

ORIGINAL ARTICLE

Asymptotics of partition functions in a fermionic matrix model and of related q -polynomials

Dan Dai¹ | Mourad E. H. Ismail² | Xiang-Sheng Wang³ ¹City University of Hong Kong, Tat Chee Avenue, Kowloon Tong, Hong Kong, China²University of Central Florida, Orlando, Florida³University of Louisiana at Lafayette, Lafayette, Louisiana**Correspondence**Xiang-Sheng Wang, University of Louisiana at Lafayette, Lafayette, LA 70503, USA.
Email: xswang@louisiana.edu**Funding information**

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Abstract

In this paper, we study asymptotics of the thermal partition function of a model of quantum mechanical fermions with matrix-like index structure and quartic interactions. This partition function is given explicitly by a Wronskian of the Stieltjes-Wigert polynomials. Our asymptotic results involve the theta function and its derivatives. We also develop a new asymptotic method for general q -polynomials.

KEYWORDS

asymptotics, matrix models, partition function, Stieltjes-Wigert polynomials, theta function

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1 | INTRODUCTION AND STATEMENT OF RESULTS

In the past few decades, matrix models have attracted a lot of research interests due to their close relations and various applications in many areas of mathematics and physics; for example, see Refs. 1 and 2. Quite recently, to better understand the physics of a large number of fermionic degrees of freedom subject to nonlocal interactions, Anninos and Silva³ studied models of quantum mechanical fermions with matrix-like index structure. Given $L, N \in \mathbb{N}$, they considered a fermionic matrix model consisting of NL complex fermions $\{\psi^{iA}, \bar{\psi}^{Ai}\}$ with $i = 1, \dots, N$ and $A = 1, \dots, L$. Note that the indices i and A transform in the bifundamental of a $U(N) \times U(L)$ symmetry. They showed that the thermal partition function is given by

$$\tilde{Z}_{L \times N} = Q \int \prod_{i < j} \sinh^2 \left(\frac{\mu_i - \mu_j}{2} \right) \prod_{i=1}^N \cosh^L \left(\frac{\mu_i}{2} \right) e^{-L\tilde{\gamma}\mu_i^2} \prod_{i=1}^N d\mu_i, \quad (1)$$

where $\tilde{\gamma} > 0$ is a positive parameter and the normalization constant Q is

$$Q = 2^{-L} \int \prod_{i < j} \sinh^2 \left(\frac{\mu_i - \mu_j}{2} \right) \prod_{i=1}^N e^{-L\tilde{\gamma}\mu_i^2} \prod_{i=1}^N d\mu_i. \quad (2)$$

It is interesting to point out that the sinh term in the above integrals also appears in the study of matrix models in Chern-Simons-matter theories; for example, see Refs. 2 and 4.

Later, Tierz⁵ realized that the partition function in (1) can be written explicitly as a Wronskian of the Stieltjes-Wigert polynomials. Let the constant \hat{C} be given as

$$\hat{C} = 2^{N(N-1)-NL} \exp \left(-\frac{N^3}{4L\tilde{\gamma}} - \frac{3NL}{16\tilde{\gamma}} - \frac{N^2}{2\tilde{\gamma}} \right),$$

Tierz showed that

$$\hat{Z}_{L \times N} = \frac{\tilde{Z}_{L \times N}}{2^L \hat{C}} = \frac{(-1)^{LN}}{\prod_{j=0}^{L-1} j!} \begin{vmatrix} S_N(\lambda) & S_{N+1}(\lambda) & \dots & S_{N+L-1}(\lambda) \\ S'_N(\lambda) & S'_{N+1}(\lambda) & \dots & S'_{N+L-1}(\lambda) \\ \vdots & \vdots & \vdots & \vdots \\ S_N^{(L-1)}(\lambda) & S_{N+1}^{(L-1)}(\lambda) & \dots & S_{N+L-1}^{(L-1)}(\lambda) \end{vmatrix}, \quad (3)$$

where the spectral parameter is

$$\lambda = -q^{-N-L/2}. \quad (4)$$

Here $q = \exp(-\frac{1}{2\tilde{\gamma}L}) \in (0, 1)$ and $S_n(x)$ is the monic Stieltjes-Wigert polynomial

$$S_n(x) = (-1)^n q^{-n^2-n/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2+k/2} (-x)^k; \quad (5)$$

see Refs. 6 and 7. After studying the partition function $\hat{Z}_{L \times N}$ for some finite L and N , Tierz⁵ raised the question of analyzing its large N limit. He discussed briefly the case $L = 1$ and pointed out a connection to the Rogers-Ramanujan identities and possibly to their m version of Garrett et al.⁸ However, he did not prove any asymptotic results rigorously.

In this paper, we further develop an asymptotic approach of Wang and Wong⁹, to find the asymptotics of $\hat{Z}_{L \times N}$ as $N \rightarrow \infty$ for general $L \in \mathbb{N}$. Our asymptotic technique applies to general q -polynomials, which are not even required to be orthogonal. To express our results, we need the following notation for the theta function:

$$\Theta(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k; \quad (6)$$

see Whittaker and Watson.¹⁰ For convenience, let us also introduce the function related to the derivatives of the theta function as follows:

$$\Theta_j(z) := z^j \Theta^{(j)}(z) = \sum_{k=-\infty}^{\infty} q^{k^2} z^k (-k)_j (-1)^j. \quad (7)$$

It is easy to see that, when $j = 0$, the above formula reduces to the theta function in (6); and when $j = 1$, we have

$$\Theta_1(z) = z\Theta'(z) = \sum_{k=-\infty}^{\infty} k q^{k^2} z^k. \tag{8}$$

Now, we are in a position to state one of our main results in the following theorem:

Theorem 1.1. *Let $q \in (0, 1)$, $L, N \in \mathbb{N}$, $m = \lfloor N/2 \rfloor$ and $\alpha = 2m - N$. With the partition function $\widehat{Z}_{L \times N}$ defined in (3), we have, for all $L \in \mathbb{N}$,*

$$q^{\frac{5LN^2}{4} + \frac{L^2N}{2}} \widehat{Z}_{L \times N} \sim \frac{q^{\frac{L(L-\alpha-1)^2}{4}}}{(q; q)_{\infty}^L \prod_{j=0}^{L-1} j!} \det(R), \quad \text{as } N \rightarrow \infty, \tag{9}$$

where R is an $L \times L$ matrix with entries involving functions $\Theta_j(z)$ in (7) as follows:

$$R_{ij} = \Theta_i \left(q^{\alpha-j-\frac{L-1}{2}} \right) = \sum_{k=-\infty}^{\infty} q^{k^2} q^{k(\alpha-j-\frac{L-1}{2})} (-k)_i (-1)^i \tag{10}$$

for $0 \leq i, j \leq L - 1$.

For simplicity, we use the symbol \sim to denote asymptotic equal; namely, we write $a(N) \sim b(N)$ as $N \rightarrow \infty$ if

$$\lim_{N \rightarrow \infty} \frac{a(N)}{b(N)} = 1.$$

When $L = 1, 2$, the asymptotic results in Theorem 1.1 may be put into a more concrete form.

Corollary 1.2. *For $L = 1, 2$, we have*

$$q^{\frac{5N^2}{4} + \frac{N}{2}} \widehat{Z}_{1 \times N} \sim \frac{q^{\frac{\alpha^2}{4}} \Theta(q^{\alpha})}{(q; q)_{\infty}} = \frac{q^{\frac{\alpha^2}{4}} (-q^{1+\alpha}; q^2)_{\infty} (-q^{1-\alpha}; q^2)_{\infty}}{(q; q^2)_{\infty}} \tag{11}$$

$$= \begin{cases} \frac{(-q; q^2)_{\infty}^2}{(q; q^2)_{\infty}}, & \text{if } N \text{ is even;} \\ \frac{q^{\frac{1}{4}} (-1; q^2)_{\infty} (-q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, & \text{if } N \text{ is odd} \end{cases}$$

and

$$q^{\frac{5N^2}{2} + 2N} \widehat{Z}_{2 \times N} \sim \frac{q^{\frac{(\alpha-1)^2}{2}}}{4} (-q; q)_{\infty} (-1; q)_{\infty}$$

$$\times \left[- (q; q)_{\infty}^2 (q; q^2)_{\infty}^2 \left(q^{\alpha-\frac{1}{2}}; q \right)_{\infty} \left(q^{\frac{3}{2}-\alpha}; q \right)_{\infty} + \left(-q^{\alpha-\frac{1}{2}}; q \right)_{\infty} \left(-q^{\frac{3}{2}-\alpha}; q \right)_{\infty} \right]. \tag{12}$$

Remark 1.3. In Ref. 5 Tierz conjectured that

$$\lim_{N \rightarrow \infty} q^{6N^2 + \frac{21N}{2}} \widehat{Z}_{1 \times N} = \frac{1}{(q; q^5)_\infty^2 (q^4; q^5)_\infty^2} - \frac{1}{(q^2; q^5)_\infty^2 (q^3; q^5)_\infty^2}. \quad (13)$$

Here, our asymptotic formula (11) is different from what Tierz conjectured in Ref. 5.

Remark 1.4. It will be of interest to evaluate the determinant of the matrix R in Theorem 1.1 for general L . We strongly believe that it has a simple close form.

The rest of this paper is organized as follows: In Section 2, we formulate a new technique to derive the asymptotics of q -polynomials. This technique is applicable to all classical q -polynomials which are orthogonal on unbounded intervals. We also prove an asymptotic symmetry property of zeros of q -polynomials with positive zeros. This property states that the product of the k th largest zero and the k th smallest zero is asymptotically independent of k . In the case of the Stieltjes-Wigert polynomials, this property is known; see Refs. 11 and 12. Based on the general asymptotic results in Section 2, the proofs of Theorem 1.1 and Corollary 1.2 are done in Section 3. We also give another proof for the particular case $L = 1$ at the end of this section. In Section 4, we continue the development of a new asymptotic technique started in Section 2 by considering the asymptotics in the nonoscillatory range.

2 | ASYMPTOTICS OF q -POLYNOMIALS AND SYMMETRY OF ZEROS

2.1 | Asymptotics of q -polynomials in the oscillatory interval

To prove Theorem 1.1, we actually develop a new asymptotic technique to study asymptotics of general q -polynomials. Consider the following general q -polynomials with real coefficients:

$$P_n(x) = \sum_{k=0}^n q^{k^2} f_n(k) (-x)^k, \quad (14)$$

and the related derivative functions

$$P_{n,j}(x) := x^j P_n^{(j)}(x) = \sum_{k=0}^n q^{k^2} f_n(k) (-x)^k (-k)_j (-1)^j \quad (15)$$

with $j \in \mathbb{N}$. Define

$$X_{j,m}(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k (-k-m)_j (-1)^j. \quad (16)$$

Note that this function is related to the functions $\Theta(z)$ and $\Theta_j(z)$ in (6) and (7) as $X_{0,m}(x) = \Theta(x)$ and $X_{j,0}(x) = \Theta_j(x)$.

Next, we state our asymptotic results for general q -polynomials in (14).

Theorem 2.1. *Assume that $f_n(k)$ is uniformly bounded for $n \geq 0$ and $0 \leq k \leq n$; moreover, for some $l \in (0, 1)$ and $0 < \delta < \min(l, 1-l)$,*

$$\sup_{n(l-\delta) \leq k \leq n(l+\delta)} |f_n(k) - 1| \leq \varepsilon(n, l, \delta) = o(n^{-j}), \quad \text{with } j \in \mathbb{N}, \quad (17)$$

as $n \rightarrow \infty$. Let $m = \lfloor nl \rfloor$, $d = \lfloor n\delta \rfloor$, and M be a fixed large number. Then, for the functions $P_{n,j}(x)$ given in (15), we have

$$P_{n,j}(q^{-2m}y) = q^{-m^2}(-y)^m [X_{j,m}(-y) + O(n^j \varepsilon(n, l, \delta)) + O(q^{d^2} M^d n^j)], \tag{18}$$

uniformly for $1/M \leq |y| \leq M$.

Proof. By a shift of variable $k \rightarrow k + m$, we have

$$P_{n,j}(q^{-2m}y) = \sum_{k=-m}^{n-m} q^{k^2-m^2} f_n(k+m)(-y)^{k+m}(-k-m)_j(-1)^j = q^{-m^2}(-y)^m(I_1 + I_2),$$

where

$$I_1 = \sum_{k=-m}^{-d} q^{k^2} f_n(k+m)(-y)^k(-k-m)_j(-1)^j + \sum_{k=d}^{n-m} q^{k^2} f_n(k+m)(-y)^k(-k-m)_j(-1)^j,$$

and

$$I_2 = \sum_{k=-d+1}^{d-1} q^{k^2} [f_n(k+m) - 1](-y)^k(-k-m)_j(-1)^j + \sum_{k=-d+1}^{d-1} q^{k^2}(-y)^k(-k-m)_j(-1)^j.$$

Note that $0 < (-k-m)_j(-1)^j < (k+m+j)^j < (d+m+j)^j(1+k-d)^j$ for all $k \geq d$. It then follows that

$$\sum_{k=d}^{\infty} q^{k^2} M^k(-k-m)_j(-1)^j \leq \sum_{k=0}^{\infty} q^{k^2+2kd+d^2} M^{k+d}(1+k)^j(d+m+j)^j = O\left(q^{d^2} M^d n^j\right),$$

which implies that $I_1 = O(q^{d^2} M^d n^j)$. Furthermore, it is easily seen that $I_2 = X_{j,m}(-y) + O(n^j \varepsilon(n, l, \delta)) + O(q^{d^2} M^d n^j)$. Consequently, we obtain

$$P_{n,j}(q^{-2m}y) = q^{-m^2}(-y)^m \left[X_{j,m}(-y) + O(n^j \varepsilon(n, l, \delta)) + O\left(q^{d^2} M^d n^j\right) \right].$$

This completes the proof. □

To illustrate the application of the above theorem, we provide asymptotics of the Stieltjes-Wigert polynomials with a scaled variable $x = q^{-nt}u$, where $t \in (0, 2)$. Let $l = t/2$ and $m = \lfloor nl \rfloor$. We obtain from (5) that

$$S_n(q^{-nt}u) = \frac{(-1)^n q^{-n^2-n/2}}{(q; q)_n} \sum_{k=0}^n q^{k^2} f_n(k)(-q^{-2m}y)^k,$$

where

$$f_n(k) = (q; q)_n \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

and $y = q^{-nt+2m+1/2}u$. For any $0 < \delta < \min(l, 1-l)$, we have

$$|f_n(k) - 1| = 1 - (q^{k+1}; q)_{n-k} (q^{n-k+1}; q)_k \leq q^{k+1} + \dots + q^n + q^{n-k+1} + \dots + q^n$$

$$\leq \frac{q^{k+1} + q^{n-k+1}}{1 - q} \leq \frac{1}{1 - q} [q^{n(l-\delta)} + q^{n(1-l-\delta)}]$$

for all $n(l - \delta) \leq k \leq n(l + \delta)$. It then follows from Theorem 2.1 (with $j = 0$) that

$$S_n(q^{-nt}u) = \frac{\Theta(-q^{-nt+2m+1/2}u) + O(q^{n(l-\delta)} + q^{n(1-l-\delta)})}{(-1)^n(q; q)_n q^{n^2-m^2+nmt+(n-m)/2} (-u)^{-m}}. \tag{19}$$

Similarly, Theorem 2.1 also gives us asymptotic results for the q^{-1} -Hermite polynomials

$$\begin{aligned} h_n(\sinh \xi) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2-nk} (-1)^k e^{(n-2k)\xi} \\ &= (-1)^n q^{n^2+n/2} e^{n\xi} S_n(q^{-n-1/2} e^{-2\xi}) \end{aligned} \tag{20}$$

and the q -Laguerre polynomials

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2+\alpha k} \frac{(-x)^k}{(q^{\alpha+1}; q)_k}. \tag{21}$$

For the q^{-1} -Hermite polynomials, let $\xi = -nt \ln q + \ln u$ with $t \in (-1/2, 1/2)$. By choosing $f_n(k) = (q; q)_n \begin{bmatrix} n \\ k \end{bmatrix}_q$, we obtain

$$h_n(\sinh \xi) = \frac{\Theta(-q^{-n(1-2t)+2m}u^{-2}) + O(q^{n(l-\delta)} + q^{n(1-l-\delta)})}{(-1)^m(q; q)_n q^{n^2t-m^2+nmt+(1-2t)u^{2m-n}}}, \tag{22}$$

where $l = 1/2 - t$, $m = \lfloor nl \rfloor$, and $\delta > 0$ is any small positive number such that $\delta < \min(l, 1 - l)$. Regarding the q -Laguerre polynomials, for $t \in (0, 2)$, by choosing $f_n(k) = (q^{\alpha+k+1}; q)_{n-k} (q; q)_n \begin{bmatrix} n \\ k \end{bmatrix}_q$, we obtain

$$L_n^{(\alpha)}(q^{-nt}u; q) = \frac{\Theta(-q^{-nt+2m+\alpha}u) + O(q^{n(l-\delta)} + q^{n(1-l-\delta)})}{(q; q)_n^2 q^{-m^2+nmt-\alpha m} (-u)^{-m}}, \tag{23}$$

where $l = t/2$, $m = \lfloor nl \rfloor$, and $\delta > 0$ is any small positive number such that $\delta < \min(l, 1 - l)$.

2.2 | Symmetry of zeros of q -polynomials

It is a well-known fact that zeros of some classical q -orthogonal polynomials satisfy nice symmetric properties. Let $x_1 < x_2 < \dots < x_n$ be the zeros of the Stieltjes-Wigert polynomial $S_n(x)$. By (5), we have

$$S_n(q^{-2n-1}/x) = q^{-n^2-n/2} (-x)^{-n} S_n(x). \tag{24}$$

It is readily seen that

$$x_j x_{n+1-j} = q^{-2n-1}, \quad j = 1, \dots, n; \tag{25}$$

see also (2.8) of Ref. 11 and (2.20) of Ref. 12. For the q^{-1} -Hermite polynomials in (20), let the zeros be denoted as $\xi_1 < \xi_2 < \dots < \xi_n$. They also satisfy a symmetric relation as follows:

$$\xi_j + \xi_{n+1-j} = 0. \tag{26}$$

Actually, a similar asymptotic symmetry property of polynomial zeros is satisfied for a general class of q -polynomials $P_n(x)$ in (14), where the coefficient $f_n(k)$ is uniformly bounded for $n \geq 0$ and $0 \leq k \leq n$. Then, for some $l \in (0, 1)$ and $x = q^{-2nl}y$, we obtain from Theorem 2.1

$$P_n(x) \sim q^{-m^2}y^m\Theta(-q^{2(m-nl)}y), \tag{27}$$

where $m = \lfloor nl \rfloor$. For each fixed $j = 1, 2, \dots$, there exists a pair $y_j^\pm = q^{\pm(2j-1)-2(m-nl)}$ such that $\Theta(-q^{2(m-nl)}y_j^\pm) = 0$. Consequently, for sufficiently large n , $P_n(x)$ has a pair of zeros $x_j^\pm \sim q^{\pm(2j-1)-2m}$; in particular, we have

$$x_j^+x_j^- \sim q^{-4m}. \tag{28}$$

We now apply the above results to the q -Laguerre polynomials in (21) where $f_n(k)$ in (14) is now given by

$$f_n(k) = \frac{(q^{\alpha+1}; q)_n(q; q)_n}{(q^{\alpha+1}; q)_k} \left[\begin{matrix} n \\ k \end{matrix} \right]_q.$$

For any $l \in (0, 1)$ and $0 < \delta < \min(l, 1 - l)$, we have

$$\sup_{n(l-\delta) \leq k \leq n(l+\delta)} |f_n(k) - 1| = O(q^{n(l-\delta)} + q^{n(1-l-\delta)}).$$

Thus, for any fixed $l \in (0, 1)$ and $j = 1, 2, 3, \dots$,

$$L_n^{(\alpha)}(x; q) = \frac{1}{(q; q)_n^2} \sum_{k=0}^n q^{k^2} f_n(k) (-xq^\alpha)^k.$$

has a pair of zeros $x_j^\pm \sim q^{\pm(2j-1)-2\lfloor nl \rfloor - \alpha}$. This implies that for any integer $k \in (1, n)$ such that k/n is bounded away from 0 and 1, $L_n^{(\alpha)}(x; q)$ has a zero $x_k \sim q^{1-2k-\alpha}$; in particular, we get

$$x_k x_{n+1-k} \sim q^{-2n-2\alpha}, \quad n \rightarrow \infty. \tag{29}$$

Let us conduct numerical computation and choose $q = 0.6$, $\alpha = 0.4$, and $n = 20$. The values of $q^{2n+2\alpha}x_k x_{n+1-k}$ for $k = 1, \dots, 10$, are given below:

$$0.45, 0.725, 0.852, 0.917, 0.952, 0.972, 0.983, 0.989, 0.993, 0.994.$$

We also take $q = 0.5$, $\alpha = 0.7$, $n = 25$ and obtain the values of $q^{2n+2\alpha}x_k x_{n+1-k}$ for $k = 1, \dots, 12$, as follows:

$$0.658, 0.861, 0.937, 0.97, 0.985, 0.993, 0.996, 0.998, 0.999, 1., 1., 1.$$

From the above computations, one can see that the asymptotic symmetry property is more significant with smaller q , larger n , or k closer to $n/2$.

3 | ASYMPTOTICS OF PARTITION FUNCTIONS

3.1 | Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 based on the general asymptotic results in Theorem 2.1.

Proof of Theorem 1.1. From the definition of $\widehat{Z}_{L \times N}$ in (3), we have

$$\widehat{Z}_{L \times N} = \det(S) \frac{(-1)^{LN}}{\prod_{j=0}^{L-1} j!} \prod_{j=0}^{L-1} (-1)^{N+j} q^{-(N+j)^2 - (N+j)/2}, \quad (30)$$

where S is an $L \times L$ matrix with ij th entry:

$$S_{ij} = \sum_{k_i=i}^{N+j} \begin{bmatrix} N+j \\ k_i \end{bmatrix}_q q^{k_i^2 + k_i/2 - (k_i-i)(N+L/2)} (-k_i)_i, \quad 0 \leq i, j \leq L-1.$$

Similar as in the proof of Theorem 2.1, one can show that the main contribution of the sum of the right-hand side comes from the items with index k_i close to $N/2$. We may ignore the (exponentially small) items with indices $k_i < L$ or $k_i > N$, and obtain

$$S_{ij} \sim \sum_{k_i=L}^N G_{ij} q^{k_i^2 + k_i/2 - (k_i-i)(N+L/2)} (-k_i)_i,$$

where $G_{ij} = \begin{bmatrix} N+j \\ k_i \end{bmatrix}_q$ is the ij th entry of a matrix G . It then follows that

$$\det(S) \sim \sum_{L \leq k_0, \dots, k_{L-1} \leq N} \det(G) q^{\sum_{i=0}^{L-1} k_i^2 - k_i(N+L/2 - 1/2) + i(N+L/2)} \prod_{i=0}^{L-1} (-k_i)_i. \quad (31)$$

A simple calculation gives us

$$\det(G) = \det(V) \prod_{i=0}^{L-1} \frac{(q; q)_{N+i}}{(q; q)_{k_i} (q; q)_{N+L-1-k_i}},$$

where V is a matrix with ij th entry

$$V_{ij} = \prod_{l=j+1}^{L-1} (1 - q^{N-k_i+l}), \quad 0 \leq i, j \leq L-1.$$

Here, when $j = L-1$, the empty product is understood to be 1. By row operations, the matrix V can be transformed to a Vandermonde one with ij th entry q^{-jk_i} , multiplied by certain constant factors. Indeed, one obtains

$$\det(V) = q^{NL(L-1)/2 + L(L-1)(2L-1)/6} \prod_{0 \leq i < j \leq L-1} (q^{-k_j} - q^{-k_i}).$$

This kind of reduction is used systematically in Krattenthaler¹³. Now, for k_i near $N/2$, we have

$$\det(G) \sim \frac{q^{NL(L-1)/2 + L(L-1)(2L-1)/6}}{(q; q)_{\infty}^L} \prod_{0 \leq i < j \leq L-1} (q^{-k_j} - q^{-k_i}).$$

Substituting the above formula into (31) gives us

$$\det(S) \sim \frac{q^{\frac{L(L-1)}{2}(2N + \frac{7L-2}{6})}}{(q; q)_\infty^L} \det(T),$$

where T is a matrix with ij th entry

$$T_{ij} = \sum_{k_i=L}^N q^{k_i^2 - k_i(N+L/2-1/2) - jk_i} (-k_i)_i, \quad 0 \leq i, j \leq L-1.$$

Let $m = \lfloor N/2 \rfloor$. Shifting the index $k_i = k + m$ yields

$$T_{ij} \sim q^{-m(N-m+L/2-1/2+j)} \sum_{k=-\infty}^{\infty} q^{k^2 - k(N-2m+L/2-1/2+j)} (-k-m)_i.$$

Denote $\beta := N - 2m + (L - 1)/2$. We have $\det(T) \sim q^{-mL(N-m+L-1)} \det(U)$, where the ij th entry of U is given by

$$U_{ij} = \sum_{k=-\infty}^{\infty} q^{k^2 - k(\beta+j)} (-k-m)_i, \quad 0 \leq i, j \leq L-1.$$

By row operations, we obtain $\det(U) = (-1)^{L(L-1)/2} \det(R)$, where R is a matrix with ij th entry

$$R_{ij} = \sum_{k=-\infty}^{\infty} q^{k^2 - k(\beta+j)} (-k)_i (-1)^i = \Theta_i(q^{-\beta-j}), \quad 0 \leq i, j \leq L-1.$$

Summarizing the above derivations, we have

$$\begin{aligned} \hat{Z}_{L \times N} &= \det(S) \frac{(-1)^{\frac{L(L-1)}{2}} q^{-LN^2 - \frac{LN}{2} - \frac{L(L-1)}{2}(2N + \frac{4L+1}{6})}}{\prod_{j=0}^{L-1} j!} \\ &\sim \det(R) \frac{q^{-LN^2 - \frac{LN}{2} + \frac{L(L-1)^2}{4} - mL(N-m+L-1)}}{(q; q)_\infty^L \prod_{j=0}^{L-1} j!}, \end{aligned}$$

where a rigorous error estimation for more general situations has been given as in Theorem 2.1. A further simplification gives us the result in (9). □

3.2 | Proof of (12)

A direct application of (9) with $L = 2$ gives

$$q^{5N^2/2+2N} \hat{Z}_{2 \times N} \sim \frac{\Theta(q^{\alpha-1/2})\Theta'(q^{\alpha-3/2}) - q\Theta'(q^{\alpha-1/2})\Theta(q^{\alpha-3/2})}{q^{1-\alpha^2/2}(q; q)_\infty^2}. \tag{32}$$

Denote $q = e^{\pi i \tau}$ and $a = \pi \tau(\alpha/2 - 3/4)$. Making use of the theta functions defined in Chapter 21 of Whittaker and Watson¹⁰, we may rewrite

$$\begin{aligned}\Theta(q^{\alpha-3/2}) &= \sum_{k=-\infty}^{\infty} q^{k^2} e^{2kia} = \vartheta_3(a; q), \\ \Theta(q^{\alpha-1/2}) &= \vartheta_3(a + \pi \tau/2; q) = q^{(1-\alpha)/2} \vartheta_2(a; q), \\ \Theta'(q^{\alpha-3/2}) &= \frac{q^{3/2-\alpha}}{2i} \vartheta_3'(a; q), \\ \Theta'(q^{\alpha-1/2}) &= \frac{q^{1-3\alpha/2}}{2i} [\vartheta_2'(a; q) - i\vartheta_2(a; q)].\end{aligned}$$

It then follows from Chapter 21 of Ref. 10 and Chapter 20 of Ref. 15 that

$$\begin{aligned}\Theta(q^{\alpha-1/2})\Theta'(q^{\alpha-3/2}) - q\Theta'(q^{\alpha-1/2})\Theta(q^{\alpha-3/2}) & \quad (33) \\ &= \frac{q^{2-3\alpha/2}}{2i} [\vartheta_2(a; q)\vartheta_3'(a; q) - \vartheta_3(a; q)\vartheta_2'(a; q) + i\vartheta_3(a; q)\vartheta_2(a; q)] \\ &= \frac{q^{2-3\alpha/2}}{2i} [\vartheta_4(0; q)^2\vartheta_4(a; q)\vartheta_1(a; q) + i\vartheta_3(a; q)\vartheta_2(a; q)] \\ &= \frac{q^{2-3\alpha/2}\vartheta_2(0; \sqrt{q})}{4i} [\vartheta_4(0; q)^2\vartheta_1(a; \sqrt{q}) + i\vartheta_2(a; \sqrt{q})].\end{aligned}$$

Recalling the Jacobi triple product identity (see Ref. 16 or (II.33) of Ref. 13)

$$\sum_{k=-\infty}^{\infty} q^{k^2/2} z^k = (q; q)_{\infty} (-z\sqrt{q}; q)_{\infty} (-\sqrt{q}/z; q)_{\infty}, \quad (34)$$

we have

$$\begin{aligned}\vartheta_1(a; \sqrt{q}) &= -iq^{\alpha/2-5/8}(q; q)_{\infty}(q^{\alpha-1/2}; q)_{\infty}(q^{3/2-\alpha}; q)_{\infty}, \\ \vartheta_2(a; \sqrt{q}) &= q^{\alpha/2-5/8}(q; q)_{\infty}(-q^{\alpha-1/2}; q)_{\infty}(-q^{3/2-\alpha}; q)_{\infty}, \\ \vartheta_4(0; q) &= (q; q)_{\infty}(q; q^2)_{\infty}, \\ \vartheta_2(0; \sqrt{q}) &= q^{1/8}(q; q)_{\infty}(-q; q)_{\infty}(-1; q)_{\infty}.\end{aligned}$$

Substituting the above formulas into (33) yields

$$\begin{aligned}\Theta(q^{\alpha-1/2})\Theta'(q^{\alpha-3/2}) - q\Theta'(q^{\alpha-1/2})\Theta(q^{\alpha-3/2}) &= \frac{q^{3/2-\alpha}}{4}(q; q)_{\infty}^2(-q; q)_{\infty}(-1; q)_{\infty} \\ &\times [-(q; q)_{\infty}^2(q; q^2)_{\infty}^2(q^{\alpha-1/2}; q)_{\infty}(q^{3/2-\alpha}; q)_{\infty} + (-q^{\alpha-1/2}; q)_{\infty}(-q^{3/2-\alpha}; q)_{\infty}],\end{aligned}$$

which, together with (32) gives (12).

3.3 | Another proof for the case $L = 1$

For the case $L = 1$, (11) is a simple application of (9) and (34). However, there is another proof due to the simple structure of $\widehat{Z}_{1 \times N}$. This new proof is based on the integral representation of the partition $\widehat{Z}_{1 \times N}$. For brevity, let us denote the summation in (34) by

$$F(z) := \sum_{k=-\infty}^{\infty} q^{k^2/2} z^k = (q; q)_{\infty} (-z\sqrt{q}; q)_{\infty} (-\sqrt{q}/z; q)_{\infty}.$$

Cauchy's residue theorem gives us

$$q^{k^2/2} = \frac{1}{2\pi i} \oint_C F(z) \frac{dz}{z^{k+1}},$$

where C is a positively oriented contour around the origin. It then follows from the definition of $\widehat{Z}_{L \times N}$ in (3) that

$$q^{N^2+N/2} \widehat{Z}_{1 \times N} = \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_q q^{k^2-kN} = \frac{1}{2\pi i} \oint_C \sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_q q^{k^2/2-kN} F(z) \frac{dz}{z^{k+1}}.$$

Using the q -binomial theorem

$$(x; q)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-x)^k, \tag{35}$$

(cf. Ref. 16 and (II.4) in Ref. 13), we have

$$q^{N^2+N/2} \widehat{Z}_{1 \times N} = \frac{1}{2\pi i} \oint_C F(z) (-q^{-N+1/2}/z; q)_N \frac{dz}{z}. \tag{36}$$

Let us introduce a change of variable $z = q^{-m}u$ with $m = \lfloor N/2 \rfloor$. Since

$$F(q^{-m}u) = \sum_{k=-\infty}^{\infty} q^{k^2/2-km} u^k = F(u) q^{-m^2/2} u^m, \tag{37}$$

and

$$\begin{aligned} (-q^{-N+1/2+m}/u; q)_N &= (-q^{-N+1/2+m}/u; q)_m (-q^{-N+1/2+2m}/u; q)_{N-m} \\ &= q^{-Nm+3m^2/2} u^{-m} (-q^{N+1/2-2m}u; q)_m (-q^{-N+1/2+2m}/u; q)_{N-m}, \end{aligned}$$

we obtain

$$\begin{aligned} q^{N^2+N/2} \widehat{Z}_{1 \times N} &= \frac{q^{-Nm+m^2}}{2\pi i} \oint_C F(u) (-q^{N+1/2-2m}u; q)_m (-q^{-N+1/2+2m}/u; q)_{N-m} \frac{du}{u} \\ &\sim \frac{q^{-Nm+m^2}}{2\pi i} \oint_C F(u) (-q^{N+1/2-2m}u; q)_{\infty} (-q^{-N+1/2+2m}/u; q)_{\infty} \frac{du}{u}. \end{aligned}$$

Note that

$$F(q^{N-2m}u) = (q; q)_{\infty} (-q^{N+1/2-2m}u; q)_{\infty} (-q^{-N+1/2+2m}/u; q)_{\infty},$$

it then follows that

$$(q; q)_{\infty} q^{N^2+N/2+Nm-m^2} \widehat{Z}_{1 \times N} \sim \frac{1}{2\pi i} \oint_C F(u) F(q^{N-2m}u) \frac{du}{u}.$$

The right-hand side equals to the constant term of the product $F(u)F(q^{N-2m}u)$:

$$\sum_{k=-\infty}^{\infty} q^{(-k)^2/2} q^{k^2/2} q^{(N-2m)k} = \Theta(q^{N-2m}) = \Theta(q^{2m-N}).$$

Thus, the above two formulas give us

$$q^{5N^2/4+N/2} \widehat{Z}_{1 \times N} \sim \frac{q^{\alpha^2/4} \Theta(q^{\alpha})}{(q; q)_{\infty}}, \quad (38)$$

where $\alpha = 2m - N$. Moreover, if $N = 2m$ is even, we have

$$\Theta(q^{2m-N}) = (q^2; q^2)_{\infty} (-q; q^2)_{\infty}^2,$$

and if $N = 2m + 1$ is odd

$$\Theta(q^{2m-N}) = (q^2; q^2)_{\infty} (-1; q^2)_{\infty} (-q^2; q^2)_{\infty}.$$

Then, (11) immediately follows from a combination of the above three formulas.

4 | ASYMPTOTICS OF q -POLYNOMIALS IN THE NONOSCILLATORY INTERVAL

Theorem 2.1 gives asymptotics of $P_{n,j}(x)$ in the oscillatory interval:

$$\frac{\ln x}{n \ln q} \in (-2, 0). \quad (39)$$

To make our asymptotic technique complete, we further provide asymptotics in the interval where $P_{n,j}(x)$ is nonoscillatory. To this end, we shall introduce the generalized theta function:

$$\Phi(z) := \sum_{k=0}^{\infty} a_k q^{k^2} z^k \quad (40)$$

and the associated functions:

$$\Phi_j(z) := z^j \Phi^{(j)}(z) = \sum_{k=0}^{\infty} a_k q^{k^2} z^k (-k)_j (-1)^j. \quad (41)$$

Here, $\{a_k\}_{k \geq 0}$ is a sequence with uniform bound. When $a_k \equiv 1$, we have $\Phi(z) + \Phi(1/z) = \Theta(z) + 1$, where $\Theta(z)$ is given in (6). It is worth pointing out that, when $a_k = (-1)^k / (q; q)_k$, the function $\Phi(z)$ is the same as the Ramanujan function (i.e., the q -Airy function); see Ref. 11.

Theorem 4.1. Assume that $f_n(k)$ is uniformly bounded for $n \geq 0$ and $0 \leq k \leq n$, and there exist uniformly bounded sequence $\{a_k\}_{k \geq 0}$ and $\delta \in (0, 1)$ such that

$$\sup_{0 \leq k \leq n\delta} |f_n(k) - a_k| \leq \varepsilon(n, \delta) = o(1) \tag{42}$$

as $n \rightarrow \infty$. Let $d = \lfloor n\delta \rfloor$ and M be a fixed large number. Then, for the functions $P_{n,j}(x)$ given in (15), we have

$$P_{n,j}(q^{nt}y) = \Phi_j(-q^{nt}y) + O(\varepsilon(n, \delta)) + O\left(q^{d^2}M^d\right), \tag{43}$$

uniformly for $|y| \leq M$ and $t \geq 0$.

Proof. We split the error term into two sums:

$$P_{n,j}(q^{nt}y) - \Phi_j(-q^{nt}y) = I_1 + I_2,$$

where

$$I_1 = \sum_{k=d}^n q^{k^2} f_n(k) (-q^{nt}y)^k (-k)_j (-1)^j - \sum_{k=d}^{\infty} a_k q^{k^2} (-q^{nt}y)^k (-k)_j (-1)^j = O\left(q^{d^2}M^d\right),$$

and

$$I_2 = \sum_{k=0}^{d-1} q^{k^2} [f_n(k) - a_k] (-q^{nt}y)^k (-k)_j (-1)^j = O(\varepsilon(n, \delta)).$$

Here, we have used the estimation:

$$\begin{aligned} \sum_{k=d}^{\infty} q^{k^2} M^k (-k)_j (-1)^j &= \sum_{k=0}^{\infty} q^{k^2+2kd+d^2} M^{k+d} (-k-d)_j (-1)^j \\ &\leq q^{d^2} M^d \sum_{k=0}^{\infty} q^{k^2} q^{2kd} (k+d+j)^j = O\left(q^{d^2}M^d\right) \end{aligned} \tag{44}$$

due to uniform boundedness of $q^{2kd} (k+d+j)^j$ for $k \geq 0$ and $d \geq 0$. This completes the proof. \square

To conduct asymptotic analysis of $P_{n,j}(q^{nt}y)$ with $t \leq -2$, we set $t = -(2+s)$ with $s \geq 0$ and change the index k to $n-k$. It follows that

$$P_{n,j}(q^{nt}y) = (-q^{n+ns}/y)^{-n} \sum_{k=0}^n q^{k^2} f_n(n-k) (-q^{ns}/y)^k (-n+k)_j (-1)^j.$$

We shall make use of the following special function:

$$\Psi_{j,n}(z) = \sum_{k=0}^n a_k q^{k^2} z^k (-n+k)_j (-1)^j. \tag{45}$$

Note that $\Psi_{0,n}(z) = \Phi_0(z) = \Phi(z)$ with $\Phi(z)$ defined in (40). Moreover, when $a_k \equiv 1$, we have the relation $\Psi_{j,n}(z) = (-1)^j X_{j,j-n-1}(z)$ with $X_{j,m}(z)$ given in (16).

Theorem 4.2. Assume that $f_n(k)$ is uniformly bounded for $n \geq 0$ and $0 \leq k \leq n$, and there exist uniformly bounded sequence $\{a_k\}_{k \geq 0}$ and $\delta \in (0, 1)$ such that

$$\sup_{0 \leq k \leq n\delta} |f_n(n-k) - a_k| \leq \varepsilon(n, \delta) = o(n^{-j}) \quad (46)$$

as $n \rightarrow \infty$. Let $d = \lfloor n\delta \rfloor$ and M be a fixed large number. Then, for the functions $P_{n,j}(x)$ given in (15), we have

$$P_{n,j}(q^{nt}y) = (-q^{n+nt}y)^n \left[\Psi_{j,n}(-q^{-2n-nt}/y) + O(n^j \varepsilon(n, \delta)) + O\left(q^{d^2} M^d n^j\right) \right], \quad (47)$$

uniformly for $|y| \geq 1/M$ and $t \leq -2$.

Proof. Again, we split the error term into two sums:

$$(-q^{n+ns}/y)^n P_{n,j}(q^{nt}y) - \Phi_j(-q^{ns}/y) = I_1 + I_2,$$

where

$$I_1 = \sum_{k=d}^n q^{k^2} f_n(n-k) (-q^{ns}/y)^k (-n+k)_j (-1)^j - \sum_{k=d}^{\infty} a_k q^{k^2} (-q^{ns}/y)^k (-n+k)_j (-1)^j,$$

and

$$I_2 = \sum_{k=0}^{d-1} q^{k^2} [f_n(n-k) - a_k] (-q^{ns}/y)^k (-n+k)_j (-1)^j = O(n^j \varepsilon(n, \delta)).$$

Since $|(-n+k)_j| \leq (n+d)^j (1+k-d)^j$ for all $k \geq d$, we have

$$\sum_{k=d}^{\infty} q^{k^2} M^k |(-n+k)_j| \leq \sum_{k=0}^{\infty} q^{k^2+2kd+d^2} M^{k+d} (1+k)^j (n+d)^j = O\left(q^{d^2} M^d n^j\right).$$

Thus, $I_1 = O(q^{d^2} M^d n^j)$. This completes the proof. \square

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ORCID

Xiang-Sheng Wang  <http://orcid.org/0000-0003-0410-4643>

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