | 9.91 |
| 17 | .868 | − .122 | 4.78 | .895 | − .170 | 6.11 | .918 | − .214 | 9.13 | .948 | − .190 | 5.44 | .948 | − .223 | 6.73 | .947 | − .248 | 9.67 |
| 18 | .865 | − .110 | 4.69 | .894 | − .157 | 5.99 | .918 | − .200 | 8.94 | .948 | − .174 | 5.33 | .948 | − .207 | 6.59 | .948 | − .233 | 9.48 |
| 19 | .863 | − .099 | 4.61 | .891 | − .144 | 5.87 | .917 | − .187 | 8.78 | .948 | − .161 | 5.23 | .948 | − .192 | 6.46 | .947 | − .217 | 9.28 |
| 20 | .859 | − .089 | 4.53 | .889 | − .134 | 5.78 | .916 | − .175 | 8.63 | .949 | − .149 | 5.14 | .948 | − .180 | 6.35 | .947 | − .205 | 9.12 |

**Table 4** continued

| $(p, \gamma)$ | Factors for two-sided TI | | | | | | | | | Factors for equal-sided TI | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | |
| $n$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ | $\gamma_e'$ | $L_e$ | $U_e$ |
| 25 | .845 | −.054 | 4.24 | .880 | −.095 | 5.40 | .912 | −.133 | 8.05 | .949 | −.105 | 4.80 | .948 | −.134 | 5.93 | .948 | −.158 | 8.51 |
| 30 | .831 | −.032 | 4.06 | .872 | −.071 | 5.15 | .908 | −.106 | 7.68 | .949 | −.076 | 4.57 | .949 | −.105 | 5.65 | .948 | −.128 | 8.10 |
| 35 | .822 | −.018 | 3.93 | .864 | −.054 | 4.98 | .905 | −.088 | 7.41 | .949 | −.057 | 4.41 | .949 | −.085 | 5.43 | .948 | −.108 | 7.81 |
| 40 | .813 | −.008 | 3.83 | .858 | −.043 | 4.84 | .902 | −.075 | 7.20 | .949 | −.043 | 4.28 | .948 | −.070 | 5.28 | .948 | −.092 | 7.58 |
| 45 | .803 | −.000 | 3.75 | .851 | −.034 | 4.74 | .900 | −.065 | 7.05 | .949 | −.033 | 4.18 | .948 | −.059 | 5.15 | .948 | −.081 | 7.40 |
| 50 | .796 | .005 | 3.69 | .843 | −.026 | 4.65 | .898 | −.057 | 6.92 | .949 | −.023 | 4.10 | .949 | −.050 | 5.05 | .948 | −.072 | 7.26 |
| 60 | .781 | .014 | 3.59 | .834 | −.017 | 4.53 | .893 | −.046 | 6.72 | .949 | −.010 | 3.98 | .949 | −.037 | 4.90 | .948 | −.058 | 7.04 |
| 70 | .771 | .020 | 3.53 | .825 | −.010 | 4.43 | .889 | −.038 | 6.57 | .949 | −.001 | 3.89 | .949 | −.028 | 4.79 | .949 | −.049 | 6.88 |
| 80 | .762 | .024 | 3.48 | .817 | −.005 | 4.36 | .885 | −.032 | 6.45 | .949 | .005 | 3.82 | .949 | −.021 | 4.70 | .948 | −.042 | 6.75 |
| 90 | .752 | .028 | 3.43 | .810 | −.002 | 4.30 | .882 | −.027 | 6.37 | .949 | .010 | 3.76 | .949 | −.016 | 4.63 | .948 | −.036 | 6.64 |
| 100 | .745 | .032 | 3.40 | .805 | .002 | 4.26 | .878 | −.024 | 6.29 | .949 | .014 | 3.71 | .949 | −.012 | 4.57 | .948 | −.032 | 6.56 |

and $\widehat{b} = \left(\sum_{i=1}^{n} X_{i*}^{\widehat{c}}/r\right)^{1/\widehat{c}}$. The solution of the above equation can be obtained using the Newton–Raphson method or using the R function [survreg(Surv($X$, $\delta$) $\sim$ 1, dist = "weibull")], where $X$ is the censored sample and $\delta$ is a vector whose $i$th element is 1 if $X_i$ is uncensored and is 0 if $X_i$ is censored.

Using the MLEs and following the approach in Sect. 3.2, we can compute the factors for finding two-sided or equal-tailed TIs. In Table 5, we present the lower $(L_t)$ and upper $(U_t)$ factors for constructing $(p, \gamma)$ two-sided TIs for a Weibull distribution based on a sample of size $n$ with $r$ uncensored observations. For easy reference, the computed factors are given for some values of $n$ and $r$ in Table 5. The readers can also use the R code provided in the supplemental file to compute the factors for a given $(n, r, p, \gamma)$.

*Two-parameter exponential distribution*

Let $X_{(1)} < \cdots < X_{(r)}$ be the ordered uncensored observations from a type II censored sample of size $n$ from a two-parameter exponential distribution. The MLEs of the location parameter $a$ and the scale parameter $b$ are given by

$$\widehat{a} = X_{(1)} \quad \text{and} \quad \widehat{b} = \frac{1}{r}\left[\sum_{i=1}^{r}(X_{(i)} - X_{(1)}) + (n-r)\left(X_{(r)} - X_{(1)}\right)\right].$$

The MLEs are independent with $\frac{2n(\widehat{a}-a)}{b} \sim \chi_2^2$ and $\frac{2r\widehat{b}}{b} \sim \chi_{2r-2}^2$. See Section 4.5.3 of Lawless (2003).

*Two-parameter Rayleigh distribution*

Let $X_{(1)} < \cdots < X_{(r)}$ be the ordered uncensored observations from a type II censored sample of size $n$ from a two-parameter Rayleigh distribution with the location parameter $\mu$ and the scale parameter $\sigma$. Krishnamoorthy et al. (2019) have derived the MLE of $\mu$ as the solution of the equation

$$g(\mu) = 2r\left[\sum_{i=1}^{n}(X_i^* - \mu)\right] \Big/ \sum_{i=1}^{n}(X_i^* - \mu)^2 - \sum_{i=1}^{r} 1/(X_i^* - \mu) = 0,$$

where $X_{i*} = X_{(i)}$, $i = 1, \ldots, r$ and is $X_{(r)}$ for $i = r+1, \ldots, n$. Let $\widetilde{\sigma}_r^2 = 2S_r^2/(4-\pi)$ and $S_r^2$ is the variance of the uncensored observations. The MLE $\widehat{\mu}_r$ of $\mu$ can be found as the root of the equation $g(\mu) = 0$ using the interval $\left(X_{(1)} - 12\widetilde{\sigma}_r/\sqrt{r},\ X_{(1)}\right)$ as the root bracketing interval. The MLE of $\sigma$ can be obtained as $\widehat{\sigma}_r = \sqrt{\sum_{i=1}^{n}(X_i^* - \widehat{\mu})^2/(2r)}$. For easy reference, the computed factors based on censored samples are given for some values of n and r in Table 6.

## 6 Examples

***Example 1*** The data in Table 7, taken from Krishnamoorthy et al. (2006), represent air lead levels collected by the National Institute of Occupational Safety and Health (NIOSH) from a work facility for health hazard evaluation purpose. The air lead levels were collected from $n = 15$ different areas within the work facility as follows.

**Table 5** Factors for constructing two-sided tolerance intervals for a Weibull distribution with censored data

| | $(p, \gamma)$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.80,.95) | | | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | |
| $r$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ | $\gamma_t'$ | $L_t$ | $U_t$ |
| $n = 10$ | | | | | | | | | | | | |
| 7 | .891 | −5.00 | 2.76 | .902 | −6.801 | 3.437 | .906 | −8.529 | 3.949 | .910 | −12.51 | 4.821 |
| 9 | .877 | −4.44 | 1.95 | .893 | −5.919 | 2.472 | .900 | −7.374 | 2.877 | .908 | −10.69 | 3.589 |
| $n = 15$ | | | | | | | | | | | | |
| 8 | .890 | −4.43 | 2.93 | .899 | −6.06 | 3.53 | .903 | −7.69 | 4.04 | .907 | −11.42 | 4.86 |
| 10 | .882 | −4.09 | 2.18 | .895 | −5.56 | 2.72 | .901 | −6.97 | 3.13 | .906 | −10.21 | 3.85 |
| 13 | .865 | −3.75 | 1.63 | .887 | −5.01 | 2.07 | .896 | −6.24 | 2.44 | .905 | −9.051 | 3.04 |
| $n = 20$ | | | | | | | | | | | | |
| 10 | .887 | −3.92 | 2.52 | .897 | −5.39 | 3.08 | .901 | −6.82 | 3.53 | .906 | −10.12 | 4.26 |
| 15 | .867 | −3.57 | 1.64 | .885 | −4.76 | 2.08 | .894 | −5.93 | 2.42 | .902 | −8.64 | 3.02 |
| 17 | .858 | −3.45 | 1.47 | .881 | −4.60 | 1.88 | .891 | −5.72 | 2.20 | .901 | −8.29 | 2.77 |
| $n = 30$ | | | | | | | | | | | | |
| 10 | .888 | −3.61 | 2.99 | .897 | −5.04 | 3.59 | .900 | −6.48 | 4.04 | .904 | −9.82 | 4.81 |
| 20 | .868 | −3.25 | 1.54 | .885 | −4.38 | 1.95 | .893 | −5.46 | 2.28 | .900 | −7.95 | 2.83 |
| 27 | .838 | −3.07 | 1.24 | .868 | −4.10 | 1.60 | .882 | −5.10 | 1.89 | .896 | −7.38 | 2.39 |

**Table 6** Factors for constructing two-sided tolerance intervals for a Rayleigh distribution based on a type II censored sample

| | $(p, \gamma)$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | (.80,.95) | | | (.90,.95) | | | (.95,.95) | | | (.99,.95) | | |
| $r$ | $\gamma_t'$ | L | U | $\gamma_t'$ | L | U | $\gamma_t'$ | L | U | $\gamma_t'$ | L | U |
| $n = 10$ | | | | | | | | | | | | |
| 7 | 0.882 | −0.198 | 3.725 | 0.900 | −0.508 | 4.487 | 0.909 | −0.729 | 5.164 | 0.919 | −1.030 | 6.497 |
| 9 | 0.869 | −0.110 | 3.201 | 0.892 | −0.38 | 3.823 | 0.904 | −0.57 | 4.381 | 0.918 | −0.832 | 5.490 |
| $n = 15$ | | | | | | | | | | | | |
| 8 | 0.873 | 0.008 | 3.582 | 0.895 | −0.264 | 4.298 | 0.905 | −0.463 | 4.924 | 0.917 | −0.738 | 6.170 |
| 10 | 0.862 | 0.049 | 3.189 | 0.889 | −0.199 | 3.803 | 0.902 | −0.376 | 4.345 | 0.916 | −0.623 | 5.420 |
| 13 | 0.848 | 0.091 | 2.879 | 0.878 | −0.135 | 3.405 | 0.895 | −0.295 | 3.876 | 0.914 | −0.514 | 4.825 |
| $n = 20$ | | | | | | | | | | | | |
| 10 | 0.862 | 0.128 | 3.271 | 0.886 | −0.111 | 3.891 | 0.9 | −0.285 | 4.445 | 0.914 | −0.526 | 5.544 |
| 15 | 0.842 | 0.169 | 2.824 | 0.874 | −0.043 | 3.334 | 0.891 | −0.194 | 3.785 | 0.91 | −0.402 | 4.688 |
| 17 | 0.835 | 0.178 | 2.724 | 0.87 | −0.027 | 3.205 | 0.888 | −0.173 | 3.634 | 0.908 | −0.372 | 4.498 |
| $n = 30$ | | | | | | | | | | | | |
| 10 | 0.856 | 0.224 | 3.348 | 0.883 | 0.000 | 3.991 | 0.897 | −0.168 | 4.557 | 0.912 | −0.404 | 5.673 |
| 20 | 0.825 | 0.254 | 2.680 | 0.861 | 0.065 | 3.144 | 0.880 | −0.072 | 3.556 | 0.904 | −0.261 | 4.386 |
| 27 | 0.807 | 0.269 | 2.529 | 0.849 | 0.087 | 2.952 | 0.872 | −0.043 | 3.329 | 0.899 | −0.220 | 4.090 |

**Table 7** Air lead levels ($\mu g/m^3$)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 200 | 120 | 15 | 7 | 8 | 6 | 48 | 61 |
| 380 | 80 | 29 | 1000 | 350 | 1400 | 110 | |

**Table 8** Numbers of revolutions (in millions) of 23 ball bearings before failure

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17.88 | 28.92 | 33.00 | 41.52 | 42.12 | 45.60 | 48.48 | 51.84 |
| 51.96 | 54.12 | 55.56 | 67.80 | 68.64 | 68.64 | 68.88 | 84.12 |
| 93.12 | 98.64 | 105.12 | 105.84 | 127.92 | 128.04 | 173.40 | |

Krishnamoorthy et al. (2006) have shown that the data fit a lognormal distribution, which belongs to the log-location-scale family of distributions. The mean and standard deviation of the log-transformed samples are $\bar{x} = 4.3329$ and $s = 1.7394$. To compute (.90, .90) two-sided TI, the adjusted confidence level $\gamma_t'$ that satisfies (19) is 0.7736 and the factor is

$$\frac{1}{\sqrt{n}} t_{n-1;(1+\gamma_t')/2}(z_p\sqrt{n}) = \frac{1}{15} t_{14;.8868}(6.3705) = 2.2855.$$

The above tolerance factor coincides with the exact factor given in Table B2 of Krishnamoorthy and Mathew (2009). The mean and standard deviation of the log-transformed data are 4.3329 and 1.7394, respectively. So, the two-sided TI $\exp(4.3329 \pm 2.2855 \times 1.7394) = (\mathbf{1.43}, \mathbf{4057.4})$. The two-sided factor reported in Yuan et al. (2016) is 2.29 and $\exp(4.3329 \pm 2.29 \times 1.7394) = (\mathbf{1.42}, \mathbf{4089.3})$. The factor 2.37 reported in Yuan et al. (2016) is based on the maximum likelihood estimate of the variance. If we use the unbiased estimate of the variance, then the factor is $2.37\sqrt{\frac{n-1}{n}} = 2.29$. The discrepancy between the TIs is that Yuan et al. used the factor as 2.29 which is accurate only up to two digits.

To compute the (.90, .90) equal-tailed TI, we calculated the adjusted confidence level $\gamma_e'$ that satisfies (20) as .8874. The tolerance factor is computed as

$$\frac{1}{\sqrt{n}} t_{n-1;(1+\gamma_e')/2}(z_p\sqrt{n}) = \frac{1}{15} t_{14;.9437}(6.3705) = 2.5260.$$

The above factor also coincides with the exact one reported in Table B3 of Krishnamoorthy and Mathew (2009). The (.90, .90) equal-tailed TI is $\exp(4.3329 \pm 2.5260 \times 1.7394) = (\mathbf{.94}, \mathbf{6164.9})$. Using their computational approach, Yuan et al. (2016) computed the equal-tailed TI as $\exp(4.3329 \pm 2.52 \times 1.7394) = (\mathbf{.95}, \mathbf{6100.9})$, which is also different from our equal-tailed TI. This discrepancy is due to the fact that Yuan et al. (2016) have used the factor 2.52 instead of 2.5260.

***Example 2*** The data in Table 8 represent the number of million revolutions before failure for each of 23 ball bearings. The data were analyzed by Thoman et al. (1970) and others using a Weibull distribution. The MLEs are $\widehat{c} = 2.102$ and $\widehat{b} = 81.874$.

**Table 9** Failure mileage of 19 military carriers

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 162 | 200 | 271 | 302 | 393 | 508 | 539 | 629 | 706 | 777 |
| 884 | 1008 | 1101 | 1182 | 1463 | 1603 | 1984 | 2355 | 2880 | |

We computed the adjusted confidence level to find (.90, .95) two-sided tolerance interval as $\gamma_t' = .862$. The lower factor is estimated by $(1 - .862)/2 = .069$ quantile of $W_{.05} = \widehat{c}^* \left(-\ln(\widehat{b}^*) + \ln(-\ln(.95))\right)$, and is $-4.20$. The upper factor is estimated by $(1 + .862)/2 = .931$ quantile of $W_{.95} = \widehat{c}^* \left(-\ln(\widehat{b}^*) + \ln(-\ln(.05))\right)$, and is 1.60. Thus, the (.90, .95) two-sided TI is

$$\left(\widehat{b}\exp(-4.20/\widehat{c}),\ \widehat{b}\exp(1.60/\widehat{c})\right) = (\mathbf{11.10}, \mathbf{175.3}).$$

The adjusted confidence level to compute a (.90, .95) equal-tailed TI is .9456, and the factors are

$$W_{.05;.0272} = .0272 \text{ quantile of } \widehat{c}^* \left(-\ln(\widehat{b}^*) + \ln(-\ln(.95))\right) = -4.62$$

and

$$W_{.95;.9728} = .9728 \text{ quantile of } \widehat{c}^* \left(-\ln(\widehat{b}^*) + \ln(-\ln(.05))\right) = 1.77$$

Thus, the (.90, .95) equal-tailed TI is

$$\left(\widehat{b}\exp(-4.62/\widehat{c}),\ \widehat{b}\exp(1.77/\widehat{c})\right) = (\mathbf{9.1}, \mathbf{190.0}).$$

To illustrate the method for the censored case, let us assume the test was terminated after the 16th failure. That is, we have a type II right-censored sample with $n = 23$ and $r = 16$. The MLEs for this right-censored sample is $\widehat{b} = 76.696$ and $\widehat{c} = 2.469$. The value of $\gamma_t'$ to find a (.9, .95) two-sided TI is .89, and so the factors are $W_{.05;.055} = -4.67$ and $W_{.95;.945} = 2.10$ we get $(76.695 \times \exp(-4.67/2.469),\ 76.695 \times \exp(2.10/2.469)) = (\mathbf{11.5},\ \mathbf{179.5})$, which is the desired (.9, .95) TI. The above factor also coincides with the exact one reported in Table B3 of Krishnamoorthy and Mathew (2009). Notice that the TI based on censored sample is very close to **(11.1, 175.3)** that is based on all 23 observations.

***Example 3*** In this example, we shall use the failure mileage data on 19 military carriers given in Grubbs (1971). The data are reproduced here in Table 9. Bain and Engelhardt (1973), Krishnamoorthy and Xia (2018), and many others considered this data for illustrating the methods for two-parameter exponential distribution.

The MLEs based on the data are $\widehat{\mu} = X_{(1)} = 162$ and $\widehat{\sigma} = 835.21$.

In failure time data analysis, it is of interest to find a lower tolerance limit to judge the minimum life span of a product. The exact (.95, .95) factor for finding lower tolerance limit is $-.1188$, and the tolerance limit is $162 - .1188 \times 835.2 = \mathbf{62.78}$.

**Table 10** Lifetime (in min) of a sample of 1.88-mm drills

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 105 | 105 | 95 | 87 | 112 | 80 | 95 | 97 | 77 | 103 | 78 | 87 | 107 | 96 | 79 |
| 91 | 108 | 97 | 80 | 76 | 92 | 85 | 76 | 96 | 77 | 80 | 100 | 94 | 82 | 104 |
| 91 | 95 | 93 | 99 | 99 | 94 | 84 | 99 | 91 | 85 | 86 | 79 | 89 | 89 | 100 |

This means that at least 95% of military carriers will last 62.78 units of miles with confidence 95%. The factor for computing (.95, .95) upper tolerance limit is 4.810, and the upper tolerance limit is $162 + 4.810 \times 835.2 = \mathbf{4179.3}$. To find a (.95, .95) two-sided TI, the factors given in Table 4 are −.144 and 5.87. The desired two-sided TI is $(162 - .144 \times 835.2, 162 + 5.87 \times 835.2) = (\mathbf{41.7}, \mathbf{5064.6})$. This means that at least 95% of military carriers have life span between 41.7 and 5064.6 units of miles with confidence 95%.

***Example 4*** The data in Table 10 represent the lifetimes of 1.88-mm drills from a supplier. Lifetime data for the drills are collected during the production process in a factory and reported in Table 1 of Chen et al. (2019). The data are reproduced here in Table 10. The p-value of the Kolmogorov-Smirnov test for checking a Rayleigh distribution is 0.642. Thus, the data fit a Rayleigh distribution quite well. See Krishnamoorthy et al. (2019) for more details.

There are $n = 45$ lifetime measurements, and the MLEs are $\widehat{\mu} = 72.84$ and $\widehat{\sigma} = 14.79$. To find a $(p, \gamma) = (.95, .95)$ two-sided TI, we find the factors from Table 3 are 0.041 and 3.11. The (.95,.95) two-sided TI is $(72.84 + .041 \times 14.79, 72.84 + 3.11 \times 14.79) = (\mathbf{73.4}, \mathbf{118.8})$. That is, at least 95% of drills survive 73.44 to 118.8 min with confidence 95%. The factors for computing a (.95, .95) equal-tailed TI are −.024 and 3.25, and the equal-tailed TI is $(72.84 - .024 \times 14.79, 72.84 + 3.25 \times 14.79) = (\mathbf{72.5}, \mathbf{120.9})$. That is, no more than 2.5% of drills survive less than 72.5 min and no more than 2.5% of drills survive more than 120.9 min.

Now we assume that the largest 14 observations of those $n = 45$ lifetime were censored and the $r = 31$ lifetimes were uncensored. The MLEs based on such censored sample are $\widehat{\mu} = 72.35$ and $\widehat{\sigma} = 15.74$. The value of $\gamma_t'$ is 0.868, and so $q^*_{.025;.066}$ along with $q^*_{.975;.934}$ are 0.020 and 3.303, respectively. Then the (.95, .95) TI is $(72.35 + 0.020 \times 15.74,\ 72.35 + 3.303 \times 15.74) = (\mathbf{72.7}, \mathbf{124.3})$.

## 7 Concluding remarks

Wald and Wolfowitz (1946) have proposed a method to find tolerance intervals for a normal distribution, and Owen (1964) has introduced equal-tailed TIs and outlined a method to find them for a normal distribution. The methods provided in these two articles can be extended to find TIs for a symmetric location-scale family of distributions (Krishnamoorthy and Xie 2011). As these methods use the symmetric property of the distributions to compute TIs, they are not applicable to find TIs for an asymmetric location-scale family of distributions. Recently, Yuan et al. (2016) have proposed methods to find TIs for any location-scale distribution. However, as noted earlier, their method involves searching for factors in a two-dimensional plane whereas our method involves searching for a confidence level within a specified interval, and thereby reducing computational complexities. The proposed method is applicable to any location-scale distribution, and is conceptually simpler than the methods given in the literature for a symmetric location-scale distribution. To help practitioners and future researchers, we have provided R functions (Tolerance.Intervals.r) in a supplemental file to compute factors for constructing one- and two-sided TIs and also equal-tailed TIs for Weibull, two-parameter exponential and Rayleigh distributions.

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