### 5: Uniform Series

- Cash flows of uniform series
  - Equal
  - Occur each compounding period
- Also known as *annuities*, even if not yearly
- Use one series factor instead of several single payment factors

### 5.1 Compound Amounts

- Two situations
  - Given the cash flows, determine the compound amount
  - Given the compound amount, determine the cash flows

#### Uniform Series Compound Amount Factor

\[
E_S = U \left( F \mid A, i, s-r \right)
\]

\[
E_S = U \left( F \mid A, 1/2\%, 36-0 \right) = 7,867.22
\]

#### Example 5.1 Uniform Series CA

- Purchase a car in 3 years
  - Save $200 per month
  - 36 end-of-month deposits
  - \( i = 1/2 \% \) per month
- Amount in savings
  - \( E_{36} = 7,867.22 = 200 \left( F \mid A, 1/2\%, 36-0 \right) \)

#### Example 5.2 Delayed CA

- 36 deposits of $200 at 1/2\% / mo
  - Beginning-of-month deposits
  - Savings at \( t = 36 \) and \( t = 48 \)?
  - CA must be at time of last flow to use \( F \mid A \), so multi-step problem
First Determine \( E_{35} \)
\[
E_{35} = \$7,867.22 = 200 \left( \frac{F}{A}, 1/2\%, 35 - (-1) \right)
\]
- Have $7,867.22 in savings at \( t = 35 \)
  - No longer need original cash flows

Then Other Equivalents
\[
E_{36} = \$7,906.56 = E_{35} \left( \frac{F}{P}, 1/2\%, 36 - 35 \right)
\]
\[
E_{48} = \$8,394.32 = E_{35} \left( \frac{F}{P}, 1/2\%, 48 - 35 \right)
\]

Sinking Funds
- Equal deposits to accumulate a given CA
  - Given CA, find prior flows
  - Relationship of CA to its prior deposits known from \( F/A \)
    \[
    CA = US \times \left[ \frac{(\gamma^s - r - 1)}{i} \right]
    \]
    \[
    US = CA \times \left[ \frac{i}{(\gamma^s - r - 1)} \right]
    \]
    \[
    E_U = c_s \left( \frac{A}{F}, i, s-r \right)
    \]

Observations
- Sinking fund factor also known as “\( A|F \)”
  - Find prior annuity given future CA, interest rate, and number of payments
  \[
  E_U = c_s \left( A|F, i, s-r \right)
  \]

Example 5.3 End-of-Month Sinking Fund
- Want $10,000 for car in 3 years
  - 36 end-of-month deposits
  - \( i = 1/2 \% / \text{mo} \)
  - Amount of each deposit?
    \[
    E_U = \$254.22 = 10,000 \left( \frac{A}{F}, 1/2\%, 36-0 \right)
    \]

Different Diagrams – Same Meaning
- Preceding diagram shows equivalents
  - Dashed, same direction
- Could show deposit flows
  - Solid lines
  - Depositor’s viewpoint
    - Deposits flow from
    - Withdrawal flows to
Example 5.4  Beginning-of-Month Sinking Fund

- Still make 36 deposits at 1/2 % per month
  - Beginning-of-month
- Need CA at \( t = 35 \) to use \( A|F \)

So ask, “How much is needed at time 35 to have $10,000 at time 36?”

If diagram does not exactly match diagram used to derive factor, then multi-step problem!

First Step

- Amount necessary at \( t = 35 \) to have $10,000 at \( t = 36 \) is its discounted amount

\[
E_{35} = $9,950.25 = 10,000 \ (P \mid F, \frac{1}{2}\%, \ 36-35)
\]

Second Step

- What deposits at 0, 1, 2, …, 35 compound to \( E_{35} = $9,950.25 \)?

\[
E_U = $252.95 = E_{35} \ (A \mid F, \frac{1}{2}\%, \ 35 – (-1))
\]

Challenges

- Position of equivalents
- Fourth parameter

5.2 Discounted Amounts

- Two situations
  - Given the cash flows, determine the discounted amount
  - Given the discounted amount, determine the cash flows

Uniform Series DA Factor

\[
E_r = U(P \mid A, i, s-r)
\]

\[
E_r = U\gamma^{-r} (r + 1 - r) + U\gamma^{-r} (r + 2 - r) + \ldots + U\gamma^{-s-r}
\]

\[
E_r = U \left[ \gamma^{-1} + \gamma^{-2} + \ldots + \gamma^{-(s-r)} \right]
\]

\[
E_r = U \left[ 1 – (1+i)^{-(s-r)} \right] / i
\]

\[
E_r = U(P/A, i, s-r)
\]
### Observations

- \( E_r = U(P | A, i, s-r) \)
- Number of flows = Last – One Before
- Uniform series discounted amount factor
- “\( P | A \)” since looking for prior given annuity

### Example 5.5 Uniform Series DA

- **Borrow for car**
  - Afford $200 / mo
  - 36 end-of-month payments
  - \( i = 3/4\% / \text{mo} \)
- **How much car?**
  - Amount loaned = Discounted payments
  - \( E_0 = 6,289.36 = 200(P | A, 3/4\%, 36-0) \)
  - Have $1,577.86 \( (7,867.22 - 6,289.36) \) more for car by saving $200 / mo instead

### Example 5.6 DA with Delayed Series

- **Still 36 notes of $200 at 3/4\% / \text{mo}**
  - Delay for 12 months
- **Determine amount borrowed**
  - \( P / A \) requires equivalent to be before first cash flow, so multi-step problem
- **First Step**
  - What must the debt be at \( t = 12 \) to require payments of $200?
    - Time 12 chosen so that \( P | A \) can be used
    - One period before first flow!
    - \( E_{12} = 6,289.36 = 200(P | A, 3/4\%, 48-12) \)
- **Second Step**
  - If the debt is \( E_{12} \) at time 12, then what was it at time 0?
    - \( E_0 = 5,749.97 = E_{12}(P | F, 3/4\%, 12-0) \)
    - Now can borrow only $5,749.97
    - The incredible shrinking car

### The Moral to These Examples Is ...

- Borrowing instead of saving and delaying payments decreases the amount available for a purchaser
**Capital Recovery Factor**

- At what rate must invested capital be recovered to earn a specified rate of return?
- Know from deriving $P/A$
  
  \[ DA = US \times \left[ 1 - (1+i)^{-(s-r)} \right] / i \]
  
  \[ US = DA \times i / \left[ 1 - (1+i)^{-(s-r)} \right] \]
  
  \[ E_U = c_r \times i / \left[ 1 - (1+i)^{-(s-r)} \right] \]
  
  \[ E_U = c_r (A/P,i,s-r) \]

**Observations**

- (Uniform series) Capital recovery factor
- “A given P” since finding an equivalent annuity given a prior cash flow

---

**Example 5.7  Capital Recovery**

- Borrow $10,000 to buy a car
  - 36 EOM notes
  - $i = 3/4\% / mo
- Payments that provide a return of 3/4\% / mo?
  \[ E_U = $318.00 = 10,000(A/P, 3/4\%, 36-0) \]
- Saved $10,000 with 36 deposits of $254.22, so $63.78 ($318.00 − 254.22) more each mo

**Example 5.8  CR with Delayed Payments**

- Same loan but delay payments by 12 months
- Determine payments
  - $A/P$ requires equivalent to be one period before first cash flow, so multi-step problem

---

**First Step**

- If $10,000 owed at time 0, then how much owed at time 12?
- Time 12 chosen one period before notes so $A/P$ can be used in step 2
  
  \[ E_{12} = $10,938.07 = 10,000(F/P, 3/4\%, 12-0) \]

**Second Step**

- Know debt at time 12 ($10,938.07), so no longer need original cash flows
  
  - Note position of $E_{12}$, one before payments
  
  \[ E_U = $347.83 = E_{12} (A/P, 3/4\%, 48-12) \]
- Delay costs $94.88 ($347.83 − 252.95) more per month than saving for the car
5.3 Multiple Series

- Series of deposits followed by series of withdrawals fairly common
- Strategy
  - Use $F|A$ or $P|A$ on known series
  - Use $F|P$ or $P|F$ on single equivalent
    - Move to either last flow or one period before first flow of unknown series
  - Use $A|F$ or $A|P$ for unknown series
- Number flows = First - One before

Ex 5.9 Known Deposits, Unknown Withdrawals

First Step a)

- How much in savings at $t = 30$?

$$E_{30} = 3,000 (F|A, 8\%, 30-1)$$

Drawn from account’s perspective

First Step b)

- How much in savings at $t = 30$?

First Step c)

- How much in savings at $t = 30$?

$$E_{30} = 3,000 (F|A, 8\%, 30-1)$$

No longer need original flows
**First Step d)**
- Original problem is now
  \[ E_{30} = 311,897.81 = 3,000(F/A, 8\%, 30-1) \]
- Next question?

**Second Step a)**
- How much in savings at \( t = 38 \)?
  \[ E_{38} = 577,301.08 = E_{30}(F/P, 8\%, 38-30) \]
- Next question?

**Second Step b)**
- How much in savings at \( t = 38 \)?
  \[ E_{38} = 577,301.08 = E_{30}(F/P, 8\%, 38-30) \]

**Second Step c)**
- How much in savings at \( t = 38 \)?
  \[ E_{38} = 577,301.08 = E_{30}(F/P, 8\%, 38-30) \]
- Next question?

**Third Step**
- Withdrawals?
  \[ X = 54,080.86 = E_{38}(A/P, 8\%, 63-38) \]
- Deposits?
  \[ X = 54,080.86 = E_{38}(A/P, 8\%, 63-38) \]
- What is the first question?

---

**Ex 5.10 Known Withdrawals, Unknown Deposits**
- As before, but want withdrawals of \$60,000
  \[ X = 54,080.86 = E_{38}(A/P, 8\%, 63-38) \]
- Deposits?
  \[ X = 54,080.86 = E_{38}(A/P, 8\%, 63-38) \]
- What is the first question?
First Step a)

- What must be in savings at \( t = 38 \)?

\[ E_{38} = 60,000 \]

First Step b)

- What must be in savings at \( t = 38 \)?

\[ E_{38} = 640,486.57 = 60,000 \left( P/A, 8\%, 63-38 \right) \]
- Do not need original flows

First Step c)

- What must be in savings at \( t = 38 \)?

\[ E_{38} = 640,486.57 = 60,000 \left( P/A, 8\%, 63-38 \right) \]
- Next question?

Second Step a)

- What must be in savings at \( t = 30 \)?

\[ E_{30} = 346,034.97 = E_{38} \left( P/F, 8\%, 38-30 \right) \]

Second Step b)

- What must be in savings at \( t = 30 \)?

\[ E_{30} = 346,034.97 = E_{38} \left( P/F, 8\%, 38-30 \right) \]

Second Step c)

- What must be in savings at \( t = 30 \)?

\[ E_{30} = 346,034.97 = E_{38} \left( P/F, 8\%, 38-30 \right) \]
- Next question?
Third Step

- Deposits?

\[ X = $3,328.35 = E^{30} (A/F, 8\%, 30-1) \]

5.4 Bonds

- Government and private industry borrow directly from the public
  - Bypass banks
- Contract to make future payments
  - Sold on open market
  - Max price = Discounted payments
- Terminology

**Terminology**

- *Maturity date*: time of final payment (10)
- *Coupon*: series paid to bondholder ($80)
- *Redemption, face, or par value*: paid at maturity date in addition to coupon ($1,000)
- *Coupon rate*: rate used to compute the series payments (8%)

\[ \text{Coupon} = \text{Face Value} \times \text{Coupon Rate} \]
\[ 80 = 1,000 \times 8\% \]

- Max to pay \( E_0 \)

**Example 5.11 Purchase of a New Bond**

- Max to pay to earn 9%?
  \[ E_0 = $935.82 = 80(P/A, 9\%, 10-0) \]
  \[ + 1,000 (P/F, 9\%, 10-0) \]
  - Offer ($935.82) is *below par* ($1,000) when IRR > Coupon Rate

**Purchase @ 8%**

- Max to pay to earn 8%?
  \[ E_0 = $1,000 = 80(P/A, 8\%, 10-0) \]
  \[ + 1,000 (P/F, 8\%, 10-0) \]
  - Offer ($1,000) is *at par* ($1,000) when IRR = Coupon Rate

**Purchase @ 7%**

- Max to pay to earn 7%?
  \[ E_0 = $1,070.24 = 80(P/A, 7\%, 10-0) \]
  \[ + 1,000 (P/F, 7\%, 10-0) \]
  - Offer ($1,070.24) is *above par* ($1,000) when IRR < Coupon Rate
**Example 5.12  Purchase of an Existing Bond**

- Max to pay at time 3 for same bond if third coupon just paid to earn an IRR of 9%

\[
E_3 = 949.67 = 80 \left( P | A, 9\%, 10 - 3 \right) + 1,000 \left( P | F, 9\%, 10 - 3 \right)
\]

- Offer ($949.67) is below par ($1,000) when IRR > Coupon Rate

---

### 5.5 Loans

- Balloon notes
  - Final payment larger or “balloons”
- Principal and interest
  - Components of each payment
- Early repayment
  - Effect on duration of loan

---

**Example 5.13  Balloon Note**

- Borrow $80,000 for house
  - 3/4% / mo for 5 years
  - M as for 30 yr loan
  - Balance B due @ t = 60

First, determine monthly note

\[
M = 80,000 \left( A | P, \frac{3}{4}\%, 360 - 0 \right) = 643.70
\]

- Second, determine balloon payment
  - At time 60, with payment 60 just made
  - Payments 61, ..., 360 remain
  - Easiest to discount remaining payments

\[
B = 76,704.11 = 643.70 \left( P | A, \frac{3}{4}\%, 360 - 60 \right)
\]

---

**Alternative Step 2**

- Could use balance equation once M known

\[
0 = 80,000 \left( F | P, \frac{3}{4}\%, 60 - 0 \right) - 643.70 \left( F | A, \frac{3}{4}\%, 60 - 0 \right) - B
\]

\[
\Rightarrow B = 76,704.11, \text{ as before}
\]

- Discounting easier

---

**Principal and Interest**

- Two parts of each note or payment

\[
\text{Loan Payment} = \text{Interest} + \text{Principal}
\]

- Principal component reduces the debt
- Interest reduces individual’s income tax
  - Home loan
  - Business loan

- Interest on several payments

\[
\Sigma \text{Interest} = \Sigma \text{Loan Payments} - \text{Debt Reduction}
\]

- Easier than period-by-period balances
Example 5.14  Principal and Interest Payment

- House loan for $100,000
  - 3/4% / mo
  - 30 years
- Notes 3, 4, …, 14 in next tax year. Interest?
- First, determine the monthly payment

\[ M = \$804.62 = \frac{100,000}{A/P, \frac{3}{4}\%, 360-0} \]

\[ \Sigma \text{Interest} = \Sigma \text{Loan Payments} - \text{Debt Reduction} \]
\[ \Sigma \text{Loan Payments} = \$9,655.44 = 804.62 \times 12 \]
\[ \text{Debt Reduction} = \text{Debt before} - \text{Debt after} \]

Step 2

Before = \( E_2 = \$99,890.35 \)
\[ = 804.62 \left( P | A, \frac{3}{4}\%, 360-2 \right) \]

After = \( E_{14} = \$99,196.86 \)
\[ = 804.62 \left( P | A, \frac{3}{4}\%, 360-14 \right) \]

Reduction = Before - After
\[ = \$693.48 \]

\[ \Sigma \text{Interest} = \$8,961.96 \]

Example 5.15  Early Loan Repayment

- Same loan
  - $100,000 @ 3/4%
  - 360 notes of $804.62
- Pay $10,000 extra at \( t = 12 \)
  - Before = $804.62 \( (P | A, \frac{3}{4}\%, 360-12) \) = \$99,316.48
  - After = $89,316.48 = 99,316.48 - 10,000
- $804.62 / mo continues until time \( s \) when $89,316.48 = DA of Remaining Payments

Step 2

89,316.48 = $804.62 \( (P | A, \frac{3}{4}\%, s-12) \)
\[ (P | A, \frac{3}{4}\%, 252-12) = 111.145 \]
\[ (P | A, \frac{3}{4}\%, 251-12) = 110.979 \]
\[ (P | A, i, m) = \frac{1 - (1+i)^{-m}}{i} \]
\[ \text{Note}_{252} = \$125.75 = 124.81 \left( F | P, \frac{3}{4}\%, 252-251 \right) \]
Step 3
- Eliminate 108 (360 - 252) notes of $804.62
- Invest savings at 1/2% / mo until $t = 360
  \[ \$116,015 = (804.62 - 252)(F/P, 1/2\%, 360-252) + 804.62 (F/A, 1/2\%, 360-252) \]
- $116,015 more @ 360 than $10,000 on car
- If invest $10,000 @ 12 at 1/2% instead of paying off 3/4% loan
  \[ \$56,727 = 10,000(F/P, 1/2\%, 360-12) \]
- Eliminate debt @ 3/4% – Invest @ 1/2%
  \[ \$59,288 = 116,015 - 56,727 \]

5.6 Multiple Interest Rates
- Same concepts as single rates
- Might not be able to use easy formulas
- Simplifications for constant regions

Example 5.16 Multiple Rate Series and CAs
- \[ U = 1,000, CA? \]
  \[ $4,417.47 = 1,000 \left[ \frac{1}{1.06}(1.065)(1.07) \right] \]
  \[ + (1.065)(1.07) + 1.07 + 1 \]
- \[ CA = 2,000, U? \]
  \[ $452.74 = 2,000 \left[ \frac{1}{1.06}(1.065)(1.07) \right] \]
  \[ + (1.065)(1.07) + 1.07 + 1 \]

Discounted Amounts
- \[ DA = U(\gamma_{r+1})^{-1} + \ldots + U(\gamma_{r+1}\gamma_{r+2}\ldots\gamma_s)^{-1} \]
- \[ DA = U[(\gamma_{r+1})^{-1} + \ldots + (\gamma_{r+1}\gamma_{r+2}\ldots\gamma_s)^{-1}] \]
- \[ U = DA/[\gamma_{r+1}]^{-1} + \ldots + (\gamma_{r+1}\gamma_{r+2}\ldots\gamma_s)^{-1} \]
**Example 5.17  Multiple Rate Series and DA**

\[ \text{U} = \$1,000, \text{DA} \]

\[ = \$3,427.19 = 1,000 \left[ (1.06)^{-1} + (1.06 \times 1.065)^{-1} + (1.06 \times 1.065 \times 1.07)^{-1} + (1.06 \times 1.065 \times 1.07 \times 1.075)^{-1} \right] \]

**Step 2**

\[ \text{DA} = \$2,000, \text{U} \]

\[ = \$583.56 = 2,000 / \left[ (1.06)^{-1} + (1.06 \times 1.065)^{-1} + (1.06 \times 1.065 \times 1.07)^{-1} + (1.06 \times 1.065 \times 1.07 \times 1.075)^{-1} \right] \]

**Regions with Constant Rates**

- Discount and compound up to boundaries
- Equation relating series and DA or CA
- Solve equation for unknown

**Example 5.18  Series CA with Regions**

\[ E_7 = \text{U} (F | A, 5\%, 7-4) \]

\[ CA = E_7 (F | P, 6\%, 12-7) + \text{U} (F | A, 6\%, 12-7) \]

\[ CA = 9.8558\text{U} \]

**Step 2**

\[ CA = 9.8558\text{U} \]

- If \( \text{U} \) equals $1,000, then
  \[ CA = 9,855.80 = 9.8558 \times 1,000 \]
- If \( CA \) equals $20,000, then
  \[ \text{U} = 2,029.26 = 20,000 / 9.8558 \]

**Example 5.19  Series DA with Regions**

\[ E_7 = \text{U} (P | A, 6\%, 12-7) \]

\[ DA = E_7 (P | F, 5\%, 7-4) + \text{U} (P | A, 5\%, 7-4) \]

\[ DA = 6.3620\text{U} \]
### Step 2

\[ DA = 6.3620U \]

- If \( U \) equals $1,000, then
  \[ DA = 6,362.00 = 6.3620 \times 1,000 \]

- If \( DA \) equals $20,000, then
  \[ U = 3,143.67 = \frac{20,000}{6.3620} \]

### Au revoir

Making money is rough work!