UNBIASED ESTIMATION IN TYPE II CENSORED SAMPLES
FROM A ONE-TRUNCATION PARAMETER DENSITY

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ABSTRACT

We consider uniform minimum variance unbiased estimation of a U-estimable function when the sample is (singly) Type II censored and comes from a one-truncation parameter density \( f(x; \theta) = h(x) q(\theta) \). An explicit expression for the estimator is derived. Shortest length confidence interval for \( c(\theta) \) is obtained.

1. INTRODUCTION

Let \( X_1, X_2, \ldots, X_n \) be independent identically distributed random variables with common probability density function (pdf) \( f(x; \theta) \)
and let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the corresponding set of order statistics. Suppose the sample is (singly) Type II censored so that one observes only $X_{1:n}, X_{2:n}, \ldots, X_{r:n}, \quad 1 \leq r \leq n$. In this paper we consider minimum variance unbiased (UMVU) estimation of a U-estimable function $g(\theta)$ when the (censored) sample comes from a one-truncation parameter pdf

$$f_1(x, \theta) = q_1(\theta) h_1(x), \quad a < \theta < x < b \quad (1.1)$$

or

$$f_2(x, \theta) = q_2(\theta) h_2(x), \quad a < x < \theta < b \quad (1.2)$$

where $-\infty < a < b < \infty$ are known. $h_1, h_2$ are positive absolutely continuous functions and, $q_1, q_2$ are everywhere differentiable. The case $r = n$, that is, when a complete sample is available was treated by Tate (1959) who showed that the UMVU estimator for $g$ for the family $\{f_1\}$ is given by

$$g(X_{1:n}) = g(X_{1:n}) \frac{g'(X_{1:n})}{n q_1(X_{1:n}) h_1(X_{1:n})}, \quad (1.3)$$

and that for the family $\{f_2\}$ is given by

$$g(X_{n:n}) = g(X_{n:n}) \frac{g'(X_{n:n})}{n q_2(X_{n:n}) h_2(X_{n:n})}, \quad (1.4)$$

where $g'(\theta) = \frac{dg(\theta)}{d\theta}$.

2. RESULT

We first consider the case when $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ is a type II censored sample from pdf $f_1$ given in (1.1). The likelihood function is given by

$$L_1(x, \theta) = \frac{n!}{(n-r)!} \left( \prod_{j=1}^{r} h_1(x_{j:n}) \right) q_1(\theta) \left( \int_{x_{r:n}}^{h} h_1(x) dx \right)^{n-r} I(x_{1:n} > \theta) \quad (2.1)$$

where $x = (x_{1:n}, x_{2:n}, \ldots, x_{r:n})$ and $I(A)$ denotes the indicator function of set $A$. It follows from (2.1) that $X_{1:n}$ is a minimal
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A sufficient statistic that is complete and hence the UMVU estimator of any U-estimable function \( g \) is given by \( \Phi(X_{1:n}) \) defined in (1.3).

Next we consider the pdf \( f_2(x; \theta) \) defined in (1.2). For convenience we write \( f_2(x; \theta) = f(x; \theta) = q(\theta) h(x) \). In this case the likelihood function is given by

\[
I(x; \theta) = \frac{n!}{(n-r)!} \left( \prod_{j=1}^{r} h(x_{1:n}) \right) \left( \int_{x_{r:n}}^B h(x)dx \right)^{n-r} I(x_{1:n} < \theta)
\]

and it follows that \( X_{r:n} \) is minimal sufficient with pdf

\[
f_{r:n}(x) = n \binom{n-1}{r-1} q^n(\theta) \int_a^x h(u)du \left( \int_a^B h(u)du \right)^{r-1} \left( \int_a^B h(u)du \right)^{n-r} h(x).
\]

If \( E_\theta \Phi(X_{r:n}) = 0 \) for all \( \theta \in (a,b) \) then one sees easily on differentiation that \( \phi(x) = 0 \) a.e. and hence \( X_{r:n} \) is a complete sufficient statistic.

Let \( g(\theta) \) be a U-estimable function. Then there exists a function \( \phi \) such that

\[
\phi(\theta) = E_\theta \Phi(X_{r:n}) = n \binom{n-1}{r-1} q^n(\theta) \int_a^x \phi(x) \left( \int_a^B h(u)du \right)^{r-1} \left( \int_a^B h(u)du \right)^{n-r} h(x)dx
\]

and this \( \phi \) is the (essentially) unique UMVU estimator by Lehmann-Scheffe theorem. We need a few simple facts. First

\[
1 = \int_a^B q(\theta) h(x)dx
\]

so that \( q(\theta) = \left( \int_a^B h(x)dx \right)^{-1} \) and, moreover, on differentiation with respect to \( \theta \)
\[ q'(\theta) = \frac{\partial q(\theta)}{\partial \theta} = -q^2(\theta)h(\theta). \quad (2.5) \]

Next, we need a result on differentiation of a function defined by integrals. Let \( \psi(x,t) \in C' \) for \( a \leq t \leq b, \ A \leq x \leq B \) and suppose \( h(x) \) that \( \psi(x) = \int_a^x \psi(x,t)dt \). Then (see Widder (1961), p. 353)

\[
\int_a^x \frac{\partial \psi(x,t)}{\partial x} \, dt = \psi(x,g(x)) \psi'(x) + \psi(x,h(x)) h'(x). \quad (2.6)
\]

We are now ready to solve the integral equation (2.4). For \( k = 0,1,...,n-r \) set

\[
l_k(\theta) = \int_a^\theta \psi(x) \left[ \int_a^x h(u)du \right]^{n-r} \left[ \int_a^x h(u)du \right] h(x)dx.
\]

Then (2.4) can be rewritten as

\[
l_0(\theta) = \frac{(r-1)!}{n!} g(\theta) q^{-n}(\theta) = \frac{(r-1)!}{n!} h^{-1}(\theta) \frac{d}{d\theta} s(\theta), \quad (2.7)
\]

where \( s(\theta) = g(\theta) q^{-n}(\theta) \). For each differentiable function \( \omega(\theta) \) on \( (a,b) \), define an operator \( D \) by

\[
(D\omega)(\theta) = h^{-1}(\theta) \frac{d}{d\theta} \omega(\theta). \quad (2.8)
\]

and for \( k \geq 2 \) define

\[
(D^k\omega)(\theta) = (DXD^{k-1}\omega)(\theta). \quad (2.9)
\]

Then \( D \) defines a linear operator on \( (a,b) \).

Clearly

\[
l_{n-r}(\theta) = \int_a^\theta \phi(x) \left[ \int_a^x h(u)du \right]^{n-r} h(x)dx,
\]

and it follows that

\[
\frac{d}{d\theta} l_{n-r}(\theta) = \phi(\theta) \left[ \int_a^\theta h(u)du \right]^{n-r} h(\theta).
\]
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In view of (2.8) we have

\[ q(\theta) = q^{-1}(\theta)(D_{n-r})(\theta). \]  

(2.10)

Using (2.8) we see easily that

\[ \frac{\partial}{\partial \theta} l_k(\theta) = (n - r - k) h(\theta) l_k(\theta), \]

and hence for \( k = 0, 1, ..., n - r - 1 \)

\[ l_{k+1}(\theta) = \frac{1}{n - r - k} (D_{n-r}l_k)(\theta). \]

(2.11)

By the linearity of \( D \) we see from (2.11) that

\[ l_{n-r}(\theta) = \frac{1}{(n - r)!} (D_{n-r}l_0)(\theta) \]

and hence from (2.10) and (2.7),

\[ q(\theta) = \frac{(r - 1)!}{n!} q^{-1}(\theta)(D_{n-r}l_s)(\theta). \]

We have thus proved the following result.

**THEOREM 1**: The UMVU estimator of any \( U \)-estimable function \( g(\theta) \) based on a type II censored sample from pdf \( f \) given in (1.2) is of the form

\[ q(X_{r:n}) = \frac{(r - 1)!}{n!} q^{-1}(X_{r:n})(D_{n-r}l_s)(\theta). \]

(2.12)

where \( s_\theta = g(\theta)\theta^{-n}(\theta) \) and \( D \) is a linear operator defined by (2.8) and (2.9).

Substituting \( r = n \) in (2.12) we get (1.4).

**EXAMPLE 2.1**: Let \( f(x;\theta) = 1/\theta, \ 0 < x < \theta \) and \( g(\theta) = e^{-\theta} \). In this case \( s(\theta) = \theta^n e^{-\theta} \) and hence

\[ (D_{n-r}l_s)(\theta) = \sum_{k=0}^{n-r-1} (-1)^{n-r+1-k} \binom{n-r+1}{k} \frac{n!}{(n-k)!} \theta^{n-k} e^{-\theta}. \]

It follows that the UMVU estimator of \( g(\theta) \) is given by
\[ \varphi(X_{r:n}) = (r-1)! \sum_{k=0}^{n-r+1} (-1)^{n-r+1-k} \binom{n-r+1}{k} \frac{1}{(n-k)!} \cdot (X_{r:n})^{n-k} e^{-X_{r:n}}. \]

**Example 2.2:** In many applications, \( g(\theta) = q(\theta) \). In this case \( s(\theta) = q^{-n+1}(\theta) \) and \( (D^{n-r+1}s)(\theta) = (n-1)! \ q^{2-r}(\theta)/(r-2)! \). It follows that for \( r > 1 \), \( \varphi(X_{r:n}) = (r-1)! \ n(n-r+1)/n \) is the UMVL estimator of \( q(\theta) \). It is easy to check that for \( r > 2 \)
\[ \mathbb{E} q^2(X_{r:n}) = \frac{n(n-r+1)}{(r-1)(r-2)} q^2(\theta) \]
so that
\[ \text{var}(q(X_{r:n})) = \frac{n(n-r+1)}{(r-1)^2 (r-2)} q^2(\theta), \quad r > 2, \]
and
\[ \text{var}(\varphi(X_{r:n})) = \frac{n-r+1}{n(r-2)} \eta^2(\alpha), \quad r > 2. \]

3. **An Application**

As an application we find the shortest length confidence interval for \( q(\theta) \) based on \( q(X_{r:n}) \). We show that the distribution of \( Y = q(\theta)/q(X_{r:n}) \) is independent of \( \theta \) and hence \( Y \) is a pivot for \( q(\theta) \) (or \( \theta \)). Indeed from (2.3) and (3.5)
\[ f_{r:n}(x) = n \left[ \binom{n-1}{r-1} \frac{q(\theta)}{q(x)} \right]^{r-1} \left[ 1 - \frac{q(\theta)}{q(x)} \right]^{n-r} q(\theta) h(x) \]
for \( a < x < \theta \). From (2.5) it is clear that \( q(x) \) is a decreasing function of \( \theta \) and hence \( q(\theta) < q(x) \), that is, \( 0 < y < 1 \). Also
\[ \frac{dx}{dy} = h(x) q(\theta) \]
and \( f_Y(y) = n \left( \binom{n-1}{r-1} y^{r-1} (1-y)^{n-r} \right) \), \( 0 < y < 1 \)
and \( f_Y(y) = 0 \) elsewhere.
TABLE 1

MULTIPLIER FOR THE SHORTEST CONFIDENCE INTERVAL
\(\alpha_1 q(X_{\alpha_1}) , \alpha_2 q(X_{\alpha_2})\).

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<th>(n)</th>
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It is now easy to construct a \(1 - \alpha\) level shortest length confidence interval for \(q(\theta)\) based on \(q(X_{r,n})\). The confidence interval is given by \((\alpha_1 q(X_{r,n}), \alpha_2 q(X_{r,n}))\) where \(\alpha_1, \alpha_2\) are determined simultaneously from

\[
\int_{\alpha_1}^{\alpha_2} f_Y(y) \, dy = 1 - \alpha, \text{ and } f_Y(\alpha_1) = f_Y(\alpha_2).
\]

In Table I we have numerically computed values of \((\alpha_1, \alpha_2)\) for selected values of \(1 - \alpha, n\) and \(r\). It should be noted that \(\alpha_1\) and \(\alpha_2\) satisfy

\[
|f_Y(\alpha_1) - f_Y(\alpha_2)| < 10^{-6}
\]

and

\[
\left| \int_{\alpha_1}^{\alpha_2} f_Y(y) \, dy - (1 - \alpha) \right| < 10^{-6}.
\]

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