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Asymptotics of orthogonal polynomials with asymptotic Freud-like weights

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Abstract

We derive uniform asymptotic expansions for polynomials orthogonal with respect to a class of weight functions that are real analytic and behave asymptotically like the Freud weight at infinity. Although the limiting zero distributions are the same as in the Freud cases, the asymptotic expansions are different due to the fact that the weight functions may have a finite or infinite number of zeros on the imaginary axis. To resolve the singularities caused by these zeros, an auxiliary function is introduced in the Riemann–Hilbert analysis. Asymptotic formulas are established in several regions covering the whole complex plane. We take the continuous dual Hahn polynomials as an example to illustrate our main results. Some numerical verifications are also given.

KEYWORDS

asymptotic approximation, asymptotic Freud-like weight, continuous dual Hahn polynomials, Riemann–Hilbert problem

1 | INTRODUCTION

In the study of a multiple zeta values identity $\zeta(2,1)=\zeta(3)$, where $\zeta(s_1,s_2,\cdots,s_l)$ is the multiple zeta value summing the products $n_1^{-s_1}\cdots n_l^{-s_l}$ for all $n_1>n_2>\cdots>n_l\ge 1$, Zudilin considered the following "biorthogonally looking" polynomial[1]:

$$B_n^{\alpha}(t)=\frac{1}{n!}\sum_{k=0}^{n}\frac{(\omega t)_k(\omega^2 t)_k(\alpha+t)_{n-k}(\alpha-t+k)_{n-k}}{k!(n-k)!},$$

where $\omega = e^{2i\pi/3}$ and the Pochhammer symbol is defined as $(\alpha)_k = \prod_{j=0}^{k-1}(\alpha + j)$. Zudilin further showed that $B_n^\alpha(t)$ satisfies the recurrence relation:

$$[(n+\alpha)^3 - t^3]B_n^\alpha(t) - (n+1)[2n^2 + 3n(\alpha+1) + \alpha^2 + 3\alpha + 1]B_{n+1}^\alpha(t)$$

$$+ (n+2)^2(n+1)B_{n+2}^\alpha(t) = 0, \qquad n \geq 0,$$

with initial conditions $B_0^\alpha(t) = 1$ and $B_1^\alpha(t) = \alpha^2$, from which it is obvious that $B_n^\alpha(t)$ is a polynomial in both t and α. Indeed, we could express $B_n^\alpha(t)$ as

$$B_n^\alpha(t) = \frac{(-1)^n}{(n!)^2} S_n(3t^2/4; -t/2, \alpha - t/2, 1 - \alpha - n - t/2),$$

or

$$B_n^\alpha(t) = \frac{1}{(n!)^2} S_n(-\alpha^2/4; t + \alpha/2, \omega t + \alpha/2, \omega^2 t + \alpha/2),$$

where

$$\frac{S_n(x^2; a, b, c)}{(a+b)_n(a+c)_n} = {}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix} \middle| 1\right) \tag{1}$$

is the continuous dual Hahn polynomials in the Askey scheme; see [Ref. 2 (9.3.1)]. It is this relation between multiple zeta values and the continuous dual Hahn polynomials that motivates us to consider asymptotic properties of $S_n(x^2; a, b, c)$ for general a, b, and c.

One of the generating functions, denoted as $G(t, x)$, for the continuous dual Hahn polynomials is given by [Ref. 2 (9.3.14)]

$$G(t, x) := \sum_{n=0}^{\infty} \frac{S_n(x; a, b, c)}{(b+c)_n n!} t^n = (1-t)^{-a+ix} {}_2F_1\left(\begin{matrix} b+ix, c+ix \\ b+c \end{matrix} \middle| t\right).$$

Making use of the linear transformation for the hypergeometric function [Ref. 3 (2.9.33)], we rewrite the generating function as $G(t, x) = g(t, x) + g(t, -x)$ when $2ix \notin \mathbb{Z}$, where

$$g(t, x) = \frac{(1-t)^{-a+ix}\Gamma(-2ix)\Gamma(b+c)}{\Gamma(b-ix)\Gamma(c-ix)} {}_2F_1\left(\begin{matrix} b+ix, c+ix \\ 1+2ix \end{matrix} \middle| 1-t\right).$$

A direct application of Darboux's method [Ref. 4 (II.6.18)] gives the following asymptotic expansion for $2ix \notin \mathbb{Z}$,

$$S_n(x^2; a, b, c) \sim \sum_{k=0}^{\infty} [\varphi_{n,k}(x; a, b, c) + \varphi_{n,k}(-x; a, b, c)], \quad n \to \infty, \tag{2}$$

where

$$\varphi_{n,k}(x; a, b, c) = \frac{\Gamma(-2ix)\Gamma(n+b+c)\Gamma(n+a-ix)(b+ix)_k(c+ix)_k(1-a+ix)_k}{\Gamma(a-ix)\Gamma(b-ix)\Gamma(c-ix)(1+2ix)_k(1-n-a+ix)_k k!}.$$

Especially, the leading-term approximation is

$$S_n(x^2; a, b, c) \sim \frac{n^{a+b+c-ix}\Gamma(n)^2\Gamma(-2ix)}{\Gamma(a-ix)\Gamma(b-ix)\Gamma(c-ix)} + \frac{n^{a+b+c+ix}\Gamma(n)^2\Gamma(2ix)}{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}. \tag{3}$$

Note that the asymptotic formula is invalid when $2ix \in \mathbb{Z}$, because the function $\Gamma(\pm 2ix)$ has simple poles at these points. Applying the main results in[5] to the recurrence relation of the continuous dual Hahn polynomials,[2] we find that, when $2ix \in \mathbb{Z}$, the asymptotic formula includes a $\log n$ term which is different from (3). One of the objectives of this paper is to derive an asymptotic formula, which is valid uniformly in the neighborhood of the simple poles of $\Gamma(\pm 2ix)$.

To achieve this, we make use of the orthogonality relation of continuous dual Hahn polynomials [Ref. 2 (9.3.2)]:

$$\int_0^{+\infty} \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)} \right|^2 S_n(x^2; a, b, c) S_m(x^2; a, b, c) dx = \ell_n^2 \delta_{nm}, \tag{4}$$

where $\ell_n^2 = 2\pi\Gamma(n+a+b)\Gamma(n+b+c)\Gamma(n+c+a)n!$, and the three parameters a, b, and c are all positive except for a possible pair of complex conjugates with positive real parts. By symmetry and making use of $\Gamma(1+2ix) = 2ix\Gamma(2ix)$, we change the interval of integration to the whole real line,

$$\int_{-\infty}^{+\infty} \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(1+2ix)} \right|^2 \cdot [xS_n(x^2; a, b, c)] \cdot [xS_m(x^2; a, b, c)] dx = (\ell_n^2/2)\delta_{nm}.$$

Thus, we consider the monic polynomial $\pi_n(x; a, b, c)$ orthogonal with respect to the weight function

$$w(x; a, b, c) = \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(1+2ix)} \right|^2, \quad x \in \mathbb{R}. \tag{5}$$

In what follows, we omit the parameters a, b, c in the expressions if no is confusion caused. Note that $xS_n(x^2)$ is orthogonal to $xS_m(x^2)$ for all $m \neq n$. Moreover, by symmetry, $xS_n(x^2)$ is also orthogonal to x^{2k} for all $0 \leq k \leq n$. So, $xS_n(x^2)$ is a constant multiple of $\pi_{2n+1}(x)$. Because the leading term of $xS_n(x^2)$ is $(-1)^n x^{2n+1}$, we then obtain $S_n(x^2) = (-1)^n \pi_{2n+1}(x)/x$. A slight analysis of the weight function shows the following:

(H1) The weight function $w(x)$ is continuous and positive for $x \in \mathbb{R}$, and it can be decomposed as $w(z) = A(iz)A(-iz)$, where $A(z)$ is meromorphic in the complex z-plane and its poles are located in the half-strip $\{\operatorname{Re} z > 0, -\mu < \operatorname{Im} z < \mu\}$ for some $\mu > 0$. Moreover, the zeros of $A(z)$ (if any) are all simple and positive. If $A(z)$ has infinitely many zeros, ordered as $0 < p_1 < p_2 < \cdots$, then $1/(p_{k+1} - p_k) = \mathcal{O}(1)$ as $k \to \infty$.

(H2) There exists $\alpha \in \mathbb{R}$ and $\beta > 0$ such that

$$A(z) = \beta(e^{-i\pi} z)^{\frac{\alpha}{2}} \exp\{\Theta(z)\} \left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right), \quad z \to \infty \tag{6}$$

uniformly for $\arg z \in [\epsilon, 2\pi - \epsilon]$, where $\epsilon > 0$ is arbitrarily small. Here, $\Theta(z)$ is an analytic function in $\mathbb{C} \backslash [0, +\infty)$ and

$$\Theta(z) + \Theta(-z) = i\pi z, \quad \arg z \in (0, \pi). \tag{7}$$

(H3) Let α be the same as given in (6). There exists $C > 0$ such that

$$\left| \frac{1}{w(z)e^{\pi z}} \right| \leq C|z^{-\alpha}| \tag{8}$$

uniformly for all z in $\{z \in \mathbb{C} : |z| \geq 1, \operatorname{Re} z \geq 0\}$ and bounded away from the zeros of $w(z)$.

Remark 1. Although the weight function $w(z)$ in (5) is defined only on the real axis, it can be analytically continued to the complex plane except for its poles. Moreover, because $\Gamma(x+iy) = \overline{\Gamma(x-iy)}$, it follows that $w(z)$ satisfies (H1). Using Stirling's formula for the gamma function, one can show that $w(z)$ also fulfills (H2). Note that (H2) characterizes the behavior of the weight function $w(z)$ for z large and bounded away from the imaginary axis. When z is near the imaginary axis, $w(z)$ may have poles. Note that $w(z) = A(iz)A(-iz)$. Hence, $A(z)$ may have poles near the positive real axis. As a complement of (H2), we make a further assumption (H3), which, as we shall see later, plays an important role in the resolution of singularities. To check (H3), we need to use the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ and make use of the fact $\frac{\Gamma(z+a_1)}{\Gamma(z+a_2)} \sim z^{a_1-a_2}$ as $z \to \infty$ with $|\arg z| < \pi$. See Section 5 for the detailed proofs of these arguments and the explicit expressions of $A(iz)$ and $\Theta(iz)$ for the continuous dual Hahn case.

The three conditions (H1)–(H3) listed above are also satisfied by some other polynomials in the Askey scheme, such as the continuous Hahn polynomials and the Wilson polynomials. This suggests us to consider a class of orthogonal polynomials whose weight functions satisfy the above three conditions, and treat the continuous dual Hahn weight in (5) as a special case. Moreover, it is readily seen from (6) and (7) that

$$w(x) = \beta^2 |x|^\alpha e^{-\pi|x|} \left[1 + \mathcal{O}\left(\frac{1}{|x|}\right)\right], \quad x \to \pm\infty. \tag{9}$$

In view of this asymptotic formula, we call a weight function satisfying (H1), (H2), and (H3) as an *asymptotic Freud-like weight*. It should be mentioned these weights are analytic at $z = 0$ with some simple poles and zeros in the complex plane, while the Freud weight function $e^{-\pi|x|}$, $x \in \mathbb{R}$ is not analytic at $x = 0$. Nevertheless, the equilibrium measures for the Freud weight and the asymptotic Freud-like weight are the same.

To conduct asymptotic analysis of orthogonal polynomials with asymptotic Freud-like weight, we make use of the Riemann–Hilbert approach first developed by Deift and Zhou[6] for modified KdV equations, and further applied to orthogonal polynomials,[7-10] random matrix theory,[11-13] and other integrable systems.[14-16] In the previous study on orthogonal polynomials, the weight functions are usually zero-free in the complex plane. However, as we have seen from the asymptotic formula (3) for the continuous dual Hahn polynomials, singularities may occur near the zeros of the asymptotic Freud-like weight. Moreover, after a rescaling $x \mapsto nx$, these singularities tend to the origin when the polynomial degree gets large. A similar problem was considered in,[17,18] where asymptotic behaviors of Hankel determinant and recurrence coefficients were obtained by introducing an n-dependent contour separating the zeros and poles of the weight function from the origin. In this paper, we are interested in developing a uniform asymptotic formula of $\pi_n(nz)$ in an n-independent neighborhood of the origin. As we shall see later, this uniform result is stronger than the one obtained by the aforementioned technique. Especially, near the zeros of the weight function, the leading terms in the asymptotic expansions obtained by both techniques coincide, but our new technique provides an additional term of order $\mathcal{O}(\log n/n^{2p_1})$ together with an error estimate of order $\mathcal{O}(1/n^{2p_1})$ for some $p_1 > 0$; thus our new uniform formula is more accurate.

We should also mention that asymptotic approximations for typical examples of the *Freud-like weights* have been derived via difference equation methods, including continuous Hahn polynomials,[19] continuous dual Hahn polynomials,[20] and Wilson polynomials.[21] However, in the difference equation approach, the asymptotic formulas are usually valid only on the real axis. On the other hand, the valid region of results presented in this paper will cover the whole complex plane.

The rest of this paper is arranged as follows. In Section 2, we provide some preliminary results on the equilibrium measure and the Szegő function corresponding to the *asymptotic Freud-like weight*.

We also introduce an auxiliary function to cancel the singularities caused by the zeros of the weight function in the Riemann–Hilbert analysis. Then, we state our main results in Section 3. The detailed Riemann–Hilbert analysis and the proof of the main results are carried out in Section 4. The main tool is the nonlinear steepest descent approach of Deift and Zhou.[6] Near the turning points $z = \pm 1$, local parametrices are constructed in terms of the Airy functions in a way similar as that in.[22] In a neighborhood of $z = 0$, we eliminate the poles in the Riemann–Hilbert problem (RHP) and build a limit parametrix to approximate the original one. Finally, we apply our main results to the continuous dual Hahn polynomials in Section 5.

2 | PRELIMINARIES

2.1 | The equilibrium measure and related functions

For the potential $V(x) = \pi|x|$ with $x \in \mathbb{R}$, we consider the energy minimization problem associated with the external field $Q(x) = \pi|x|/2$, with the notation used in.[23] A simple calculation shows that the equilibrium measure is

$$d\mu(x) = \psi(x)dx, \quad \psi(x) = \frac{1}{\pi}\log\frac{1+\sqrt{1-x^2}}{|x|}, \quad x \in [-1,1]; \tag{10}$$

see Ref. 23 Chapter IV, Theorem 5.1. Note that the equilibrium measure has a logarithmic singularity at $x = 0$. The corresponding g-function is

$$\begin{aligned} g(z) &= \int_{-1}^{1} \log(z-x)d\mu(x) \\ &= -iz\log\left(\frac{\sqrt{z^2-1}+i}{z}\right) + \log\left(z+\sqrt{z^2-1}\right) + \frac{l}{2}, \end{aligned} \tag{11}$$

for $z \in \mathbb{C}\backslash(-\infty, 1]$, where $l = -2 - 2\log 2$ is the Lagrange multiplier. The branches* of $\log z$ and $(z \pm 1)^\alpha$ (with $\alpha \notin \mathbb{N}$) are chosen such that $\arg z \in (-\pi, \pi)$ and $\arg(z \pm 1) \in (-\pi, \pi)$.

It is easy to calculate that

$$g'(z) = -i\log\left(\frac{\sqrt{z^2-1}+i}{z}\right) = -i\log\left(\frac{1+i\sqrt{(-z)^2-1}}{i(-z)}\right)$$

for $\pm\operatorname{Im} z > 0$. Next, we define

$$\phi(z) := \int_1^z \left(-g'(s) + \frac{\pi}{2}\right)ds = iz\log\left(\frac{\sqrt{z^2-1}+i}{z}\right) - \log\left(z+\sqrt{z^2-1}\right) + \frac{\pi z}{2}, \tag{12}$$

*Throughout the paper, the branches of $\log z$ and z^α (with $\alpha \notin \mathbb{N}$) are chosen such that $\arg z \in (-\pi, \pi)$. Especially, $\sqrt{z^2-1} = (z+1)^{1/2}(z-1)^{1/2}$ is analytic in $\mathbb{C}\backslash[-1,1]$ such that $\sqrt{z^2-1} \sim z$ as $z \to \infty$. For $\pm\operatorname{Im} z > 0$, we have $-z = e^{\mp i\pi}z$ and $\sqrt{(-z)^2} = \sqrt{e^{\mp 2i\pi}z^2} = e^{\mp i\pi}z = -z$. Similarly, we note that

$$\sqrt{(-z)^2-1} = \sqrt{(-z-1)(-z+1)} = \sqrt{e^{\mp 2i\pi}(z+1)(z-1)} = -\sqrt{z^2-1},$$

$$\sqrt{1-z^2} = \sqrt{(1-z)(z+1)} = \sqrt{e^{\mp i\pi}(z-1)(z+1)} = \mp i\sqrt{z^2-1},$$

$$\sqrt{1-(-z)^2} = \mp i\sqrt{(-z)^2-1} = \mp i\sqrt{z^2-1}.$$

for $z \in \mathbb{C}\backslash(-\infty, 1]$, and

$$\widetilde{\phi}(z) := \phi(-z) = iz\log\left(\frac{1 + i\sqrt{(-z)^2-1}}{-z}\right) + \log\left(-z - \sqrt{(-z)^2-1}\right), \tag{13}$$

for $z \in \mathbb{C}\backslash[-1, \infty)$.

It follows from (11), (12), and (13) that

$$g_\pm(x) = \begin{cases} -ix\log\left(\frac{i\pm i\sqrt{1-x^2}}{x}\right) + \log\left(x \pm i\sqrt{1-x^2}\right) + \frac{l}{2}, & x \in (0,1) \\ -ix\log\left(\frac{1\pm\sqrt{1-(-x)^2}}{i(-x)}\right) - \log\left(-x \pm i\sqrt{1-(-x)^2}\right) \pm i\pi + \frac{l}{2}, & x \in (-1,0) \end{cases} \tag{14}$$

and

$$\phi_\pm(x) = \begin{cases} ix\log\left(\frac{1\pm\sqrt{1-x^2}}{x}\right) - \log\left(x \pm i\sqrt{1-x^2}\right), & x \in (0,1) \\ ix\log\left(\frac{1\pm\sqrt{1-(-x)^2}}{-x}\right) + \log\left(-x \pm i\sqrt{1-(-x)^2}\right), & x \in (-1,0). \end{cases} \tag{15}$$

Hence, we have

$$\phi_+(x) = -\phi_-(x) = -\tilde{\phi}_+(-x) = \tilde{\phi}_-(-x) \tag{16}$$

for all $x \in (0, 1)$. A combination of (14), (15), and (16) yields that

$$\phi(z) + g(z) = (l + \pi z)/2, \qquad z \in \mathbb{C}\backslash(-\infty, 1], \tag{17}$$

$$\tilde{\phi}(z) + g(z) = (l - \pi z)/2 \pm i\pi, \qquad \pm\,\mathrm{Im}\, z > 0. \tag{18}$$

The last two equations also lead to

$$\phi(z) - \tilde{\phi}(z) = \pi z \mp \pi i, \qquad \pm\,\mathrm{Im}\, z > 0. \tag{19}$$

Some other properties of $g(z)$, $\phi(z)$, and $\tilde{\phi}(z)$ are stated in the following lemma, which can be derived by the above arguments; see also Ref. 22 Prop. 3.6.

Lemma 1.

(g1) *As* $z \to \infty$, $e^{ng(z)}z^{-n} = 1 + \mathcal{O}(|z|^{-1})$.

(g2) $g_+(x) - g_-(x) = \begin{cases} 2i\pi, & x \le -1, \\ -2\phi_+(x), & -1 < x < 1. \end{cases}$

(g3) $g_+(x) + g_-(x) - \pi|x| - l = \begin{cases} -2\tilde{\phi}(x), & x \le -1, \\ 0, & -1 < x < 1, \\ -2\phi(x), & x \ge 1. \end{cases}$

(ϕ1) *As* $z \to 0$,

$$\phi(z) = iz\log(2/z) + iz - i\pi/2 + \mathcal{O}(|z|^3), \quad \mathrm{Im}\, z > 0. \tag{20}$$

(ϕ2) The local behavior of $\phi(z)$ as $z \to 1$ and $\tilde{\phi}(z)$ as $z \to -1$ is given as follows:

$$\phi(z) = \frac{2\sqrt{2}}{3}(z-1)^{\frac{3}{2}} + \mathcal{O}\big(|z-1|^2\big), \quad z \to 1;$$

$$\tilde{\phi}(z) = \frac{2\sqrt{2}}{3}(-z-1)^{\frac{3}{2}} + \mathcal{O}\big(|z+1|^2\big), \quad z \to -1. \tag{21}$$

(ϕ3) $\phi(ix) = -\log|x| \pm i\pi|x|/2 + \mathcal{O}(1)$ *as* $x \to \pm\infty$.

2.2 | The Szegő function

We consider the so-called Szegő function $D(z,n)$, which satisfies the following RHP:

(D1) $D(z,n)$ is analytic in $\mathbb{C}\backslash[-1,1]$ and has at most weak (*i.e.* integrable) singularities at ± 1;
(D2) $D_+(x,n)D_-(x,n) = w(nx)e^{n\pi|x|}$ for $x \in (-1,1)$;
(D3) $0 < D(\infty,n) < \infty$.

The explicit solution of this RHP is given by

$$D(z,n) := \exp\left\{ \frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log w(nx) + n\pi|x|}{\sqrt{1-x^2}} \frac{dx}{z-x} \right\}, \quad z \in \mathbb{C}\backslash[-1,1]. \tag{22}$$

Lemma 2. *The Szegő function $D(z,n)$ and its limit value $D(\infty,n)$ both depend on n and*

(i) as $n \to \infty$, the following two approximations

$$D(z,n) = h(\pm nz)e^{-\frac{\alpha}{2}\log(z+\sqrt{z^2-1}) \pm \frac{\alpha\pi i}{4}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \pm \operatorname{Im} z > 0, \tag{23}$$

hold uniformly for all $z \in \mathbb{C}\backslash\mathbb{R}$, where $h(z)$ is defined by

$$h(z) = A(iz)\exp\left\{-\Theta_0(iz)\right\}, \tag{24}$$

with $\Theta_0(z)$ being the regular part of $\Theta(z)$; namely, $\Theta_0(z)$ is an analytic function in $\mathbb{C}\backslash[0,+\infty)$ with at most a weak singularity at $z=0$ and it satisfies

$$\begin{cases} \Theta_0(z) + \Theta_0(-z) = \pi i z, & \arg z \in (0,\pi) \\ \Theta(z) - \Theta_0(z) = \mathcal{O}\left(\frac{1}{|z|}\right) & \text{as } z \to \infty. \end{cases} \tag{25}$$

Moreover, (23) *is also valid for the upper and lower edges of the real axis in the sense of taking limits from upper and lower sides, respectively.*

(ii) $D(\infty,n) = 2^{-\frac{\alpha}{2}}\beta n^{\frac{\alpha}{2}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$ *as* $n \to \infty$.

Remark 2. Because (6) is only the asymptotic expansion of $A(z)$ for large z, the choice of $\Theta(z)$ is not unique and $\Theta(z)$ may have a (nonintegrable) singularity at $z=0$. In Lemma 2, we set $\Theta_0(z)$ to be the regular part of $\Theta(z)$. Especially, if $\Theta(z)$ is integrable near $z=0$, then $\Theta_0(z) = \Theta(z)$.

The proof of Lemma 2 is left to Appendix A. Using the definition of $h(z)$ in (24) and the asymptotic behavior of $A(z)$ in (6), we find that, when $n \to \infty$ and z bounded away from the origin, the two approximations in (23) are asymptotically equal to each other, and

$$D(z,n) = \beta(nz)^{\frac{\alpha}{2}}(z+\sqrt{z^2-1})^{-\frac{\alpha}{2}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{26}$$

Combining the asymptotic behavior of $A(iz)$ in (6), the symmetry condition (7) and Lemma 2, we conclude the following corollary, which will be used later.

Corollary 1. *As $n \to \infty$ with $nz \gg 1$*

$$\frac{D(z,n)^2}{w(nz)e^{n\pi z}} = \left(z+\sqrt{z^2-1}\right)^{-\alpha}(1+o(1)) \tag{27}$$

holds uniformly for $\arg z \in [-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon]$*, where $\epsilon > 0$ is given in (H2).*

2.3 | Resolution of singularities

Note that the poles of $1/w(z)$ on the imaginary axis correspond to the zeros of $A(z)$ and $A(-z)$ on the real line. Denote by $0 < p_1 < p_2 < \cdots < p_K$ the zeros of $A(z)$, where $K = \infty$ if $A(z)$ has infinitely many zeros. From (19), it is evident that

$$W(z,n) := \frac{D(z,n)^2 e^{2n\phi(z)}}{w(nz)e^{n\pi z}} = \frac{D(z,n)^2 e^{2n\tilde{\phi}(z)}}{w(nz)e^{-n\pi z}} \tag{28}$$

for all $\operatorname{Im} z \neq 0$. Moreover, from (19), it can be further derived that $W_+(x,n) = W_-(x,n)$ for all $x \in (-\infty,-1) \cup (1,+\infty)$. This implies that $W(z,n)$ is analytic in $\mathbb{C}\backslash[-1,1]$ except for the simple poles $\{ip_k/n, k = \pm1, \pm2, \cdots, \pm K\}$ on the imaginary axis, where $p_{-k} = -p_k$ for $k \geq 1$. To cancel the pole singularities of $W(z,n)$ in the Riemann–Hilbert analysis, we introduce

$$Q(z,n) := \sum_{k=1}^{K}\left[\frac{r_k}{z-z_k}+\frac{r_{-k}}{z-z_{-k}}\right], \quad z_{\pm k} = \frac{-ip_{\pm k}}{n} = \frac{\mp ip_k}{n}, \tag{29}$$

where

$$r_{\pm k} = \operatorname*{Res}_{z=z_{\pm k}} W(z,n) = \frac{1}{n}D(z_{\pm k},n)^2 e^{2n\phi(z_{\pm k})}\operatorname*{Res}_{\eta=-ip_{\pm k}}\left(\frac{1}{w(\eta)e^{\pi\eta}}\right). \tag{30}$$

Lemma 3. *For any fixed n, $Q(z,n)$ is well defined for any $z \in \mathbb{C}\backslash\{z_k, k = \pm1, \pm2, \cdots\}$ and*

$$Q(z,n) = \mathcal{O}\left(\frac{1}{n^{2p_1}}\right) \qquad as \quad n \to \infty \tag{31}$$

uniformly for all z satisfying $\inf\{|nz \pm ip_k|, k = 1,2,\cdots\} \geq d$*, where $d > 0$ is fixed but arbitrarily small.*

Although $Q(z,n)$ and $W(z,n)$ both have poles lying on the imaginary axis, $W(z,n) - Q(z,n)$ is analytic in $\mathbb{C}\backslash(-\infty,1]$. Moreover, we have the following lemma.

Lemma 4. *Let*

$$\Omega_\theta^\pm = \left\{\frac{\tilde{c}}{n} \leq |\operatorname{Im} z| \leq \tilde{M};\ \pm\frac{\pi}{2}-\theta \leq \arg z \leq \pm\frac{\pi}{2}+\theta\right\},$$

where θ satisfies $\epsilon < \theta < \frac{\pi}{2}$, $\tilde{M}$ is any positive constant, $\tilde{c} \neq p_k, k = 1, 2, \cdots$ fixed. Then, $W(z,n) - Q(z,n)$ is analytic for $z \in \Omega_\theta^+ \cup \Omega_\theta^-$ and

$$W(z,n) - Q(z,n) = \mathcal{O}\left(\frac{1}{n^{2\tilde{c}}}\right), \quad z \in \Omega_\theta^+ \cup \Omega_\theta^-, \tag{32}$$

as $n \to \infty$.

When z lies in a $o(\frac{1}{n})$ neighborhood of $z_{\pm 1} = \frac{\mp i p_1}{2n}$, the nearest two poles of $W(z,n)$, $W(z,n) - Q(z,n)$ is also uniformly small. However, the error bound should be modified when $p_1 \leq \frac{1}{2}$.

Lemma 5. *For any n, the two limit values $\lim_{z \to i p_{\pm 1}/n} (W(z,n) - Q(z,n)) = B_n^\pm$ both exist. Moreover, if $p_1 \leq \frac{1}{2}$, we have*

$$B_n^\pm = \frac{ih(\pm i p_1)^2}{w'(\pm i p_1) e^{\pm \pi i p_1}} \frac{\log n}{n^{2p_1}} + \mathcal{O}\left(\frac{1}{n^{2p_1}}\right), \quad n \to \infty. \tag{33}$$

The proof of Lemmas 3, 4, and 5 are left to Appendices B, C, and D, respectively.

3 | MAIN RESULTS

Using the Riemann–Hilbert analysis, we obtain the uniform asymptotics of polynomials orthogonal with respect to $w(z)$ satisfying the three assumptions (H1)–(H3). As illustrated in Figure 1, we divide the complex plane into several regions. All the curves in Figure 1 are allowed to make slight deformations. To make our results concise, we do not list the asymptotics in Ω_{in}^l and $\tilde{U}_\delta$, which can be derived from the results in Ω_{in}^r and U_δ, respectively. This is due to the fact that the weight function is even on $\mathbb{R}$, and the polynomials satisfy the symmetry relation $\pi_n(z) = (-1)^n \pi_n(-z)$.

For convenience, we denote

$$\gamma_n = 2^{-n-\frac{\alpha+1}{2}} e^{-n} n^{n+\frac{\alpha}{2}} \beta \tag{34}$$

and

$$c = \min\{1/2, p_1\}. \tag{35}$$

When z lies outside the neighborhood of the origin, we have the following results.

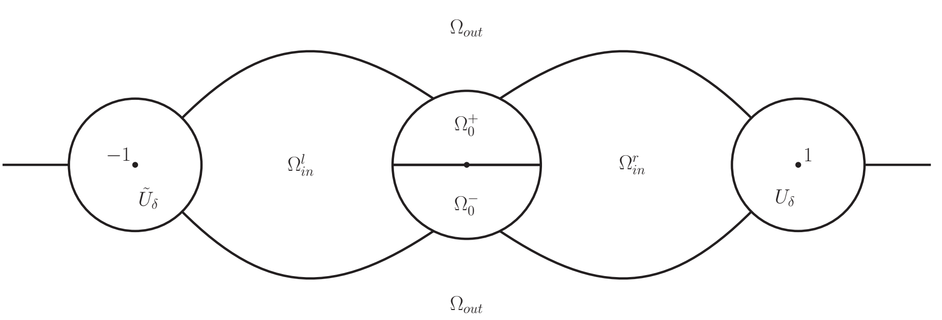

FIGURE 1 Regions for uniform asymptotic approximations of $\pi_n(z)$

Theorem 1. *Let $\phi(z)$ be defined in* (12). *As $n \to \infty$, we have*

(i) for $z \in \Omega_{out}$,

$$\pi_n(nz) = \frac{\gamma_n e^{\frac{n\pi z}{2} - n\phi(z)}}{\beta(nz)^{\frac{\alpha}{2}}} \frac{(z + \sqrt{z^2-1})^{\frac{\alpha+1}{2}}}{(z^2-1)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right); \tag{36}$$

(ii) for $z \in \Omega_{in}^r$,

$$\begin{aligned} \pi_n(nz) = \frac{\gamma_n e^{\frac{n\pi z}{2}}}{\beta(nz)^{\frac{\alpha}{2}}} \Bigg[& e^{-n\hat{\phi}(z) - \frac{\pi i}{4}} \frac{(z + i\sqrt{1-z^2})^{\frac{\alpha+1}{2}}}{(1-z^2)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) \\ & + e^{n\hat{\phi}(z) + \frac{\pi i}{4}} \frac{(z - i\sqrt{1-z^2})^{\frac{\alpha+1}{2}}}{(1-z^2)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) \Bigg], \end{aligned} \tag{37}$$

where $\hat{\phi}(z)$ is analytic on $\mathbb{C} \backslash ((-\infty, 0] \cup [1, +\infty))$ with $\hat{\phi}(x) = \phi_+(x)$ for all $x \in (0, 1)$;

(iii) for $z \in U_\delta$

$$\begin{aligned} \pi_n(nz) = \frac{2\sqrt{\pi}\gamma_n e^{\frac{n\pi z}{2}}}{\beta(nz)^{\frac{\alpha}{2}}} \Bigg\{ & \frac{\cos\left[\frac{\alpha+1}{2} \arccos z\right]}{(z^2-1)^{\frac{1}{4}}} \zeta^{\frac{1}{4}} \mathrm{Ai}(\zeta) \left(1 + \mathcal{O}\left(\tfrac{1}{n^{2c}}\right)\right) \\ & - \frac{i \sin\left[\frac{\alpha+1}{2} \arccos z\right]}{(z^2-1)^{\frac{1}{4}}} \zeta^{-\frac{1}{4}} \mathrm{Ai}'(\zeta) \left(1 + \mathcal{O}\left(\tfrac{1}{n^{2c}}\right)\right) \Bigg\}; \end{aligned} \tag{38}$$

where $\zeta = [\frac{3}{2} n\phi(z)]^{\frac{2}{3}}$ is a conformal mapping near $z = 1$.

Remark 3. The formula (37) can be written in terms of trigonometric functions, which implies that the polynomials $\pi_n(nx)$ have zeros in Ω_{in}. Indeed, all zeros of $\pi_n(nx)$ locate in the interval $[-1, 1]$. See the limiting zero distribution $d\mu$ in (10).

Note that, in the above three formulas, there is a common factor $e^{\frac{n\pi z}{2}} \beta(nz)^{-\frac{\alpha}{2}}$, which, according to (H1) and (H2), is asymptotically equal to $w(nz)^{-1/2}$ when n is large. It is usual that such a factor appears in asymptotic expansions of orthogonal polynomials; for example, see Ref. 9 Theorem 2.2.

As discussed in the previous section, due to the zeros of the weight function on the imaginary axis, some special considerations need to be taken when z is in the neighborhood of the origin. When $z \in \Omega_0$, we have the following result.

Theorem 2. *Let γ_n, c and $Q(z, n)$ be given in* (34), (35)*, and* (29)*. respectively. Then, as $n \to \infty$, we have*

$$\pi_n(nz) = \gamma_n \left[\Phi_1(z, n) \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) + \left(\Phi_2(z, n) - \Phi_3(z, n) Q(z, n)\right) \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) \right] \tag{39}$$

uniformly for $z \in \Omega_0^+ \cup \Omega_0^-$, where $\Phi_1(z, n)$, $\Phi_2(z, n)$, and $\Phi_3(z, n)$ are defined by

$$\Phi_1(z, n) = \frac{e^{ng(z) - \frac{nl}{2}}}{D(z, n)} \frac{(z + \sqrt{z^2-1})^{\frac{1}{2}}}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}, \qquad z \in \Omega_0^+ \cup \Omega_0^-, \tag{40}$$

$$\Phi_2(z,n) = \pm \frac{D(z,n)e^{ng(z)-\frac{nl}{2}+2n\phi(z)}}{w(nz)e^{n\pi z}} \frac{i(z+\sqrt{z^2-1})^{-\frac{1}{2}}}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}, \quad z \in \Omega_0^{\pm}, \tag{41}$$

and

$$\Phi_3(z,n) = \pm \frac{e^{ng(z)-\frac{nl}{2}}}{D(z,n)} \frac{i(z+\sqrt{z^2-1})^{-\frac{1}{2}}}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}, \qquad z \in \Omega_0^{\pm}, \tag{42}$$

and $g(z)$, $\phi(z)$, and $D(z,n)$ are given in (11), (12), *and* (22), *respectively.*

Remark 4. Both $\Phi_2(z,n)$ and $Q(z,n)$ possess poles at the zeros of $w(nz)$ in $\Omega_0^{\pm}$; see (29) and (41). Nevertheless, according to Lemma 4, the difference $\Phi_2(z,n) - \Phi_3(z,n)Q(z,n) = \Phi_3(z,n)[W(z,n) - Q(z,n)]$ is indeed analytic in Ω_0^+ and Ω_0^-.

By (H1), the poles of $w(nz)$ are all in the strip $\{|\operatorname{Im} z| > 0, -\mu < \operatorname{Re}(nz) < \mu\}$ for some $\mu > 0$. At these poles, we have $\Phi_2(z,n) = 0$, while $\Phi_1(z,n)$ and $\Phi_3(z,n)$ are nonzero and of the same order as $n \to \infty$. Due to Lemma 3, we obtain

$$\pi_n(nz) \sim \gamma_n \frac{e^{ng(z)-\frac{nl}{2}}}{D(z,n)} \frac{(z+\sqrt{z^2-1})^{\frac{1}{2}}}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}, \tag{43}$$

when $1/w(nz) = 0$ and $z \in \Omega_0^+ \cup \Omega_0^-$. This implies that the poles of $w(nz)$ do not play a significant role in the asymptotic properties of the polynomials.

Remark 5. Let us take a look at how the asymptotic expansions change from one region to another. When z is on the real axis, $\Phi_1(z,n)$ and $\Phi_2(z,n)$ are of the same order, while $\Phi_3(z,n)Q(z,n)$ is asymptotically smaller than $\Phi_1(z,n)$ and $\Phi_2(z,n)$; see Lemma 3. Actually, $\Phi_3(z,n)Q(z,n)$ is of the same order of $\Phi_1(z,n) + \Phi_2(z,n)$ only when z is close to the zeros of the weight function. Therefore, when z moves to Ω_{in}, the expansion (39) becomes (37). When z moves from the real axis to the upper half plane, $\Phi_2(z,n) - \Phi_3(z,n)Q(z,n)$ is asymptotically smaller than $\Phi_1(z,n)$; see Lemma 4. Then, when z moves to Ω_{out}, the expansion (39) reduces to (36). A similar phenomenon happens when z moves from the real axis to the lower half plane.

Remark 6. One may follow the ideas in[17,18] to introduce an n-dependent contour separating the zeros of $w(z)$ from the origin and then obtain asymptotic formulas of $\pi_n(nz)$ near the origin. However, that formula is weaker and less accurate than (39). For instance, when z is close to the zeros of $w(z)$, the technique in[17,18] only gives the leading term $\gamma_n\Phi_1(z,n)$, while our formula includes an additional term $\gamma_n[\Phi_2(z,n) - \Phi_3(z,n)Q(z,n)] = \gamma_n\Phi_3(z,n)[W(z,n) - Q(z,n)]$, which, according to Lemma 5, (40) and (42), is of order $|\Phi_1(z,n)| \cdot \mathcal{O}(\log n/n^{2p_1})$ under the condition that $p_1 \le 1/2$.

The asymptotics in Theorems 1 and 2 are for $\pi_n(nz)$, the orthogonal polynomials with scaled variable. For the nonscaled case $\pi_n(z)$, the corresponding asymptotics can be obtained from Theorem 2 by letting $z \to \frac{z}{n}$. This is summarized in the following corollary.

Corollary 2. *Let γ_n and $h(z)$ be defined in* (34) *and* (24)*, and* $0 < \epsilon < \theta < \frac{\pi}{2}$. *Then, as $n \to \infty$, we have for fixed z that,* *(i) if* $|\operatorname{Im} z| < p_1$ *or* $|\arg z| \le \frac{\pi}{2} - \epsilon$*, then*

$$\pi_n(z) = \gamma_n \left[\frac{e^{\frac{n\pi i}{2} - iz + iz\log\frac{z}{2n}}}{h(z)e^{-\frac{\pi z}{2}}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) + \frac{e^{-\frac{n\pi i}{2} + iz - iz\log\frac{z}{2n}}}{h(-z)e^{-\frac{\pi z}{2}}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right)\right], \tag{44}$$

especially, for all $\pm x \ge 0$*,*

$$\pi_n(x) = 2\gamma_n w(x)^{-\frac{1}{2}} \cos\left[\pm x \mp x\log\frac{\pm x}{2n} \mp \frac{n\pi}{2} + \arg h(\pm x)\right] + \mathcal{O}\left(\frac{1}{n^{2c}}\right); \tag{45}$$

(ii) if $|\operatorname{Im} z| > p_1$ *and* $|\arg z \pm \frac{\pi}{2}| < \theta$*, then*

$$\pi_n(z) = \gamma_n \frac{e^{\frac{n\pi i}{2} - iz + iz\log z - iz\log n - iz\log 2}}{h(z)e^{-\frac{\pi z}{2}}}\left[1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right], \quad \text{if } \operatorname{Im} z > p_1, \tag{46}$$

$$\pi_n(z) = \gamma_n \frac{e^{-\frac{n\pi i}{2} + iz - iz\log z + iz\log n + iz\log 2}}{h(-z)e^{-\frac{\pi z}{2}}}\left[1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right], \quad \text{if } \operatorname{Im} z < -p_1; \tag{47}$$

(iii) if $z = ip_{\pm 1}$ *and* $|p_{\pm 1}| > \frac{1}{2}$*, then* (46) *and* (47) *also hold;*

(iv) if $z = ip_{\pm 1}$ *and* $|p_{\pm 1}| \le \frac{1}{2}$*, then*

$$\pi_n(z) = \gamma_n \frac{e^{\pm\frac{n\pi i}{2} \mp iz \pm iz\log\frac{z}{2n}}}{h(\pm z)e^{-\frac{\pi z}{2}}}\left(1 + \frac{ih(\pm ip_1)^2}{w'(\pm ip_1)e^{\pm\pi i p_1}}\frac{\log n}{n^{2p_1}} + \mathcal{O}\left(\frac{1}{n^{2p_1}}\right)\right). \tag{48}$$

Remark 7. It should be noted that if $A(z)$ has no zero, the function $Q(z,n)$ is regarded as a zero function and $c = \frac{1}{2}$.

4 | RH ANALYSIS AND PROOF OF THE MAIN RESULTS

In this section, the nonlinear steepest descent method for RHPs is applied to derive the uniform asymptotics of the monic orthogonal polynomial $\pi_n(z)$ with respect to a weight function $w(z)$ on $\mathbb{R}$. which satisfies the three assumptions (H1), (H2) and (H3). Because some of the analyses are similar to those of the Freud weight, we omit some details in the proof and refer the readers to.[22]

The analysis starts from the following RHP for a 2×2 matrix valued function $Y(z)$:

(Y1) $Y(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$;

(Y2) $Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for any $x \in \mathbb{R}$;

(Y3) $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

By the well-known theorem of Fokas, Its, and Kitaev,[24] the unique solution of the above RHP is given by

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\pi_n(t)w(t)}{t-z}dt \\ -2\pi i \ell_{n-1}^2 \pi_{n-1}(z) & -\ell_{n-1}^2 \int_{\mathbb{R}} \frac{\pi_{n-1}(t)w(t)}{t-z}dt \end{pmatrix}, \tag{49}$$

where $\pi_n(z) = z^n + \cdots$ denotes the n-th monic orthogonal polynomial with respect to the measure $w(x)dx$ on $\mathbb{R}$, and $\ell_n > 0$ denotes the leading coefficient of the n-th orthonormal polynomial $p_n(z) = \ell_n \pi_n(z)$.

4.1 | The scaling and normalization, $Y(z) \to U(z) \to T(z)$

The first transformation $Y(z) \to U(z)$ is a scaling transformation. Define $U(z) = n^{-n\sigma_3} Y(nz)$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, $U(z)$ satisfies the following RHP:

(U1) $U(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$;

(U2) $U_+(x) = U_-(x) \begin{pmatrix} 1 & w(nx) \\ 0 & 1 \end{pmatrix}$ for any $x \in \mathbb{R}$;

(U3) $U(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

To normalize condition (U3) in the RHP for U, we introduce the second transformation, $U(z) \to T(z)$, as follows,

$$T(z) = D(\infty, n)^{-\sigma_3} e^{-\frac{l}{2}n\sigma_3} U(z) e^{-n(g(z)-\frac{l}{2})\sigma_3} D(z,n)^{\sigma_3}, \qquad z \in \mathbb{C}\backslash\mathbb{R},$$

where $g(z)$, $D(z,n)$, and $D(\infty, n)$ are defined in Section 2. Then, according to (16) and Lemma 1 and the fact $D_+(x,n)D_-(x,n) = w(nx)e^{n\pi|x|}$ for all $x \in (-1,1)$, we see that $T(z)$ satisfies the following RHP:

(T1) $T(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$;

(T2)

$$T_+(x) = T_-(x) \begin{cases} \begin{pmatrix} \frac{D_+(x,n)}{D_-(x,n)} e^{2n\phi_+(x)} & 1 \\ 0 & \frac{D_-(x,n)}{D_+(x,n)} e^{2n\phi_-(x)} \end{pmatrix}, & x \in (0,1), \\ \begin{pmatrix} \frac{D_+(x,n)}{D_-(x,n)} e^{2n\tilde{\phi}_+(x)} & 1 \\ 0 & \frac{D_-(x,n)}{D_+(x,n)} e^{2n\tilde{\phi}_-(x)} \end{pmatrix}, & x \in (-1,0), \\ \begin{pmatrix} 1 & \frac{1}{W(x,n)} \\ 0 & 1 \end{pmatrix}, & x > 1, \\ \begin{pmatrix} 1 & \frac{1}{W(x,n)} \\ 0 & 1 \end{pmatrix}, & x < -1; \end{cases} \tag{50}$$

(T3) $T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

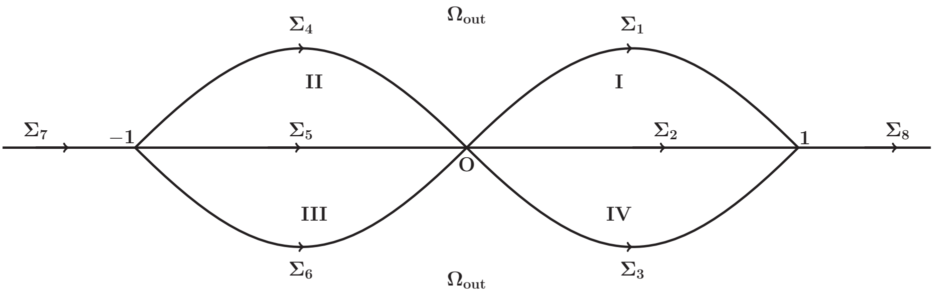

FIGURE 2 The jump contour Σ_S

4.2 | The steepest descent factorization, $T(z) \to S(z)$

Because the jump matrix of $T(z)$ on $(-1, 1)$ is rapid oscillatory, the steepest descent factorization is needed. Define

$$S(z) = \begin{cases} T(z), & z \in \Omega_{\text{out}} \\ T(z)\begin{pmatrix} 1 & 0 \\ -W(z,n) & 1 \end{pmatrix}, & z \in \mathrm{I} \cup \mathrm{II}, \\ T(z)\begin{pmatrix} 1 & 0 \\ W(z,n) & 1 \end{pmatrix}, & z \in \mathrm{III} \cup \mathrm{IV}, \end{cases} \tag{51}$$

where $W(z, n)$ is defined in (28), I, II, III, IV, and Ω_{out} are regions described in Figure 2. Then, $S(z)$ satisfies the following RHP.

(S1) $S(z)$ is analytic in $\mathbb{C}\backslash\Sigma_S$, where $\Sigma_S = \bigcup_{k=1}^{8} \Sigma_k$;

(S2) $S(z)$ has the following jump conditions on Σ_S

$$S_+(z) = S_-(z)\begin{cases} \begin{pmatrix} 1 & 0 \\ W(z,n) & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_6, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Sigma_2 \cup \Sigma_5, \\ \begin{pmatrix} 1 & \frac{1}{W(z,n)} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_7, \\ \begin{pmatrix} 1 & \frac{1}{W(z,n)} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_8; \end{cases} \tag{52}$$

(S3) $S(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

4.3 | The parametrix for the outer region

According to (27) and the facts that $\operatorname{Re}\phi(z) < 0$ ($\operatorname{Re}\tilde{\phi}(z) < 0$) on Σ_1, Σ_3 (Σ_4, Σ_6) and $\operatorname{Re}\phi(z) > 0$ ($\operatorname{Re}\tilde{\phi}(z) > 0$) on Σ_8 (Σ_7), we know that the jump matrices in (52) tend to the identity matrix as $n \to \infty$ when $z \in \mathbb{C}\backslash[-1,1]$. Hence, we expect that, in the outer region Ω_{out}, the asymptotic behaviors of $S(z)$ can be approximated by a matrix valued function $N(z)$ which satisfies the following RHP:

(N1) $N(z)$ is analytic in $\mathbb{C}\backslash[-1,1]$ and only has weak singularities at ± 1;

(N2) For any $x \in (-1,1)$, $N_+(x) = N_-(x)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

(N3) $N(z) = I + \mathcal{O}(\frac{1}{z})$ as $z \to \infty$.

According to Ref. 25 Prop. 5.2, the solution of the above RHP is given by

$$N(z) = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{a(z) + a(z)^{-1}}{2} & \dfrac{a(z) - a(z)^{-1}}{2i} \\ \dfrac{a(z) - a(z)^{-1}}{-2i} & \dfrac{a(z) + a(z)^{-1}}{2} \end{pmatrix}, \tag{53}$$

where $a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}$.

4.4 | A local parametrix near $z = 0$

According to (52), the jump matrices of $S(z)$ on $\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_6$ do not uniformly tend to identity when $z \to 0$. Moreover, the function $W(z,n)$ may have infinitely many poles on the imaginary axis. which accumulate at the origin as $n \to \infty$. Hence, to derive the uniform asymptotic behavior of the RHP for $S(z)$, we introduce a new function $Q(z,n)$—see (29)—to eliminate the residues of the poles of $W(z,n)$ and construct a local parametrix near $z = 0$. By a careful analysis, we find that it is not necessary to construct the exact parametrix using special functions, instead an asymptotic parametrix would serve the purpose.

Denote $\Omega_0 = \{|z| \leq \rho\}$ and let U_0 be a neighborhood of Ω_0, where $\rho > 0$ is any fixed constant. Moreover, we divide Ω_0 into six regions Ω_{0j}, $j = 1, \dots, 6$, whose boundaries are denoted by Γ_j and Σ_j; see Figure 3.

Set

$$P^{(0)}(z) = \begin{cases} N(z)\begin{pmatrix} 1 & 0 \\ W(z,n) - Q(z,n) & 1 \end{pmatrix}, & z \in \Omega_{02}; \\ N(z)\begin{pmatrix} 1 & 0 \\ -Q(z,n) & 1 \end{pmatrix}, & z \in \Omega_{01} \cup \Omega_{03}; \\ N(z)\begin{pmatrix} 1 & 0 \\ Q(z,n) & 1 \end{pmatrix}, & z \in \Omega_{04} \cup \Omega_{06}; \\ N(z)\begin{pmatrix} 1 & 0 \\ -W(z,n) + Q(z,n) & 1 \end{pmatrix}, & z \in \Omega_{05}. \end{cases} \tag{54}$$

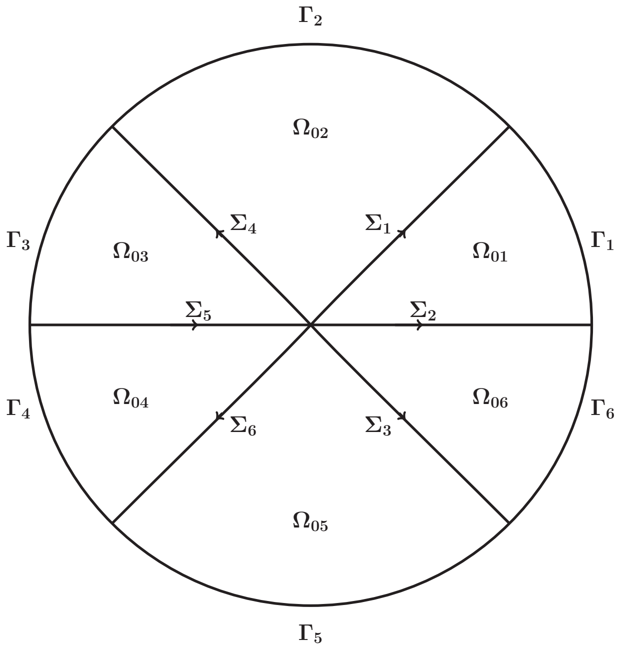

FIGURE 3 The jump contour Σ_p for $P^{(0)}(z)$

Then, $P^{(0)}(z)$ is analytic in $U_0 \backslash \Sigma_p$ and satisfies the following jump conditions,

$$P_+^{(0)}(z) = P_-^{(0)}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ W(z,n) & 1 \end{pmatrix}, & \text{if } z \in \left(\Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_6\right) \cap U_0, \\ \begin{pmatrix} -Q(z,n) & 1 \\ -1 + Q(z,n)^2 & -Q(z,n) \end{pmatrix}, & \text{if } z \in \left(\Sigma_2 \cup \Sigma_5\right) \cap U_0, \end{cases} \tag{55}$$

where $\Sigma_p = (\bigcup_{j=1}^6 \Gamma_j) \cup (\bigcup_{j=1}^6 (\Sigma_j \cap \Omega_0))$. Although the jump of $S(z)$ on $(\Sigma_2 \cup \Sigma_5) \cap \Omega_0$ cannot be completely canceled by the parametrix $P^{(0)}(z)$, it can be rigorously proved that $S(z)$ can be approximated by $P^{(0)}(z)$ as $n \to \infty$ uniformly for $z \in \Omega_0$. Indeed, when $z \in \Sigma_0$, the jump matrix of $S(z)P^{(0)}(z)^{-1}$ is

$$\begin{aligned} J_0 &= N_-(z) \begin{pmatrix} 1 & 0 \\ Q(z,n) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q(z,n) & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} N_-(z)^{-1} \\ &= N_-(z) \begin{pmatrix} 1 & -Q(z,n) \\ Q(z,n) & 1 - Q(z,n)^2 \end{pmatrix} N_-(z)^{-1}. \end{aligned}$$

It follows from Lemma 3 and (53) that

$$J_0 = I + \mathcal{O}\left(\frac{1}{n^{2p_1}}\right), \quad n \to \infty \tag{56}$$

uniformly for all $z \in \Sigma_0$. Furthermore, by Lemmas 3 and 4, we immediately have

$$P^{(0)}(z)N(z)^{-1} = I + \mathcal{O}\left(\frac{1}{n^{2p_1}}\right) \tag{57}$$

as $n \to \infty$ uniformly for $z \in \partial\Omega_0 = \bigcup_{j=1}^{6} \Gamma_j$.

4.5 | The local parametrix near $z = \pm 1$

Note that the jump matrix for $S(z)$ does not uniformly tend to the identity as $n \to \infty$ when $z \in \Sigma_k$, $k = 1, 3, 4, 6$ and near ± 1. This implies that $S(z)N(z)^{-1}$ does not uniformly tend to the identity in any neighborhood of ± 1. Hence, we need to construct two local parametrices near ± 1, respectively. For the sake of convenience, we denote a small but fixed neighborhood $\{z;\ |z-1| < \delta\}$ of $z = 1$ by U_δ, and the corresponding neighborhood of $z = -1$ by $\tilde{U}_\delta$.

First, we introduce a parametrix $P^{(1)}(z)$ near $z = 1$ by the following RHP:

(P1) $P^{(1)}(z)$ is analytic in the $U_{\delta_0} \backslash \Sigma$, where $\delta_0 > \delta$;

(P2) $P^{(1)}_{+}(z) = P^{(1)}_{-}(z) J^{(1)}_{P}(z)$ on $U_\delta \cap \Sigma$, where

$$J_{P^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ W(z,n) & 1 \end{pmatrix}, & \text{if } z \in U_\delta \cap (\Sigma_1 \cup \Sigma_3), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } z \in U_\delta \cap \Sigma_2, \\ \begin{pmatrix} 1 & \frac{1}{W(z,n)} \\ 0 & 1 \end{pmatrix}, & \text{if } z \in U_\delta \cap \Sigma_8. \end{cases} \tag{58}$$

(P3) On $\partial\Omega_0$, $P^{(1)}(z)N(z)^{-1} = I + \mathcal{O}(n^{-1})$ as $n \to \infty$.

Set $P^{(1)}(z) = P(z)\left(\frac{D(z,n)^2}{w(nz)e^{\pi nz}}\right)^{-\frac{\sigma_3}{2}}$, then $P_{+}(z) = P_{-}(z)J_P(z)$ on $U_\delta \cap \Sigma$, where

$$J_{P}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix}, & \text{if } z \in U_\delta \cap (\Sigma_1 \cup \Sigma_3), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{if } z \in U_\delta \cap \Sigma_2, \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix}, & \text{if } z \in U_\delta \cap \Sigma_8. \end{cases} \tag{59}$$

Denote $\omega = e^{2i\pi/3}$ and $\zeta = (\frac{3}{2}n\phi(z))^{\frac{2}{3}}$. We follow the arguments in Ref. 22 Sec. 5 to obtain $P(z) = E_n(z)\Psi(\zeta)e^{n\phi(z)\sigma_3}$, where

$$E_n(z) = N(z)\left(\frac{D(z,n)^2}{w(nz)e^{\pi nz}}\right)^{\frac{\sigma_3}{2}}\sqrt{\pi}e^{\frac{i\pi}{6}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\zeta^{\frac{1}{4}\sigma_3}, \tag{60}$$

and

$$\Psi(\zeta) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\mathrm{Ai}'(\omega^2) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}, & \text{if } \arg\zeta \in \left(0, \frac{2\pi}{3}\right), \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\mathrm{Ai}'(\omega^2) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{if } \arg\zeta \in \left(\frac{2\pi}{3}, \pi\right), \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega^2) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{if } \arg\zeta \in \left(-\pi, -\frac{2\pi}{3}\right), \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \mathrm{Ai}'(\omega^2) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3}, & \text{if } \arg\zeta \in \left(-\frac{2\pi}{3}, 0\right). \end{cases} \tag{61}$$

Lemma 6. *The function $E_n(z)$ given in* (60) *is an analytic function in U_δ, and*

$$P^{(1)}(z)N(z)^{-1} = I + \mathcal{O}(n^{-1}) \quad \text{as} \quad n \to \infty \tag{62}$$

uniformly for all $z \in \partial U_\delta$.

Proof. First, we show that $E_n(z)$ is analytic in U_δ. In fact, from its definition in (60), $E_n(z)$ is analytic in $U_\delta \backslash (-\infty, 1]$, and for $z \in (0,1)$

$$\begin{aligned} (E_n)_+(z) &= N_+(z)\left(\frac{w(nz)e^{n\pi z}}{D_+(z,n)^2}\right)^{\frac{\sigma_3}{2}}\sqrt{\pi}e^{\frac{i\pi}{6}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\zeta_+^{\frac{1}{4}\sigma_3} \\ &= N_-(z)\left(\frac{w(nz)e^{n\pi z}}{D_-(z,n)^2}\right)^{\frac{\sigma_3}{2}}\sqrt{\pi}e^{\frac{i\pi}{6}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\zeta_-^{\frac{1}{4}\sigma_3}\zeta_-^{-\frac{1}{4}\sigma_3}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}^{-1} \\ &\quad \times \left(\frac{w(nz)e^{n\pi z}}{D_-(z,n)^2}\right)^{-\frac{\sigma_3}{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\frac{w(nz)e^{n\pi z}}{D_+(z,n)^2}\right)^{\frac{\sigma_3}{2}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\zeta_+^{\frac{1}{4}\sigma_3} \\ &= (E_n)_-(z)\zeta_-^{-\frac{1}{4}\sigma_3}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\zeta_+^{\frac{1}{4}\sigma_3}. \end{aligned} \tag{63}$$

Recalling that $\phi(z) \sim \frac{2\sqrt{2}}{3}(z-1)^{\frac{3}{2}}$ as $z \to 1$ (see (21) in Lemma 1) and $\zeta = (\frac{3}{2}n\phi(z))^{\frac{2}{3}}$, we have $\zeta_+^{\frac{1}{4}} = i\zeta_-^{\frac{1}{4}}$ for $z \in (0,1)$. Substituting this into (63), we immediately get $(E_n)_+(z) = (E_n)_-(z)$ for $z \in (0,1)$. In addition, $N(z), \zeta^{\frac{1}{4}\sigma_3} = \mathcal{O}((z-1)^{-\frac{1}{4}})$ as $z \to 1$, then $E_n(z) = \mathcal{O}((z-1)^{-\frac{1}{2}})$. Therefore, we conclude that $z = 1$ is a removable singularity of $E_n(z)$.

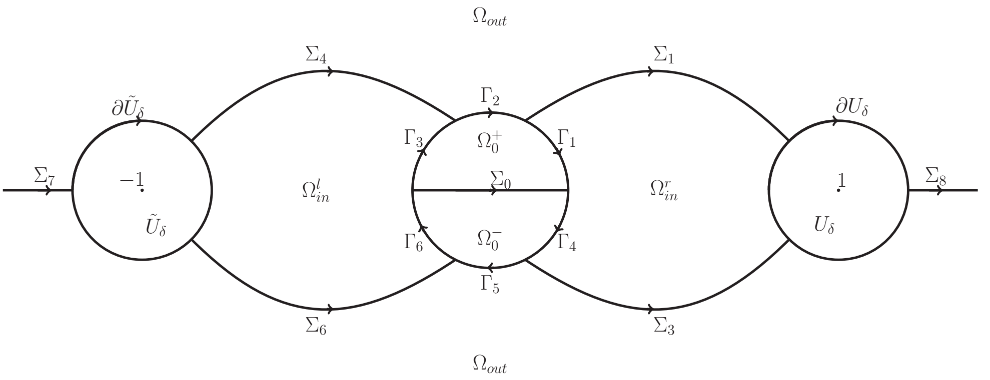

FIGURE 4 The reduced jump contour Σ_R for $R(z)$

Now we show that $P^{(1)}(z)N(z)^{-1} = I + \mathcal{O}(n^{-1})$ as $n \to \infty$ uniformly for all $z \in \partial U_\delta$. According to Ref. 22 Lemma 5.12, it is readily seen that

$$P^{(1)}(z)N(z)^{-1} = N(z)\left(\frac{w(nz)e^{n\pi z}}{D(z,n)^2}\right)^{\frac{\sigma_3}{2}}[I + \mathcal{O}(n^{-1})]\left(\frac{w(nz)e^{n\pi z}}{D(z,n)^2}\right)^{\frac{-\sigma_3}{2}} N(z)^{-1}. \tag{64}$$

This implies (62) by noting that $N(z)(\frac{w(nz)e^{n\pi z}}{D(z,n)^2})^{\frac{\sigma_3}{2}} = \mathcal{O}(1)$ uniformly for $z \in \partial U_\delta$. ■

Because $\tilde{\phi}(z) = \phi(-z)$ and $W(z,n)$ is an even function (see (28)), the parametrix $P^{(-1)}(z)$ in $\tilde{U}_\delta$ can be constructed by the symmetry relation $P^{(-1)}(z) = P^{(1)}(-z)$.

4.6 | The final transformation, $S(z) \to R(z)$

In the final transformation, we define

$$R(z) = \begin{cases} S(z)P^{(1)}(z)^{-1}, & z \in U_\delta\backslash\Sigma, \\ S(z)P^{(-1)}(z)^{-1}, & z \in \tilde{U}_\delta\backslash\Sigma, \\ S(z)P^{(0)}(z)^{-1}, & z \in \Omega_0\backslash\Sigma, \\ S(z)N(z)^{-1}, & z \in \mathbb{C}\backslash\left(U_\delta \cup \tilde{U}_\delta \cup \Omega_0 \cup \Sigma\right), \end{cases} \tag{65}$$

where $\Sigma = \bigcup_{k=1}^{8} \Sigma_k$; see Figure 3. Then, $R(z)$ satisfies the following RHP:

(R1) $R(z)$ is analytic in $\mathbb{C}\backslash\Sigma_R$, where Σ_R is the reduced jump contour of $R(z)$ described in Figure 4. To avoid the abuse of notations, we denote $\Sigma_0 = (\Sigma_2 \cup \Sigma_5) \cap (\Omega_0)$.

(R2) $R_+(z) = R_-(z)J_R(z)$, where

$$J_R(z) = \begin{cases} P^{(1)}(z)N(z)^{-1}, & z \in \partial U_\delta, \\ P^{(-1)}(z)N(z)^{-1}, & z \in \partial \tilde{U}_\delta, \\ N(z)\begin{pmatrix} 1 & 0 \\ W(z,n) & 1 \end{pmatrix}N(z)^{-1}, & z \in \left(\Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_6\right) \backslash \left(U_\delta \cup \tilde{U}_\delta\right), \\ P^{(0)}(z)N(z)^{-1} & z \in \bigcup_{k=1}^{6} \Gamma_k, \\ J_0, & z \in \Sigma_0, \\ N(z)\begin{pmatrix} 1 & \frac{1}{W(z,n)} \\ 0 & 1 \end{pmatrix}N(z)^{-1}, & z \in \Sigma_7 \backslash \tilde{U}_\delta, \\ N(z)\begin{pmatrix} 1 & \frac{1}{W(z,n)} \\ 0 & 1 \end{pmatrix}N(z)^{-1}, & z \in \Sigma_8 \backslash U_\delta. \end{cases} \tag{66}$$

(R3) $R(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

Lemma 7. *As $n \to \infty$, there exists $\sigma > 0$ such that*

$$J_R(z) = \begin{cases} I + \mathcal{O}\left(\frac{1}{n}\right), & z \in \partial U_\delta \cup \partial \tilde{U}_\delta, \\ I + \mathcal{O}\left(\frac{1}{n^{2p_1}}\right), & z \in \left(\bigcup_{k=1}^{6} \Gamma_k\right) \cup \Sigma_0, \\ I + \mathcal{O}(n^{|\alpha|}e^{-\sigma n}), & z \in \left(\Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_6 \cup \Sigma_7 \cup \Sigma_8\right) \backslash \left(U_\delta \cup \tilde{U}_\delta \cup \Omega_0\right), \end{cases} \tag{67}$$

where $c = \min\{\frac{1}{2}, p_1\}$ as given previously.

Proof. Note that $N(z)$ is independent of n. In view of (56), (57), and (62), it is left to show that

$$J_R(z) = I + \mathcal{O}(n^{|\alpha|}e^{-\sigma n}), \quad z \in \left(\Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_6 \cup \Sigma_7 \cup \Sigma_8\right) \backslash \left(U_\delta \cup \tilde{U}_\delta \cup \Omega_0\right). \tag{68}$$

From the definition of $\phi(z)$ (resp. $\tilde{\phi}(z)$) in (12) (resp. in (13)), and its asymptotic behavior near $z = 1$(resp. $z = -1$) in (21), it is readily seen that $\operatorname{Re}\phi(z) < 0$ (resp. $\operatorname{Re}\tilde{\phi}(z) < 0$) on Σ_1, Σ_3 (resp. Σ_4, Σ_6) and $\operatorname{Re}\phi(z) > 0$ (resp. $\operatorname{Re}\tilde{\phi}(z) > 0$) on Σ_8 (resp. Σ_7). Meanwhile, according to (27) and the definition of $W(z,n)$ in (28), there exists $\sigma > 0$ such that as $n \to \infty$

$$W(z,n) = \frac{D(z,n)^2 e^{2n\phi(z)}}{w(nz)e^{\pi nz}} = \mathcal{O}\left(n^{|\alpha|}e^{-\sigma n}\right), \quad z \in \left(\Sigma_1 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_6\right) \backslash \left(U_\delta \cup \tilde{U}_\delta \cup \Omega_0\right),$$

$$\frac{1}{W(z,n)} = \frac{w(nz)e^{\pi nz}}{D(z,n)^2 e^{2n\phi(z)}} = \mathcal{O}\left(n^{|\alpha|}e^{-\sigma n}\right), \quad z \in \left(\Sigma_7 \cup \Sigma_8\right) \backslash \left(U_\delta \cup \tilde{U}_\delta \cup \Omega_0\right),$$

which immediately implies (68). ∎

Note from (12) and (13) that $\phi(x) = \tilde{\phi}(-x) \sim \pi x/2$ as $x \to +\infty$. Hence, the Stieltjes transform of $J_R(z)$ is integrable on Σ_7 and Σ_8. By[9] and (67), we have

$$R(z) = I + \mathcal{O}\left(\frac{1}{n^{2c}}\right), \quad \text{as} \quad n \to \infty, \tag{69}$$

uniformly for all $z \in \mathbb{C}\backslash\Sigma_R$, where $c = \min\{\frac{1}{2}, p_1\}$.

4.7 | Proof of the main results

The main idea to prove the main theorems is tracing back the transformations $Y \to U \to T \to S \to R$ and making use of the fact $\pi_n(nz) = Y_{11}(nz)$.

Proof of Theorem 1. For $z \in \Omega_{out}$, by tracing back the transformations, it can be seen that

$$Y(nz) = n^{n\sigma_3} e^{\frac{nl}{2}\sigma_3} D(\infty, n)^{\sigma_3} R(z) N(z) D(z, n)^{-\sigma_3} e^{n(g(z)-\frac{l}{2})\sigma_3}.$$

Because $R(z) = I + \mathcal{O}\left(\frac{1}{n^{2c}}\right)$, we have

$$\pi_n(nz) = Y_{11}(nz) = \frac{n^n D(\infty, n) N_{11} e^{ng(z)}}{D(z, n)}\left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right), \quad n \to \infty. \tag{70}$$

Substituting (26) into (70), noting (17), and observing that

$$D(\infty, n) = 2^{-\frac{\alpha}{2}} \beta n^{\frac{\alpha}{2}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) \tag{71}$$

and

$$N_{11} = \frac{1}{2}\left(\frac{(z-1)^{\frac{1}{4}}}{(z+1)^{\frac{1}{4}}} + \frac{(z+1)^{\frac{1}{4}}}{(z-1)^{\frac{1}{4}}}\right) = \frac{(z + \sqrt{z^2-1})^{\frac{1}{2}}}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}, \tag{72}$$

we get (36).

For $z \in \Omega_{in}^r$ and $\operatorname{Im} z \geq 0$, we have in a similar way that

$$Y(nz) = n^{n\sigma_3} e^{\frac{nl}{2}\sigma_3} D(\infty, n)^{\sigma_3} R(z) N(z) \begin{pmatrix} 1 & 0 \\ \frac{D(z,n)^2 e^{2n\phi(z)}}{w(nz)e^{n\pi z}} & 1 \end{pmatrix} D(z, n)^{-\sigma_3} e^{n(g(z)-\frac{l}{2})\sigma_3}.$$

Taking the (1,1) entry of $Y(nz)$, we obtain

$$\pi_n(nz) = \frac{n^n D(\infty, n) e^{ng(z)}}{D(z, n)}\left[N_{11}\left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right) + N_{12}\frac{D(z, n)^2 e^{2n\phi(z)}}{w(nz)e^{n\pi z}}\left(1 + \mathcal{O}\left(\frac{1}{n^{2c}}\right)\right)\right]. \tag{73}$$

Hence, (37) follows by substituting (17), (26), (27), (71), and (72) into (73) and noting the fact that

$$N_{12} = \frac{1}{2i}\left(\frac{(z-1)^{\frac{1}{4}}}{(z+1)^{\frac{1}{4}}} - \frac{(z+1)^{\frac{1}{4}}}{(z-1)^{\frac{1}{4}}}\right) = \frac{i(z + \sqrt{z^2-1})^{-\frac{1}{2}}}{\sqrt{2}(z^2-1)^{\frac{1}{4}}}. \tag{74}$$

For $z \in \Omega_{in}^r$ and $\operatorname{Im} z \leq 0$, a sign in the formula (73) is changed, which becomes

$$\pi_n(nz) = \frac{n^n D(\infty, n) e^{ng(z)}}{D(z, n)} \left[N_{11} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) - N_{12} \frac{D(z, n)^2 e^{2n\phi(z)}}{w(nz) e^{n\pi z}} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right].$$

Hence, (37) also holds in this case because $(N_{11})_+ = -(N_{12})_-$ and $(N_{12})_+ = (N_{11})_-$.

When $z \in U_\delta$, tracing back the transformations again, we have

$$\begin{aligned} Y(nz) &= n^{n\sigma_3} e^{\frac{l}{2} n\sigma_3} D_\infty R(z) P^{(1)}(z) D(z, n)^{-\sigma_3} e^{n(g - \frac{l}{2})\sigma_3} \\ &= n^{n\sigma_3} e^{\frac{l}{2} n\sigma_3} D_\infty R(z) N(z) \left(\frac{w(nz) e^{n\pi z}}{D(z, n)^2} \right)^{\frac{\sigma_3}{2}} \sqrt{\pi} e^{\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \zeta^{\frac{1}{4}\sigma_3} \\ &\quad \times \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2 \mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} e^{n\phi(z)\sigma_3} \left(\frac{w(nz) e^{n\pi z}}{D(z, n)^2} \right)^{-\frac{\sigma_3}{2}} D(z, n)^{-\sigma_3} e^{n(g(z) - \frac{l}{2})\sigma_3}. \end{aligned}$$

Because $R(z) = I + \mathcal{O}\left(\frac{1}{n^{2c}} \right)$ as $n \to \infty$, we obtain

$$\begin{aligned} \pi_n(nz) &= Y_{11}(nz) \\ &= \frac{\sqrt{\pi} n^n D(\infty, n) e^{n(\phi(z) + g(z))}}{D(z, n)} \left\{ \zeta^{\frac{1}{4}} \left[N_{11} - i \frac{D(z, n)^2}{w(nz) e^{n\pi z}} N_{12} \right] \mathrm{Ai}(\zeta) \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right. \\ &\quad \left. - \zeta^{-\frac{1}{4}} \left[N_{11} + i \frac{D(z, n)^2}{w(nz) e^{n\pi z}} N_{12} \right] \mathrm{Ai}'(\zeta) \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right\}. \end{aligned} \tag{75}$$

Substituting the explicit formulas of $\phi(z), g(z), N_{11}, N_{12}$, given in (12), (11), (72), (74), and the asymptotic behaviors of $D(z, n)$, $\frac{D(z,n)^2}{w(nz) e^{n\pi z}}$, stated in (26), (27), into (75), we obtain (38). ■

Proof of Theorem 2. When $z \in \Omega_0^+$,

$$Y(nz) = n^{n\sigma_3} e^{\frac{nl}{2}\sigma_3} D(\infty, n)^{\sigma_3} R(z) N(z) \begin{pmatrix} 1 & 0 \\ W(z, n) - Q(z, n) & 1 \end{pmatrix} D(z, n)^{-\sigma_3} e^{n(g(z) - \frac{l}{2})\sigma_3}.$$

Hence, in this case

$$\begin{aligned} \pi_n(nz) = \frac{n^n D(\infty, n) e^{ng(z)}}{D(z, n)} &\left[N_{11} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right. \\ &\left. + N_{12} \left(\frac{D(z, n)^2 e^{2n\phi(z)}}{w(nz) e^{n\pi z}} - Q(z, n) \right) \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right]. \end{aligned} \tag{76}$$

Recall the definitions of $\Phi_1(z, n), \Phi_2(z, n), \Phi_3(z, n)$ in (40), (41), (42), and N_{11}, N_{12} in (72), (74). Then, we immediately get (39) by substituting (71) into (76). Similarly, when $z \in \Omega_0^-$, we have

$$\begin{aligned} \pi_n(nz) = \frac{n^n D(\infty, n) e^{ng(z)}}{D(z, n)} &\left[N_{11} \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right. \\ &\left. - N_{12} \left(\frac{D(z, n)^2 e^{2n\phi(z)}}{w(nz) e^{n\pi z}} - Q(z, n) \right) \left(1 + \mathcal{O}\left(\frac{1}{n^{2c}} \right) \right) \right]. \end{aligned} \tag{77}$$

We recall the definitions of $\Phi_1(z,n), \Phi_2(z,n), \Phi_3(z,n)$ in (40), (41), (42), and N_{11}, N_{12} in (72), (74) again. A combination of (71) and (77) also yields (39). ■

Proof of Corollary 2. As mentioned above, the function $\Phi_2(z,n) - \Phi_3(z,n)Q(z,n)$ is analytic in Ω_0^+. Moreover, according to Lemma 4, we see that $\Phi_2(z,n) - \Phi_3(z,n)Q(z,n)$ is asymptotically smaller than $\Phi_1(z,n)$ in Ω_θ^+ and Ω_0^-. Hence, we obtain (46) and (47) by replacing z by $\frac{z}{n}$ in (39), and noting the definitions of $g(z), \phi(z)$ in (11), (12), respectively, and the asymptotic behavior of $D(z,n)$ in (23).

When $|nz| < p_1$ or $\arg z \in [-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon]$, according to Lemma 3, we have

$$Q(z,n) = \mathcal{O}\left(\frac{1}{n^{2p_1}}\right) \text{ uniformly for } z \in \left\{\arg(\pm z) \in \left[-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon\right]\right\} \cup \left\{|z| \le \frac{p_1-d}{n}\right\}$$

with arbitrary small $d > 0$. This implies that $\Phi_3(z,n)Q(z,n)$ is smaller than $\Phi_1(z,n) + \Phi_2(z,n)$ in this case. Particularly, replacing z by $\frac{z}{n}$ with fixed z satisfying $|\operatorname{Im} z| < p_1$ or $\arg z \in [-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon]$ and noting (23), we get (44). Note that $D_+(x,n) = \overline{D_-(x,n)}$ for all $x \in (-1,1)$. Putting $z = \frac{x}{n}$ in (23), and then taking the limit $n \to \infty$, we obtain that $|h(x)| = |h(-x)|$ for all $x \in \mathbb{R}$. Hence, (45) is a direct implication of (44) by noting that $\arg h(-x) = -\arg h(x)$ for all $x \in \mathbb{R}$.

Finally, we note that if $nz \to ip_{\pm 1}$ and $p_1 > \frac{1}{2}$, (46) and (47) still hold by choosing $1/2 < \tilde{c} < p_1$ in Lemma 4. However, if $nz \to ip_{\pm 1}$ and $p_1 \le \frac{1}{2}$, in view of Lemma 5, we only obtain (48). ■

5 | THE CONTINUOUS DUAL HAHN POLYNOMIALS

Returning back to the special weight function in (5), we will derive the uniform asymptotic behavior of the continuous dual Hahn polynomials in the whole plane. In this special case,

$$A(z) = \Gamma(a-z)\Gamma(b-z)\Gamma(c-z)/\Gamma(1-2z) \tag{78}$$

whose zeros are $p_k = k/2$ with $k = 1, 2, \ldots$. Hence, applying Stirling's approximation, we can obtain that $\beta = \sqrt{2}\pi$, $\alpha = 2(a+b+c) - 4$ and $\Theta(iz) = iz - iz\log(-iz) + 2iz\log 2$ in (6). It follows that $h(z)$ in (24) is given by

$$h(z) = \frac{\Gamma(a-iz)\Gamma(b-iz)\Gamma(c-iz)}{\Gamma(1-2iz)} \exp\{-iz + iz\log(-iz) - 2iz\log 2\}. \tag{79}$$

It is readily seen that $|h(x)| = |h(-x)|$ for all $x \in \mathbb{R}$ because the parameters a, b, c are positive except for a possible pair of complex conjugates with positive real parts. Now we check that the weight function in (5) fulfills (H3). When $|\arg z \pm \frac{\pi}{2}| > \epsilon$, this can be immediately derived from (H2). However, attention should be paid when z is near the imaginary axis. One should use the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}$ and the ratio asymptotics of the gamma functions (see Ref. 26 (5.11.13)). We only need to check it when $\operatorname{Im} z > 0$ and the case when $\operatorname{Im} z < 0$ is similar. When $\operatorname{Im} z > 0$, $A(iz) \neq 0$ and

$$\frac{1}{w(z)} = \frac{\sin[\pi(a+iz)]\sin[\pi(b+iz)]\sin[\pi(c+iz)]\Gamma(1-a-iz)\Gamma(1-b-iz)\Gamma(1-c-iz)}{A(iz)\pi^2\Gamma(-2iz)\sin[2\pi iz]}.$$

By Ref. 26, (5.11.13), it is easy to see that (H3) holds for the above weight function. Therefore, substituting (79) into the results in Theorems 1, 2, and Corollary 2, one can obtain the asymptotics of the

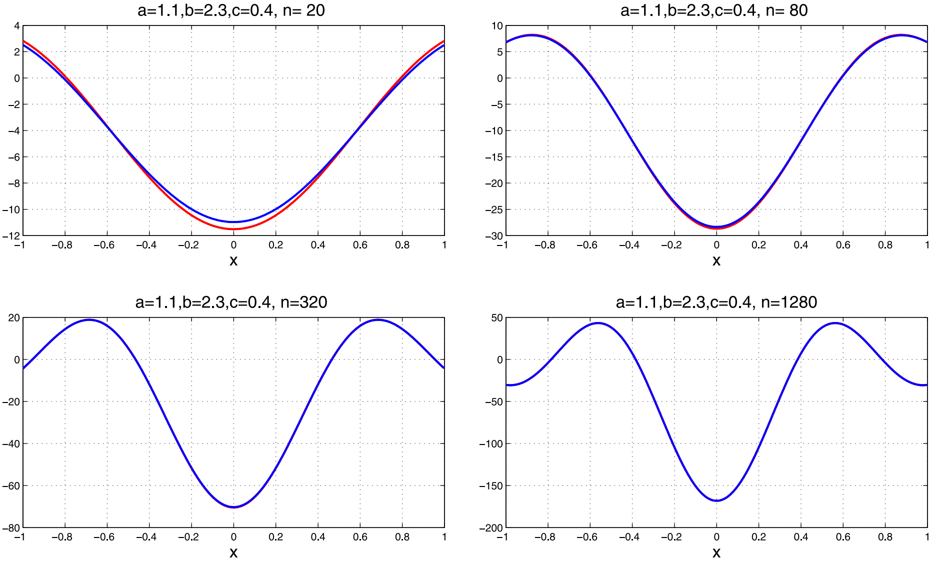

FIGURE 5 Comparisons of the exact values and the asymptotic approximations of $\mathcal{F}_n(x)$. The blue curves represent the asymptotic values of $\mathcal{F}_n(x)$ and the red ones are the exact values

continuous dual Hahn polynomials. For example, when $x \in \mathbb{R}$ is fixed, substituting (79) into (45) and noting that $S_n(x^2) = (-1)^n \frac{\pi_{2n+1}(x)}{x}$, we get

$$S_n(x^2) = \frac{2\gamma_{2n+1} \cos\left(x \log n + \arg A(ix) - \frac{\pi}{2}\right)}{x\sqrt{w(x)}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{80}$$

where γ_n and $A(z)$ are given in (34) and (78), respectively. Similarly, when z is on the imaginary axis, we apply Theorem 2 to get the corresponding asymptotic behavior.

To compare our asymptotic results with the exact values, we recall the three-term recurrence relation[2] for the continuous dual Hahn polynomials. Set $S_n(x^2; a, b, c) = (-1)^n K_n \mathcal{F}_n(x)$ with

$$K_n = 2^{2n} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+a+b}{2}\right)\Gamma\left(\frac{n+b+c}{2}\right)\Gamma\left(\frac{n+c+a}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{b+c}{2}\right)\Gamma\left(\frac{c+a}{2}\right)}.$$

Then, $\mathcal{F}_n(x)$ satisfies

$$\mathcal{F}_{n+1}(x) = \frac{K_n}{K_{n+1}}\left[x^2 - 2n^2 - (2a + 2b + 2c - 1)n - ab - bc - ac\right]\mathcal{F}_n(x) - \mathcal{F}_{n-1}(x) \tag{81}$$

with the initial conditions $\mathcal{F}_0(x) = 1$ and $\mathcal{F}_1(x) = \frac{K_0}{K_1}(x^2 - ab - bc - ac)$. Hence, we can compute the exact values of $\mathcal{F}_n(x)$ via the above recurrence relation. Figures 5 and 6 show the comparisons of the exact values and the asymptotic approximations of $\mathcal{F}_n(x)$ on the real and imaginary axes, respectively.

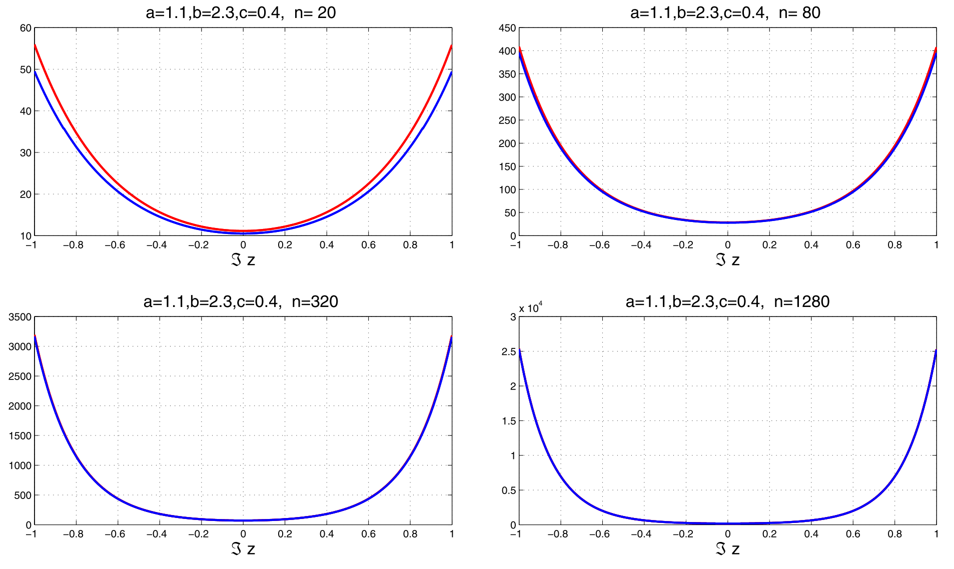

FIGURE 6 Comparisons of the exact values and the asymptotic approximations of $\mathcal{F}_n(ix)$. The blue curves represent the asymptotic values of $\mathcal{F}_n(ix)$ and the red ones are the exact values

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APPENDIX A: PROOF OF LEMMA 2

For any fixed n, set

$$d(z,n) = D(z,n)/h(\pm nz), \quad \pm \operatorname{Im} z > 0. \tag{A.1}$$

Recalling that $D(z,n)$ is analytic in $\mathbb{C}\backslash[-1,1]$ and $D_+(x,n)D_-(x,n) = w(nx)e^{n\pi|x|}$, $x \in (-1,1)$, and using the definitions of $h(z)$ in (24) and $\Theta_0(z)$ in (25), we see that $d(z,n)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$, and

$$\begin{cases} d_+(x,n)d_-(x,n) = e^{\pi|x|+\Theta_0(ix)+\Theta_0(-ix)} = 1, & x \in (-1,0) \cup (0,1), \\ d_+(x,n)h(nx) = d_-(x,n)h(-nx), & x \in (-\infty,-1) \cup (1,+\infty). \end{cases} \tag{A.2}$$

Because $\Theta_0(z)$ has at most a weak singularity at 0, the function

$$\tilde{d}(z,n) = \frac{\log d(z,n)}{\sqrt{z^2-1}}, \tag{A.3}$$

is integrable near 0, and $\tilde{d}_+(x,n) - \tilde{d}_-(x,n) = 0$ for all $x \in (-1,0) \cup (0,1)$. Hence, $\tilde{d}(z,n)$ is analytic in $\mathbb{C}\backslash((-\infty,-1] \cup [1,+\infty))$ and

$$\tilde{d}_+(x,n) - \tilde{d}_-(x,n) = \pm\frac{\log h(-nx) - \log h(nx)}{\sqrt{x^2-1}}, \quad \pm x > 1. \tag{A.4}$$

Meanwhile, because $\lim_{z\to\infty} D(z,n) = D(\infty,n) \neq 0$, we see that $d(z,n) = \mathcal{O}(z^{-\frac{\alpha}{2}})$ as $z \to \infty$ by using the definition of $h(nz)$ in (24) and (25) and the asymptotic behavior of $A(inz)$ in (6). Hence, $\lim_{z\to\infty} \tilde{d}(z,n) = 0$. According to the Plemelj formula, we get

$$\tilde{d}(z,n) = \frac{1}{2\pi i}\int_{-\infty}^{-1} \frac{\log h(nt) - \log h(-nt)}{\sqrt{t^2-1}(t-z)}dt + \frac{1}{2\pi i}\int_1^{\infty} \frac{\log h(-nt) - \log h(nt)}{\sqrt{t^2-1}(t-z)}dt. \tag{A.5}$$

Making use of (24), (25), and (6) again, we see that

$$\log h(nt) - \log h(-nt) = \mp\frac{\alpha\pi i}{2}\left(1 + \mathcal{O}\left(\frac{1}{nt}\right)\right), \quad |nt| \to \infty \tag{A.6}$$

uniformly for $\arg(\pm t) \in [-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon]$, where $\epsilon > 0$ is given in (H2). Hence, the two integrals in (A.5) are both integrable.

Now, we claim that as $n \to \infty$,

$$\tilde{d}(z,n) = \frac{-\frac{\alpha}{2}\log(z+\sqrt{z^2-1}) \pm \frac{\alpha\pi i}{4}}{\sqrt{z^2-1}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \pm \operatorname{Im} z > 0 \tag{A.7}$$

which holds uniformly for all $z \in \mathbb{C}\backslash\mathbb{R}$, or by taking limits to the real axis from the upper and lower half plane, respectively. In fact, it suffices to show that

$$\Delta := \frac{\sqrt{z^2-1}}{2\pi i}\int_1^{\infty} \frac{\log h(-nt) - \log h(nt) + \frac{\alpha\pi i}{2}}{\sqrt{t^2-1}(t-z)}dt = \mathcal{O}\left(\frac{1}{n}\right), \quad n \to \infty \tag{A.8}$$

uniformly for all z. When z is away from $[1,+\infty)$, this can be directly obtained by (A.6). When $z \to [1,+\infty)$, without loss of generality, we assume $z \in \{|\arg(z-\frac{1}{2})| < \frac{\pi}{2}-\epsilon\}$. Let C be the two radials $\{|\arg(z-\frac{1}{2})| = \frac{\pi}{2}-\epsilon\}$. Then, by the Cauchy theorem,

$$2i\Delta = \frac{\sqrt{z^2-1}}{2\pi i}\int_C \frac{\log\frac{h(-nt)}{h(nt)}+\frac{\alpha\pi i}{2}}{\sqrt{1-t^2}(t-z)}dt \pm i\left(\log h(-nz)-\log h(nz)+\frac{\alpha\pi i}{2}\right). \tag{A.9}$$

Substituting (A.6) into (A.9), we get (A.8). A combination of (A.7), (A.3), and (A.1) immediately leads to (23).

Finally, letting $z\to\infty$ in (23) and making use of the asymptotic behavior of $A(inz)$ in (6), we obtain $D(\infty,n) = 2^{-\frac{\alpha}{2}}\beta n^{\frac{\alpha}{2}}(1+\mathcal{O}(\frac{1}{n}))$ as $n\to\infty$. This completes the proof of Lemma 2.

APPENDIX B: PROOF OF LEMMA 3

If $K<\infty$, $Q(z,n)$ is obviously well-defined in $\mathbb{C}\backslash\{z_k,\ k=\pm1,\pm2,\dots\}$. If $K=\infty$, we need to show that the two series in (29) are both convergent. In view of (H1), $1/(p_{k+1}-p_k)=\mathcal{O}(1)$ as $k\to\infty$. This means the distance between two consecutive zeros of $w(z)$ is bounded below by a positive constant. Moreover, it can be derived that $1/p_k=\mathcal{O}(1/k)$ as $k\to\infty$. Let C_k be a small circle around p_k that contains only one zero of $w(\eta)$ inside it. Then, (H3) implies

$$\left|\operatorname*{Res}_{\eta=ip_k}\left(\frac{1}{w(\eta)e^{\pi\eta}}\right)\right| = \left|\oint_{C_k}\frac{1}{w(\eta)e^{\pi\eta}}d\eta\right| \le C'|p_k|^{-\alpha}. \tag{B.1}$$

For any fixed n and any $z\in\mathbb{C}\backslash\{z_k,\ k=\pm1,\pm2,\dots\}$,

$$\left|\frac{r_{\mp k}}{z-z_{\mp k}}\right| \le M|r_{\mp k}| \le M|D(z_{\mp k},n)|^2 e^{2n\operatorname{Re}\phi(z_{\mp k})}\left|\operatorname*{Res}_{\eta=ip_{\mp k}}\left(\frac{1}{w(\eta)e^{\pi\eta}}\right)\right| = \mathcal{O}\left(\frac{1}{k^{2n+\alpha}}\right)$$

as $k\to\infty$, where $M>0$ is independent of k. This approximation can be shown by combining (ϕ3) of Lemma 1 and (B.1). Consequently, for all $n>\frac{-\alpha+1}{2}$, the two series in (29) are convergent. Hence, $Q(z,n)$ is well defined in $\mathbb{C}\backslash\{z_{\pm k},\ k=1,2,\cdots\}$.

Now we begin to prove (31). If $\inf\{|nz+\pm ip_k|,\ k=1,2,\dots\}\ge d$, we have

$$\begin{aligned}|Q(z,n)| \le{}& \tfrac{n}{d}|r_1| + \frac{1}{d}e^{2n\operatorname{Re}\phi(z_1)}\sum_{k=2}^{+\infty}|D(z_k)|^2 e^{2n\operatorname{Re}(\phi(z_k)-\phi(z_1))}\left|\operatorname*{Res}_{\eta=ip_k}\left(\frac{1}{w(\eta)e^{\pi\eta}}\right)\right| \\ &+ \frac{n}{d}|r_{-1}| + \frac{1}{d}e^{2n\operatorname{Re}\phi(z_{-1})}\sum_{k=2}^{\infty}|D(z_{-k})|^2 e^{2n\operatorname{Re}(\phi(z_{-k})-\phi(z_{-1}))}\left|\operatorname*{Res}_{\eta=ip_{-k}}\left(\frac{1}{w(\eta)e^{\pi\eta}}\right)\right|.\end{aligned} \tag{B.2}$$

According to (ϕ1) of Lemma 1, we know that $n\operatorname{Re}\phi(z_{\pm1}) = -p_1\log n(1+o(1))$ as $n\to\infty$, then $r_{\pm1} = \frac{1}{n^{2p_1+1}}(1+o(1))$ as $n\to\infty$.

Now, we claim that for any $k\ge 2$,

$$|D(z_{\pm k})|^2 e^{2n\operatorname{Re}(\phi(z_{\pm k})-\phi(z_{\pm1}))}\left|\operatorname*{Res}_{\eta=-ip_{\pm k}}\left(\frac{1}{w(\eta)e^{\pi\eta}}\right)\right| = \mathcal{O}(1), \quad n\to\infty. \tag{B.3}$$

Set $\mu=|\alpha|+1$. If $|nz_{\pm k}|\le\mu$ and fulfills the assumptions in this lemma, then $D(z_{\pm k})$ and $\operatorname{Res}_{\eta=-ip_{\pm k}}(\frac{1}{w(\eta)e^{\pi\eta}})$ are both bounded. Meanwhile, from the definition of $\phi(z)$ in (12), we see that

$\operatorname{Re}\phi(\pm ix)$, regarded as functions of x, are both negative and monotonically decreasing when $x \in (0, +\infty)$. Hence, $\operatorname{Re}\phi(z_{\pm k}) < \operatorname{Re}\phi(z_{\pm 1}) < 0$, $k = 2, 3, \cdots$. It implies that (B.3) holds in this case. For $|nz_{\pm k}| > \mu$, according to Lemma 2 and the definition of $h(nz)$ in (24), (25), and the asymptotic behavior of $A(z)$ in (6), we know that

$$D(z_{\pm k}) \leq C n^{\frac{\alpha}{2}} \tag{B.4}$$

for some constant $C > 0$. Meanwhile, from (B.1), we know that

$$\left| \operatorname*{Res}_{\eta = ip_k} \left(\frac{1}{w(\eta) e^{\pi\eta}} \right) \right| \leq C' |p_k|^{-\alpha} = C' |nz_k|^{-\alpha}. \tag{B.5}$$

And, in this case, $e^{2n\phi(z_k) - 2n\phi(z_{\pm 1})} = \mathcal{O}(n^{-2\mu+1})$. Hence, combining (B.4) and (B.5), we get (B.3).

Finally, making use of (B.3) and the fact that the two series in (B.2) are convergent, we obtain the desired estimate in (31). This completes the proof of Lemma 3.

APPENDIX C: PROOF OF LEMMA 4

We only need to prove (32) for $z \in \Omega_\theta^+$, and the proof for $z \in \Omega_\theta^-$ is similar. To this end, we choose a suitable constant $d' > 0$ such that the line

$$\Gamma_{d'} = \left\{ |\operatorname{Im} z| = \tilde{M} + d'; \ \frac{\pi}{2} - \theta \leq \arg z \leq \frac{\pi}{2} + \theta \right\}$$

crosses the imaginary axis in the middle of two poles of $W(z, n)$ on the positive imaginary axis. This ensures that, on this line, $\inf\{|nz - ip_k| \ \ k = 1, 2, \cdots, \infty\} \geq d$ for some constant $d > 0$.

With a little abuse of notations, we also denote

$$\Omega_\theta^+ = \left\{ \frac{\tilde{c}}{n} \leq |\operatorname{Im} z| \leq \tilde{M} + d'; \ \frac{\pi}{2} - \theta \leq \arg z \leq \frac{\pi}{2} + \theta \right\}$$

for the moment. Define

$$H(z) = \frac{1}{2\pi i} \int_{\partial\Omega_\theta^+} \frac{W(s, n) - Q(s, n)}{s - z} ds, \quad z \in \mathbb{C} \backslash \partial\Omega_\theta^+.$$

Applying the Cauchy theorem, we know that $H(z)$ is analytic in $\mathbb{C} \backslash \partial\Omega_\theta^+$ and

$$H(z) = \begin{cases} W(z, n) - Q(z, n), & z \in \Omega_\theta^+; \\ 0, & z \in \mathbb{C} \backslash \Omega_\theta^+. \end{cases}$$

Hence, to obtain (32), it is sufficient to show that as $n \to \infty$

$$W(z, n) - Q(z, n) = \mathcal{O}\left(\frac{1}{n^{2\tilde{c}}} \right), \quad z \in \partial\Omega_\theta^+. \tag{C.1}$$

To this end, we split $\partial\Omega_\theta^+$ into three parts. First, when $z \in \partial\Omega_\theta^+$ and $|n \operatorname{Im} z| = \tilde{c}$, we note from (20) that

$$\left| e^{2n\phi(z)} \right| \leq C' n^{-2n|\operatorname{Im} z|} \leq \frac{C'}{n^{2\tilde{c}}}. \tag{C.2}$$

Moreover, $|\frac{D(z,n)^2}{w(nz)e^{n\pi z}}|$ and $Q(z,n)$ are both bounded because nz is bounded on this curve. This implies (C.1) in this case. Second, when $z \in \partial\Omega_\theta^+$ and $\arg z = \frac{\pi}{2} \pm \theta$, thanks to (23) and (27), we still have the boundedness of $|\frac{D(z,n)^2}{w(nz)e^{\pm n\pi z}}|$. Moreover, (C.2) also holds in this case. Hence, we get (C.1) again. Finally, on the arc $\Gamma_{d'}$, combining the results of Lemmas 2, 3, and (H3), and making use of the facts that $e^{2n\phi(z)}$ and $e^{2n\tilde{\phi}(z)}$ are exponentially small on the right and left half of $\Gamma_{d'}$, we also have $W(z,n) - Q(z,n) = \mathcal{O}(n^{-2\tilde{c}})$ as $n \to \infty$ and for $z \in \Gamma_{d'}$.

APPENDIX D: PROOF OF LEMMA 5

For any n, from the definition of $W(z,n)$ and $Q(z,n)$ in (28) and (29), we see that $W(z,n) - Q(z,n)$ is analytic in the upper and the lower half planes. This obviously implies that $\lim_{z\to ip_{\pm 1}/n}(W(z,n) - Q(z,n))$ both exist.

To prove (33), we only need to consider the case of B_n^+. The case of B_n^- is similar. Set $\eta = nz$. According to the definition of $W(z,n)$ and $Q(z,n)$ again, we first have

$$B_n^+ = \lim_{z\to ip_1/n}(W(z,n) - Q(z,n)) = \lim_{\eta\to ip_1} \frac{W(z,n)(\eta - ip_1) - nr_1}{\eta - ip_1} + E_n,$$

which can be further derived as

$$\begin{aligned} B_n^+ = &\lim_{\eta\to ip_1}\left[\frac{D(z,n)^2(\eta - ip_1)}{w(\eta)e^{\pi\eta}}\frac{d}{d\eta}e^{2n\phi(z)}\right] + \lim_{\eta\to ip_1}\left[\frac{(\eta - ip_1)e^{2n\phi(z)}}{w(\eta)e^{\pi\eta}}2D(z,n)\frac{dD(z,n)}{d\eta}\right] \\ &+ \lim_{\eta\to ip_1}\left[D(z,n)^2 e^{2n\phi(z)}\frac{d}{d\eta}\left(\frac{(\eta - ip_1)}{w(\eta)e^{\pi\eta}}\right)\right] + E_n, \end{aligned} \tag{D.1}$$

where

$$E_n = \sum_{k=1}^{\infty}\frac{nr_{-k}}{ip_1 - ip_{-k}} + \sum_{k=2}^{\infty}\frac{nr_k}{ip_1 - ip_k}.$$

By (30) and (B.3), it is easily seen that $nr_k = \mathcal{O}(\frac{1}{n^{2p_1}})$ as $n \to \infty$ for all $k = \pm 2, \pm 3, \cdots$. Meanwhile, we proved that $nr_{-1} = \mathcal{O}(\frac{1}{n^{2p_1}})$ as $n \to \infty$ in a previous argument. A combination of these facts yields $E_n = \mathcal{O}(\frac{1}{n^{2p_1}})$ as $n \to \infty$.

From Lemma 2, we know that $D(z_1,n)$ and $\lim_{\eta\to ip_1}\frac{dD(z,n)}{d\eta}$ are both bounded as $n \to \infty$. Because $\eta = ip_1$ is a simple zero of $w(\eta)$, we find that $\frac{(\eta - ip_1)}{w(\eta)e^{\pi\eta}}$, as a function of η, is analytic at $\eta = ip_1$. Hence, $\frac{(\eta - ip_1)}{w(\eta)e^{\pi\eta}}$ and $\frac{d}{d\eta}(\frac{(\eta - ip_1)}{w(\eta)e^{\pi\eta}})$ are also bounded at $\eta = ip_1$. Combining these facts and (ϕ1) in Lemma 1, we find that the second and the third limits in (D.1) are both $\mathcal{O}(\frac{1}{n^{2p_1}})$ as $n \to \infty$.

Attention should be paid to the value of $\frac{d}{d\eta}e^{2n\phi(z)}$ at $\eta = ip_1$, which is not of order $\mathcal{O}(\frac{1}{n^{2p_1}})$ but $\mathcal{O}(\frac{\log n}{n^{2p_1}})$ as $n \to \infty$, because

$$\frac{d}{d\eta}e^{2n\phi(z)} = 2ne^{2n\phi(z)}\frac{d\phi(z)}{d\eta} = 2e^{2n\phi(z)}\frac{d\phi(z)}{dz}$$

and $\frac{d\phi(z)}{dz} = -i\log z + \mathcal{O}(1)$ as $z \to 0$; see (12) for the definition of $\phi(z)$. Based on the above arguments, we get

$$B_n^+ = \frac{iD(\frac{ip_1}{n}, n)^2 e^{2n\phi(ip_1/n)}}{w'(ip_1)e^{\pi i p_1}} \log n + \mathcal{O}\left(\frac{1}{n^{2p_1}}\right), \quad n \to \infty.$$

From (23) and (20), one can further obtain $D(\frac{ip_1}{n}, n) = h(ip_1)(1 + \mathcal{O}(\frac{1}{n}))$ and $e^{2n\phi(ip_1/n)} = \frac{1}{n^{2p_1}}(1 + \mathcal{O}(\frac{1}{n}))$, respectively, as $n \to \infty$. A combination of the above arguments yields (33).