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BIVARIATE LAGRANGE INTERPOLATION AT THE CHECKERBOARD NODES

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Abstract. In this paper, we derive an explicit formula for the bivariate Lagrange basis polynomials of a general set of checkerboard nodes. This formula generalizes existing results of bivariate Lagrange basis polynomials at the Padua nodes, Chebyshev nodes, Morrow-Patterson nodes, and Geronimus nodes. We also construct a subspace spanned by linearly independent bivariate vanishing polynomials that vanish at the checkerboard nodes and prove the uniqueness of the set of bivariate Lagrange basis polynomials in the quotient space defined as the space of bivariate polynomials with a certain degree over the subspace of bivariate vanishing polynomials.

1. Introduction

Given $x_0 > x_1 > \cdots > x_n$ and $y_0 > y_1 > \cdots > y_{n+\sigma}$ where n and σ are nonnegative integers, we define a rectangular set of nodes:

$$S = \{(x_r, y_u) :\ 0 \le r \le n,\ 0 \le u \le n+\sigma\}, \tag{1.1}$$

which consists of $(n+1)(n+\sigma+1)$ distinct points in $\mathbf{R}^2$. The set S can be divided into two checkerboard sets S_0 and S_1 such that $(x_r, y_u) \in S_0$ if and only if $r+u$ is even while $(x_r, y_u) \in S_1$ if and only if $r+u$ is odd. Our objective is to develop existence and uniqueness theory of bivariate Lagrange basis polynomials for the checkerboard set S_τ with $\tau = 0$ or $\tau = 1$. The special case $\sigma = 0$ was considered in [6]. The special case $\sigma = 1$ was studied in [5, 7]. If $\sigma = 1$, $x_r = \cos(r\pi/n)$ and $y_u = \cos[u\pi/(n+1)]$, then S_τ is the set of Padua points and the corresponding set of bivariate Lagrange basis polynomials is unique in $\mathbf{P}_n(x, y)$; see [2, 3]. In [9], Xu derived bivariate Lagrange basis polynomials when $\sigma = 0$ and $x_k = y_k = \cos[(2k-1)\pi/(2n)]$ are zeros of Chebyshev polynomial of first kind T_n; see also [1, 4]. The checkerboard nodes S_τ also generalize the Morrow-Patterson nodes [8] and Geronimus nodes [5]. In [7], a recursive formula of bivariate Lagrange basis polynomials for all integers $\sigma \ge 1$ was defined, where the case $\sigma = 1$ was proven and the other cases $\sigma \ge 2$ were proposed as conjectures. In this paper, we will prove these conjectures by deriving an equivalent and explicit formula of

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bivariate Lagrange basis polynomials for any nonnegative integer σ. This formula generalizes the aforementioned results of bivariate Lagrange basis polynomials at the Padua nodes, Chebyshev nodes, Morrow-Patterson nodes and Geronimus nodes. Moreover, we will prove that the set of bivariate Lagrange basis polynomials is unique in a certain quotient space of bivariate polynomials.

Let $\mathbf{P}_d(x, y)$ be the linear space of bivariate polynomials of degree no more than d, which can be generated by the monomials $x^j y^k$ with $j + k \leq d$. It is easily seen that the dimension of $\mathbf{P}_d(x, y)$ is $(d+1)(d+2)/2$. If $f_1(x, y), \cdots, f_M(x, y)$ are linearly independent polynomials in $\mathbf{P}_d(x, y)$, namely, $c_1 f_1(x, y)+\cdots+c_M f_M(x, y) = 0$ for all $(x, y) \in \mathbf{R}^2$ implies $c_1 = \cdots = c_M = 0$, then we can define the quotient space $\mathbf{P}_d(x, y)/\{f_1(x, y), \cdots, f_M(x, y)\}$ in the sense that two polynomials in this quotient space are identical if and only if their difference can be expressed as a linear combination of $f_1(x, y), \cdots, f_M(x, y)$. Clearly, the dimension of $\mathbf{P}_d(x, y)/\{f_1(x, y), \cdots, f_M(x, y)\}$ is $(d + 1)(d + 2)/2 - M$.

Given a set of nodes $(x_1, y_1), \cdots, (x_N, y_N) \in \mathbf{R}^2$, we say $\{L_1(x, y), \cdots, L_N(x, y)\}$ with each $L_j(x, y) \in \mathbf{P}_d(x, y)$ is a set of bivariate Lagrange basis polynomials if $L_j(x_k, y_k) = 0$ for $1 \leq j \neq k \leq N$ and $L_k(x_k, y_k) = 1$ for $1 \leq k \leq N$. For convenience, we also define the *bivariate vanishing polynomial* as a bivariate polynomial $f(x, y) \in \mathbf{P}_d(x, y)$ that vanishes at all of the given nodes; namely, $f(x_k, y_k) = 0$ for all $1 \leq k \leq N$.

The rest of this paper is organized as follows. In Section 2, we state some preliminary results on one-to-one map between a sequence of univariate nodes and a sequence of difference equations. We also give a necessary and sufficient condition for the uniqueness of bivariate Lagrange basis polynomials. In Section 3, we construct the bivariate Lagrange basis polynomials for S_τ with general σ. In Section 4, we prove the uniqueness of the bivariate Lagrange basis polynomials for S_τ in a certain quotient space of bivariate polynomials.

2. Preliminary results

We first rephrase the results in [6, Lemma 2] and [7, Theorem A.1 & Lemma A.2] as Lemma 2.1.

Lemma 2.1. *Given any $x_0 > x_1 > \cdots > x_n$, there exists a sequence of orthogonal polynomials $\{p_k(x)\}_{k=0}^n$ determined by two sequences $\{a_k\}_{k=0}^{n-1}$ and $\{b_k\}_{k=0}^{n-1}$ such that $p_0(x) = 1$, $p_1(x) = a_0 x + b_0$, and*

$$p_{k+1}(x) + p_{k-1}(x) = (a_k x + b_k)p_k(x), \tag{2.1}$$

for $1 \leq k \leq n - 1$, and the following properties hold.

1. *(Positivity) $a_k > 0$ for all $0 \leq k \leq n - 1$.*
2. *(Reflection) $a_k = a_{n-k}$ and $b_k = b_{n-k}$ for all $1 \leq k \leq n - 1$.*
3. *(Alternation) $p_{n-k}(x_j) = (-1)^j p_k(x_j)$ for all $0 \leq j, k \leq n$.*

It is natural to ask whether the map from the set of distinct nodes $X = \{x_k\}_{k=0}^n$ to the set of coefficients $(A, B) = \{(a_k, b_k)\}_{k=0}^{n-1}$ satisfying the positivity and reflection conditions is invertible and unique. It is easy to show that the alternation condition implies the invertibility of the map. When n is odd, the dimension of (A, B) is $n + 1$, which suggests that the map might be unique. However, when n is even, the dimension of (A, B) becomes $n + 2$, which indicates that there is one more degree of freedom in (A, B). We shall prove that this additional degree of freedom can be removed by a normalization condition $a_0 = 1$.

Theorem 2.2. *Given any sequences $\{a_k\}_{k=0}^{n-1}$ and $\{b_k\}_{k=0}^{n-1}$ satisfying the positivity and reflection conditions, namely, $a_k > 0$ for all $0 \le k \le n-1$ and $a_k - a_{n-k} = b_k - b_{n-k} = 0$ for all $1 \le k \le n-1$, there exists a unique set of nodes $x_0 > x_1 > \cdots > x_n$ satisfying the alternation condition $p_{n-k}(x_j) = (-1)^j p_k(x_j)$ for all $0 \le j, k \le n$, where $\{p_k(x)\}_{k=0}^{n}$ is the sequence of orthogonal polynomials determined by the difference equation $p_{k+1}(x) + p_{k-1}(x) = (a_k x + b_k)p_k(x)$ for $1 \le k \le n-1$ with initial conditions $p_0(x) = 1$ and $p_1(x) = a_0 x + b_0$.*

Proof. When $n = 2m-1$ is odd, we denote by $u_1 > \cdots > u_m$ the zeros of $p_m(x)$ and $v_1 > v_2 > \cdots > v_{m-1}$ the zeros of $p_{m-1}(x)$. By alternation property of zeros of orthogonal polynomials, we have $u_1 > v_1 > u_2 > v_2 > \cdots > v_{m-1} > u_m$. Since both $p_m(x)$ and $p_{m-1}(x)$ have positive leading coefficients, the difference function $p_m(x) - p_{m-1}(x)$ has at least one zero at each of the intervals $(u_1, \infty), (u_2, v_1), \cdots, (u_m, v_{m-1})$ because the difference takes opposite signs when the variable x approaches the two ends of each interval. Similarly, the sum function $p_m(x) + p_{m-1}(x)$ has at least one zero at each of the intervals (v_1, u_1), $\cdots$, (v_{m-1}, u_{m-1}), $(-\infty, u_m)$. Thus, we can order the zeros of $p_m(x) - p_{m-1}(x)$ and $p_m(x) + p_{m-1}(x)$ as $x_0 > x_1 > \cdots > x_n$ such that $p_m(x_j) - (-1)^j p_{m-1}(x_j) = 0$ for all $0 \le j \le n$. Actually, we have $x_0 > u_1 > x_1 > v_1 > x_2 > v_2 > \cdots > v_{m-1} > x_{n-1} > u_m > x_n$. It then follows from the difference equation and the reflection condition that $p_{n-k}(x_j) = (-1)^j p_k(x_j)$ for all $0 \le j, k \le n$.

When $n = 2m$ is even, the zeros of $p_m(x)$ divide the real line into $m+1$ intervals, each of which contains at least one zero of $p_{m+1}(x) - p_{m-1}(x)$, thanks to the alternation property of orthogonal polynomials. Hence, we can order the zeros of $p_m(x)$ and $p_{m+1}(x) - p_{m-1}(x)$ as $x_0 > x_1 > \cdots > x_n$ such that $p_m(x_j) = 0$ for all odd $j = 1, 3, \cdots, n-1$ and $p_{m+1}(x_j) = p_{m-1}(x_j)$ for all even $j = 0, 2, \cdots, n$. It then follows from the difference equation and the reflection condition that $p_{n-k}(x_j) = (-1)^j p_k(x_j)$ for all $0 \le j, k \le n$. This completes the proof. □

Theorem 2.3. *Given any $x_0 > x_1 > \cdots > x_n$, if there are two sets of orthogonal polynomials $\{p_k(x)\}_{k=0}^{n}$ and $\{\tilde{p}_k(x)\}_{k=0}^{n}$, which are determined by two sets of coefficients $(A, B) = \{(a_k, b_k)\}_{k=0}^{n-1}$ and $(\tilde{A}, \tilde{B}) = \{(\tilde{a}_k, \tilde{b}_k)\}_{k=0}^{n-1}$, such that the difference equation and three properties (positivity, reflection and alternation) in Lemma 2.1 are satisfied, then we have the following results depending on whether n is odd or even.*

1. *If n is odd, then $\tilde{a}_k = a_k$ and $\tilde{b}_k = b_k$ for $0 \le k \le n-1$.*
2. *If n is even, then there exists a positive constant γ such that $\tilde{a}_k / a_k = \tilde{b}_k / b_k = \gamma$ for even $k = 0, 2, \cdots, n$ and $a_k / \tilde{a}_k = b_k / \tilde{b}_k = \gamma$ for odd $k = 1, 3, \cdots, n-1$.*

Proof. When $n = 2m-1$ is odd, it follows from the alternation condition that the zeros of $p_m(x) + p_{m-1}(x)$ and $\tilde{p}_m(x) + \tilde{p}_{m-1}(x)$ are the same while the zeros of $p_m(x) - p_{m-1}(x)$ and $\tilde{p}_m(x) - \tilde{p}_{m-1}(x)$ are the same. The positivity condition implies that there exist two positive constants γ_1 and γ_2 such that $\tilde{p}_m(x) + \tilde{p}_{m-1}(x) = \gamma_1[p_m(x) + p_{m-1}(x)]$ and $\tilde{p}_m(x) - \tilde{p}_{m-1}(x) = \gamma_2[p_m(x) - p_{m-1}(x)]$. Comparing the leading coefficients yields $\gamma_1 = \gamma_2$. A simple combination of these two equations then gives $\tilde{p}_m(x) = \gamma_1 p_m(x)$ and $\tilde{p}_{m-1}(x) = \gamma_1 p_{m-1}(x)$. In view of the difference equations $\tilde{p}_m(x) + \tilde{p}_{m-2}(x) = (\tilde{a}_{m-1}x + \tilde{b}_{m-1})\tilde{p}_{m-1}(x)$ and $p_m(x) + p_{m-2}(x) = (a_{m-1}x + b_{m-1})p_{m-1}(x)$, we obtain $\tilde{a}_{m-1} = a_{m-1}$, $\tilde{b}_{m-1} = b_{m-1}$ and $\tilde{p}_{m-2}(x) = \gamma_1 p_{m-2}(x)$. Repeating this argument implies $\tilde{a}_k = a_k$ and $\tilde{b}_k = b_k$ for $0 \le k \le m-1$.

Moreover, $\tilde{p}_0(x) = \gamma_1 p_0(x)$, which yields $\gamma_1 = 1$. The reflection condition then gives $\tilde{a}_k = a_k$ and $\tilde{b}_k = b_k$ for $0 \le k \le n-1$.

When $n = 2m$ is even, it follows from the alternation condition that the zeros of $p_m(x)$ and $\tilde{p}_m(x)$ are the same while the zeros of $p_{m+1}(x) - p_{m-1}(x)$ and $\tilde{p}_{m+1}(x) - \tilde{p}_{m-1}(x)$ are the same. The positivity condition implies that there exist two positive constants γ_1 and γ_2 such that $\tilde{p}_m(x) = \gamma_1 p_m(x)$ and $\tilde{p}_{m+1}(x) - \tilde{p}_{m-1}(x) = \gamma_2[p_{m+1}(x) - p_{m-1}(x)]$. On account of the difference equations $\tilde{p}_{m+1}(x) + \tilde{p}_{m-1}(x) = (\tilde{a}_m x + \tilde{b}_m)\tilde{p}_m(x)$ and $p_{m+1}(x) + p_{m-1}(x) = (a_m x + b_m)p_m(x)$, we obtain $\gamma_2 a_m = \gamma_1 \tilde{a}_m$, $\gamma_2 b_m = \gamma_1 \tilde{b}_m$, and $\tilde{p}_{m+1}(x) + \tilde{p}_{m-1}(x) = \gamma_2[p_{m+1}(x) + p_{m-1}(x)]$. Consequently, $\tilde{p}_{m+1}(x) = \gamma_2 p_{m+1}(x)$ and $\tilde{p}_{m-1}(x) = \gamma_2 p_{m-1}(x)$. It then follows from the difference equations $\tilde{p}_m(x) + \tilde{p}_{m-2}(x) = (\tilde{a}_{m-1}x + \tilde{b}_{m-1})\tilde{p}_{m-1}(x)$ and $p_m(x) + p_{m-2}(x) = (a_{m-1}x + b_{m-1})\tilde{p}_{m-1}(x)$ that $\gamma_1 a_{m-1} = \gamma_2 \tilde{a}_{m-1}$, $\gamma_1 b_{m-1} = \gamma_2 \tilde{b}_{m-1}$ and $\tilde{p}_{m-2}(x) = \gamma_1 p_{m-2}(x)$. Repeating this argument gives

$$\gamma_1 a_{m-j} = \gamma_2 \tilde{a}_{m-j},\ \gamma_1 b_{m-j} = \gamma_2 \tilde{b}_{m-j},\ \tilde{p}_{m-j}(x) = \gamma_2 p_{m-j}(x)$$

for odd $j \le m$, and

$$\gamma_2 a_{m-j} = \gamma_1 \tilde{a}_{m-j},\ \gamma_2 b_{m-j} = \gamma_1 \tilde{b}_{m-j},\ \tilde{p}_{m-j}(x) = \gamma_1 p_{m-j}(x)$$

for even $j \le m$. Since $\tilde{p}_0(x) = p_0(x) = 1$, either $\gamma_1 = 1$ (when m is even) or $\gamma_2 = 1$ (when m is odd). We denote $\gamma = \gamma_2$ if m is even and $\gamma = \gamma_1$ if m is odd. It then follows that

$$\tilde{a}_{2j} = \gamma a_{2j},\ \tilde{b}_{2j} = \gamma b_{2j},\ \tilde{p}_{2j}(x) = p_{2j}(x),$$

for $j = 0, \cdots, m$, and

$$\gamma \tilde{a}_{2j+1} = a_{2j+1},\ \gamma \tilde{b}_{2j+1} = b_{2j+1},\ \tilde{p}_{2j+1}(x) = \gamma p_{2j}(x),$$

for $j = 0, \cdots, m-1$. This completes the proof. □

For the univariate case, the set of Lagrange basis polynomials for any set of distinct points exists and is uniquely determined because the corresponding Vandermonde matrix is invertible. Theorem 2.4 gives criteria for uniqueness of bivariate Lagrange basis polynomials.

Theorem 2.4. *Given any distinct points $(x_1, y_1), \cdots, (x_N, y_N) \in \mathbf{R}^2$ and any positive integer d such that $(d+1)(d+2)/2 \ge N$, there exist at least $M = (d+1)(d+2)/2 - N$ linear independent bivariate vanishing polynomials, denoted by $f_1(x,y), \cdots, f_M(x,y)$, in $\mathbf{P}_d(x,y)$. Let*

$$V = \begin{pmatrix} 1 & x_1 & \cdots & x_1^d & y_1 & x_1y_1 & \cdots & x_1^{d-1}y_1 & \cdots & y_1^d \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 1 & x_N & \cdots & x_N^d & y_N & x_Ny_N & \cdots & x_N^{d-1}y_N & \cdots & y_N^d \end{pmatrix} \tag{2.2}$$

be the bivariate Vandermonde matrix of dimension N by $(d+1)(d+2)/2$. The following statements are equivalent.

(i) *There exists a unique set of bivariate Lagrange interpolation polynomials in the quotient space $\mathbf{P}_d(x,y)/\{f_1(x,y), \cdots, f_M(x,y)\}$.*
(ii) *There exists a set of bivariate Lagrange interpolation polynomials in $\mathbf{P}_d(x,y)$.*
(iii) *Any bivariate vanishing polynomial in $\mathbf{P}_d(x,y)$ can be expressed as a linear combination of $f_1(x,y), \cdots, f_M(x,y)$.*
(iv) *The rank of V is N.*

Proof. The coefficients of any bivariate vanishing polynomial in $\mathbf{P}_d(x,y)$ correspond a vector $z \in \mathbf{R}^{(d+1)(d+2)/2}$ satisfying $Vz = 0$. The existence of $f_1(x,y), \cdots, f_M(x,y)$ follows from the fact that the rank of V is no more than N. Moreover, we have (iii) $\Leftrightarrow$ (iv). Any set of bivariate Lagrange interpolation polynomials in $\mathbf{P}_d(x,y)$ can be represented by the matrix L of dimension $(d+1)(d+2)/2$ by N such that VL is the identity matrix in $\mathbf{R}^{N\times N}$. Hence, (ii) $\Leftrightarrow$ (iv). It is obvious that (i) $\Rightarrow$ (ii). Finally, coupling (ii) and (iii) gives (i). The proof is complete. $\square$

3. Existence of bivariate Lagrange basis polynomials

Given $x_0 > x_1 > \cdots > x_n$ and $y_0 > y_1 > \cdots > y_{n+\sigma}$ where n and σ are nonnegative integers, we define two sets of checkerboard nodes

$$S_0 = \{(x_r, y_u):\ 0 \le r \le n,\ 0 \le u \le n+\sigma,\ r+u \text{ even}\}, \tag{3.1}$$

$$S_1 = \{(x_r, y_u):\ 0 \le r \le n,\ 0 \le u \le n+\sigma,\ r+u \text{ odd}\}, \tag{3.2}$$

which consist of N_0 and N_1 nodes, respectively. It is easily seen that $N_0+N_1 = (n+1)(n+\sigma+1)$. Moreover, we have $N_0 - N_1 = 1$ and $N_\tau = [(n+1)(n+\sigma+1)+1]/2-\tau$ if both n and σ are even, while $N_0 = N_1 = (n+1)(n+\sigma+1)/2$ if either n or σ is odd. We need to find a set of bivariate Lagrange basis polynomials for S_τ with $\tau = 0$ or $\tau = 1$. According to Lemma 2.1, there exist orthogonal polynomials $\{p_j(x)\}_{j=0}^{n}$ and $\{q_k(y)\}_{k=0}^{n+\sigma}$ such that $p_0(x) = 1$, $p_1(x) = a_0 x + b_0$, $q_0(y) = 1$, $q_1(x) = c_0 y + d_0$, and

$$p_{j+1}(x) + p_{j-1}(x) = (a_j x + b_j)p_j(x),\ 1 \le j \le n-1, \tag{3.3}$$

$$q_{k+1}(y) + q_{k-1}(y) = (c_k y + d_k)q_k(y),\ 1 \le k \le n+\sigma-1, \tag{3.4}$$

where $a_j > 0$ for $0 \le j \le n-1$, and $c_k > 0$ for $0 \le k \le n+\sigma-1$, and

$$a_j = a_{n-j}, b_j = b_{n-j},\ 1 \le j \le n-1, \tag{3.5}$$

$$c_k = c_{n+\sigma-k}, d_k = d_{n+\sigma-k},\ 1 \le k \le n+\sigma-1, \tag{3.6}$$

$$p_{n-j}(x_r) = (-1)^r p_j(x_r),\ 0 \le j, r \le n, \tag{3.7}$$

$$q_{n+\sigma-k}(y_u) = (-1)^u q_k(y_u),\ 0 \le k, u \le n+\sigma. \tag{3.8}$$

For any $(x_r, y_u) \in S_\tau$ and $(x_s, y_v) \in S_\tau$, where τ is either 0 or 1, it is easily seen that $r+u+s+v$ is even. Hence, we have from the above two equations

$$p_j(x_r)p_i(x_s)q_k(y_u)q_l(y_v) = p_{n-j}(x_r)p_{n-i}(x_s)q_{n+\sigma-k}(y_u)q_{n+\sigma-l}(y_v), \tag{3.9}$$

for all $0 \le j, i \le n$ and $0 \le k, l \le n+\sigma$. Moreover, we have the following Christoffel-Darboux formulas:

$$(x_r - x_s)\sum_{j=0}^{i} a_j p_j(x_r)p_j(x_s) = p_{i+1}(x_r)p_i(x_s) - p_i(x_r)p_{i+1}(x_s), \tag{3.10}$$

$$(y_u - y_v)\sum_{k=0}^{l} c_k q_k(y_u)q_k(y_v) = q_{l+1}(y_u)q_l(y_v) - q_l(y_u)q_{l+1}(y_v), \tag{3.11}$$

for $0 \le i \le n-1$ and $0 \le l \le n+\sigma-1$. Given any integers $0 \le \delta \le \sigma-1$, $0 \le s \le n$ and $0 \le v \le n+\sigma$, we define the bivariate polynomial

$$K_\delta(x, y; x_s, y_v) = \sum_{j=0}^{n-1} a_j p_j(x)p_j(x_s) \sum_{k=0}^{n-j+\delta} c_k q_k(y)q_k(y_v) \in \mathbf{P}_{n+\delta}(x,y). \tag{3.12}$$

It is readily seen from (3.9) and (3.11) that

$$\begin{aligned}
&(y_u - y_v)K_\delta(x_r, y_u; x_s, y_v)\\
=&\sum_{j=0}^{n-1} a_j p_j(x_r)p_j(x_s)[q_{n-j+\delta+1}(y_u)q_{n-j+\delta}(y_v) - q_{n-j+\delta}(y_u)q_{n-j+\delta+1}(y_v)]\\
=&\sum_{j=0}^{n-1} a_j p_{n-j}(x_r)p_{n-j}(x_s)[q_{j+\sigma-\delta-1}(y_u)q_{j+\sigma-\delta}(y_v) - q_{j+\sigma-\delta}(y_u)q_{j+\sigma-\delta-1}(y_v)]\\
=&\sum_{j=1}^{n} a_{n-j} p_j(x_r)p_j(x_s)[q_{n-j+\sigma-\delta-1}(y_u)q_{n-j+\sigma-\delta}(y_v)\\
&- q_{n-j+\sigma-\delta}(y_u)q_{n-j+\sigma-\delta-1}(y_v)],
\end{aligned}$$

and

$$\begin{aligned}
&(y_u - y_v)K_{\sigma-\delta-1}(x_r, y_u; x_s, y_v)\\
=&\sum_{j=0}^{n-1} a_j p_j(x_r)p_j(x_s)[q_{n-j+\sigma-\delta}(y_u)q_{n-j+\sigma-\delta-1}(y_v)\\
&- q_{n-j+\sigma-\delta-1}(y_u)q_{n-j+\sigma-\delta}(y_v)].
\end{aligned}$$

Adding the above two equations and making use of (3.5), (3.9) and (3.11) yield

$$\begin{aligned}
&(y_u - y_v)[K_\delta(x_r, y_u; x_s, y_v) + K_{\sigma-\delta-1}(x_r, y_u; x_s, y_v)]\\
=&a_0 p_n(x_r)p_n(x_s)[q_{\sigma-\delta-1}(y_u)q_{\sigma-\delta}(y_v) - q_{\sigma-\delta}(y_u)q_{\sigma-\delta-1}(y_v)]\\
&+ a_0 p_0(x_r)p_0(x_s)[q_{n+\sigma-\delta}(y_u)q_{n+\sigma-\delta-1}(y_v) - q_{n+\sigma-\delta-1}(y_u)q_{n+\sigma-\delta}(y_v)]\\
=&a_0 p_n(x_r)p_n(x_s)[q_{\sigma-\delta-1}(y_u)q_{\sigma-\delta}(y_v) - q_{\sigma-\delta}(y_u)q_{\sigma-\delta-1}(y_v)]\\
&+ a_0 p_n(x_r)p_n(x_s)[q_\delta(y_u)q_{\delta+1}(y_v) - q_{\delta+1}(y_u)q_\delta(y_v)]\\
=&-(y_u - y_v)a_0 p_n(x_r)p_n(x_s)[\sum_{k=0}^{\sigma-\delta-1} c_k q_k(y_u)q_k(y_v) + \sum_{k=0}^{\delta} c_k q_k(y_u)q_k(y_v)],
\end{aligned}$$

which can be written as

$$(y_u - y_v)[K_\delta(x_r, y_u; x_s, y_v) + K_{\sigma-\delta-1}(x_r, y_u; x_s, y_v) + J(x_r, y_u; x_s, y_v)] = 0, \tag{3.13}$$

where

$$J(x, y; x_s, y_v) = a_0 p_n(x)p_n(x_s)[\sum_{k=0}^{\sigma-\delta-1} c_k q_k(y)q_k(y_v) + \sum_{k=0}^{\delta} c_k q_k(y)q_k(y_v)]. \tag{3.14}$$

Interchanging the double sum in (3.12) gives another expression

$$K_\delta(x, y; x_s, y_v) = \sum_{k=0}^{n+\delta} c_k q_k(y)q_k(y_v) \sum_{j=0}^{\min\{n-1, n+\delta-k\}} a_j p_j(x)p_j(x_s).$$

It is readily seen from (3.6), (3.9) and (3.10) that

$$\begin{aligned}&(x_r - x_s)K_\delta(x_r, y_u; x_s, y_v)\\ &= \sum_{k=1+\delta}^{n+\delta} c_k q_k(y_u)q_k(y_v)[p_{n-k+\delta+1}(x_r)p_{n-k+\delta}(x_s) - p_{n-k+\delta}(x_r)p_{n-k+\delta+1}(x_s)]\\ &\quad + \sum_{k=0}^{\delta} c_k q_k(y_u)q_k(y_v)[p_n(x_r)p_{n-1}(x_s) - p_{n-1}(x_r)p_n(x_s)]\\ &=: A_\delta + B_\delta,\end{aligned}$$

where

$$\begin{aligned}&A_\delta\\ &= \sum_{k=1+\delta}^{n+\delta} c_{n+\sigma-k}q_{n+\sigma-k}(y_u)q_{n+\sigma-k}(y_v)[p_{k-\delta-1}(x_r)p_{k-\delta}(x_s) - p_{k-\delta}(x_r)p_{k-\delta-1}(x_s)]\\ &= \sum_{k=\sigma-\delta}^{n+\sigma-\delta-1} c_k q_k(y_u)q_k(y_v)[p_{n-k+\sigma-\delta-1}(x_r)p_{n-k+\sigma-\delta}(x_s)\\ &\quad - p_{n-k+\sigma-\delta}(x_r)p_{n-k+\sigma-\delta-1}(x_s)]\\ &= - A_{\sigma-\delta-1},\end{aligned}$$

and

$$\begin{aligned}B_\delta &= \sum_{k=0}^{\delta} c_k q_{n-k+\sigma}(y_u)q_{n-k+\sigma}(y_v)[p_0(x_r)p_1(x_s) - p_1(x_r)p_0(x_s)]\\ &= - (x_r - x_s)\sum_{k=0}^{\delta} c_k q_{n-k+\sigma}(y_u)q_{n-k+\sigma}(y_v)a_0p_0(x_r)p_0(x_s)\\ &= - (x_r - x_s)\sum_{k=0}^{\delta} c_k q_k(y_u)q_k(y_v)a_0p_n(x_r)p_n(x_s).\end{aligned}$$

Hence,

$$(x_r - x_s)[K_\delta(x_r, y_u; x_s, y_v) + K_{\sigma-\delta-1}(x_r, y_u; x_s, y_v)] = B_\delta + B_{\sigma-\delta-1}.$$

Recall the definition of J in (3.14), we obtain

$$(x_r - x_s)[K_\delta(x_r, y_u; x_s, y_v) + K_{\sigma-\delta-1}(x_r, y_u; x_s, y_v) + J(x_r, y_u; x_s, y_v)] = 0. \tag{3.15}$$

Finally, we choose $\delta = \lfloor \sigma/2 \rfloor$ and define the bivariate polynomial

$$\begin{aligned}G_n^\sigma(x, y; x_s, y_v) &= K_\delta(x, y; x_s, y_v) + K_{\sigma-\delta-1}(x, y; x_s, y_v) + J(x, y; x_s, y_v)\\ &= \sum_{j=0}^{n-1} a_j p_j(x)p_j(x_s)[\sum_{k=0}^{n-j+\delta} c_k q_k(y)q_k(y_v) + \sum_{k=0}^{n-j+\sigma-\delta-1} c_k q_k(y)q_k(y_v)]\\ &\quad + a_0p_n(x)p_n(x_s)[\sum_{k=0}^{\sigma-\delta-1} c_k q_k(y)q_k(y_v) + \sum_{k=0}^{\delta} c_k q_k(y)q_k(y_v)].\end{aligned} \tag{3.16}$$

It is obvious that $G_n^\sigma \in \mathbf{P}_{n+\delta}(x, y)$ and $G_n^\sigma(x_s, y_v; x_s, y_v) > 0$. Coupling (3.13) and (3.15) gives $G_n^\sigma(x_r, y_u; x_s, y_v) = 0$ if either $x_r \neq x_s$ or $y_u \neq y_v$. We summarize our results in Theorem 3.1.

Theorem 3.1. *Given any $x_0 > x_1 > \cdots > x_n$ and $y_0 > y_1 > \cdots > y_{n+\sigma}$ where n and σ are nonnegative integers. Let S_τ with either $\tau = 0$ or $\tau = 1$ be defined in* (3.1) *or* (3.2). *Set $\delta = \lfloor \sigma/2 \rfloor$. There exists a set of bivariate Lagrange basis polynomials in $\mathbf{P}_{n+\delta}(x,y)$ for S_τ which can be defined as*

$$L(x,y;x_s,y_v) = G_n^\sigma(x,y;x_s,y_v)/G_n^\sigma(x_s,y_v;x_s,y_v), \tag{3.17}$$

for each $(x_s,y_v) \in S_\tau$, where G_n^σ is defined in (3.16).

Now, we are ready to prove [7, Conjecture 3]: $\tilde{G}_n^\sigma(x,y;x_s,y_v)/\tilde{G}_n^\sigma(x_s,y_v;x_s,y_v)$ is a set of bivariate Lagrange basis polynomials in $\mathbf{P}_{n+\lfloor\sigma/2\rfloor}(x,y)$, where $\tilde{G}_n^\sigma$ is recursively defined as

$$\tilde{G}_n^\sigma = \begin{cases} \tilde{K}_{n-1} + \Gamma_n^0, & \sigma = 1, \\ \tilde{G}_n^{\sigma-1} + \Gamma_n^{\lfloor\sigma/2\rfloor}/2, & \sigma \geq 2, \end{cases} \tag{3.18}$$

with

$$\begin{aligned} \tilde{K}_{n-1}(x,y;x_s,y_v) =& \frac{1}{a_0H_0c_0\tilde{H}_0}\sum_{i=0}^{n-1}\sum_{j=0}^{i} a_jp_j(x)p_j(x_r)c_{i-j}q_{i-j}(y)q_{i-j}(y_v) \\ =& \frac{1}{a_0H_0c_0\tilde{H}_0}\sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1} a_jp_j(x)p_j(x_r)c_kq_k(y)q_k(y_v), \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \Gamma_n^l(x,y;x_s,y_v) =& \frac{1}{a_0H_0c_0\tilde{H}_0}\Bigg[a_0p_n(x)p_n(x_r)c_lq_l(y)q_l(y_v) \\ &+ \sum_{j=0}^{n-1} a_jp_j(x)p_j(x_r)c_{n+l-j}q_{n+l-j}(y)q_{n+l-j}(y_v)\Bigg]. \end{aligned} \tag{3.20}$$

Here, H_0 and $\tilde{H}_0$ are any given positive numbers, which are canceled in the ratio $\tilde{G}_n^\sigma(x,y;x_s,y_v)/\tilde{G}_n^\sigma(x_s,y_v;x_s,y_v)$. In other words, the set of bivariate Lagrange basis polynomials is independent of the choices of the positive numbers H_0 and $\tilde{H}_0$. In [7, Theorem 5], it was chosen that $H_0 = \mu(\mathbf{R})$ and $\tilde{H}_0 = \tilde{\mu}(\mathbf{R})$, where μ and $\tilde{\mu}$ are the measures of orthogonality for the polynomials p_j and q_k, respectively. However, if we multiply H_0 and μ by any positive number, and multiply $\tilde{H}_0$ and $\tilde{\mu}$ by another positive number, the statement in [7, Theorem 5] is still valid because an equation does not change if we multiply both sides by the same positive number. Here, without loss of generality, we may choose $H_0 = 1/a_0$ and $\tilde{H}_0 = 1/c_0$. It is easily seen that the fraction $1/(a_0H_0c_0\tilde{H}_0)$ in the definitions (3.19) and (3.20) is simplified to be 1.

Proposition 3.2. *Given any $x_0 > x_1 > \cdots > x_n$ and $y_0 > y_1 > \cdots > y_{n+\sigma}$ where n and σ are nonnegative integers. Let G_n^σ, $\tilde{G}_n^\sigma$ and Γ_n^l be defined in* (3.16), (3.18) *and* (3.20), *respectively. We have*

$$G_n^\sigma(x,y;x_s,y_v) - G_n^{\sigma-1}(x,y;x_s,y_v) = \Gamma_n^{\lfloor\sigma/2\rfloor}(x,y;x_s,y_v), \tag{3.21}$$

for $\sigma \geq 2$, and

$$G_n^\sigma(x,y;x_s,y_v) = 2\tilde{G}_n^\sigma(x,y;x_s,y_v), \tag{3.22}$$

for $\sigma \geq 1$.

Proof. If $\sigma = 2\delta + 1$ is odd with $\delta \geq 1$, we obtain from (3.16) and $\sigma - \delta - 1 = \delta$ that

$$\begin{aligned}&G_n^{\sigma}(x, y; x_s, y_v) - G_n^{\sigma-1}(x, y; x_s, y_v)\\ =&\sum_{j=0}^{n-1} a_j p_j(x) p_j(x_s) c_{n-j+\delta} q_{n-j+\delta}(y) q_{n-j+\delta}(y_v) + a_0 p_n(x) p_n(x_s) c_\delta q_\delta(y) q_\delta(y_v).\end{aligned}$$

If $\sigma = 2\delta$ is even with $\delta \geq 1$, the above equation also holds. Hence, (3.21) follows from (3.20). Next, we note from (3.16), (3.18), (3.19) and (3.20) that

$$\begin{aligned}\tilde{G}_n^1(x, y; x_s, y_v) =& \sum_{j=0}^{n-1}\sum_{k=0}^{n-j} a_j p_j(x) p_j(x_r) c_k q_k(y) q_k(y_v) + a_0 p_n(x) p_n(x_r) c_0\\ =& G_n^1(x, y; x_s, y_v)/2.\end{aligned}$$

This proves (3.22) with $\sigma = 1$. A simple induction argument together with (3.21) gives (3.22) for all $\sigma \geq 1$. □

Coupling Theorem 3.1 and (3.22) proves [7, Conjecture 3].

4. Uniqueness of bivariate Lagrange basis polynomials

We use the same notations as in the previous section. To prove that the set of bivariate Lagrange basis polynomials constructed in (3.17) is unique in a certain quotient space of $\mathbf{P}_{n+\delta}(x, y)$, we only need to find $M = (n + \delta + 1)(n + \delta + 2)/2 - N_\tau$ linearly independent bivariate vanishing polynomials in $\mathbf{P}_{n+\delta}(x, y)$, where $\delta = \lfloor \sigma/2 \rfloor$ and N_τ is the number of nodes in S_τ with $\tau = 0$ or $\tau = 1$. In other words, we shall construct a linear subspace Q of bivariate vanishing polynomials in $\mathbf{P}_{n+\delta}(x, y)$ with dimension M and then apply Theorem 2.4. First, we introduce the linear subspace of bivariate vanishing polynomials:

$$V = \operatorname{span}\{(x - x_0)\cdots(x - x_n)x^j y^k,\ j \geq 0,\ k \geq 0,\ j + k \leq \delta - 1\}. \tag{4.1}$$

It is obvious that V is a subspace of $\mathbf{P}_{n+\delta}(x, y)$ with dimension $\delta(\delta + 1)/2$. Furthermore, we have Lemma 4.1.

Lemma 4.1. *Given $\tau = 0$ or 1. If $n = 2m - 1$ is odd, then the polynomials $(x - x_0)\cdots(x - x_n)x^j y^k$ with $j, k \geq 0$ and $j + k \leq \delta - 1$ and the polynomials $p_{n-l}(x)q_{l+\delta}(y) - (-1)^\tau p_l(x)q_{n+\delta-l}(y)$ with $0 \leq l \leq m - 1$ are linearly independent. If $n = 2m$ is even, then the polynomials $(x - x_0)\cdots(x - x_n)x^j y^k$ with $j, k \geq 0$ and $j + k \leq \delta - 1$ and the polynomials $p_{n-l}(x)q_{l+\delta}(y) - (-1)^\tau p_l(x)q_{n+\delta-l}(y)$ with $0 \leq l \leq m - 1 + \tau$ are linearly independent.*

Proof. We only consider the case when $n = 2m - 1$ is odd. The case when $n = 2m$ is even can be proved in a similar manner. Assume that

$$\begin{aligned}&\sum_{j\geq 0, k\geq 0, j+k\leq \delta-1} a_{j,k}(x - x_0)\cdots(x - x_n)x^j y^k\\ &+ \sum_{l=0}^{m-1} b_l[p_{n-l}(x)q_{l+\delta}(y) - (-1)^\tau p_l(x)q_{n+\delta-l}(y)] = 0\end{aligned}$$

for all $(x, y) \in \mathbf{R}^2$. We will prove that the coefficients $a_{j,k}$ and b_l vanish. First, for any $j \geq 1$ and $k \geq 0$ with $j + k \leq \delta - 1$, the coefficient of $x^{n+j}y^k$ on the left-hand side vanishes, which implies that $a_{j,k} = 0$ for $j \geq 1$ and $k \geq 0$ with $j + k \leq \delta - 1$.

Secondly, the coefficient of $x^n y^\delta$ on the left-hand side vanishes, which implies that $b_0 = 0$. Thirdly, for any $0 \le k \le \delta - 1$, the coefficient of $x^n y^k$ on the left-hand side vanishes, which implies that $a_{0,k} = 0$ for $0 \le k \le \delta - 1$. Finally, for any $l = 1, \cdots, m-1$, the coefficient of $x^{n-l} y^{l+\delta}$ on the left-hand side vanishes, which together with a successive argument implies that $b_l = 0$ for $l = 1, \cdots, m-1$. This completes the proof. □

We shall consider the following three cases respectively.

Case I. $\sigma = 2\delta + 1$ is odd.

We have $N_0 = N_1 = (n+1)(n+\sigma+1)/2$ and

$$M = \frac{(n+\delta+1)(n+\delta+2)}{2} - \frac{(n+1)(n+\sigma+1)}{2} = \frac{\delta(\delta+1)}{2}$$

for either $\tau = 0$ or $\tau = 1$. We simply set

$$Q = V. \tag{4.2}$$

Case II. $\sigma = 2\delta$ is even and $n = 2m - 1$ is odd.

We have $N_0 = N_1 = (n+1)(n+\sigma+1)/2$ and

$$M = \frac{(n+\delta+1)(n+\delta+2)}{2} - \frac{(n+1)(n+\sigma+1)}{2} = \frac{\delta(\delta+1)}{2} + m$$

for either $\tau = 0$ or $\tau = 1$. Note from (3.7) and (3.8) that

$$p_{n-j}(x_r)q_{j+\delta}(y_u) = (-1)^{r+u} p_j(x_r) q_{n+\delta-j}(y_u) = (-1)^\tau p_j(x_r) q_{n+\delta-j}(y_u),$$

for any $(x_r, y_u) \in S_\tau$. We define

$$Q = V + \operatorname{span}\{p_{n-j}(x)q_{j+\delta}(y) - (-1)^\tau p_j(x) q_{n+\delta-j}(y),\ 0 \le j \le m-1\}, \tag{4.3}$$

which, in view of Lemma 4.1, is a subspace of bivariate vanishing polynomials in $\mathbf{P}_{n+\delta}(x,y)$ with dimension $M = \delta(\delta+1)/2 + m$.

Case III. $\sigma = 2\delta$ is even and $n = 2m$ is even.

We have $N_\tau = [(n+1)(n+\sigma+1)+1]/2 - \tau$ and

$$M = \frac{(n+\delta+1)(n+\delta+2)}{2} - \frac{(n+1)(n+\sigma+1)+1}{2} + \tau = \frac{\delta(\delta+1)}{2} + m + \tau$$

for $\tau = 0, 1$. We define

$$Q = V + \operatorname{span}\{p_{n-j}(x)q_{j+\delta}(y) - (-1)^\tau p_j(x) q_{n+\delta-j}(y),\ 0 \le j \le m-1+\tau\}, \tag{4.4}$$

which, in view of Lemma 4.1, is a subspace of bivariate vanishing polynomials in $\mathbf{P}_{n+\delta}(x,y)$ with dimension $M = \delta(\delta+1)/2 + m + \tau$.

On account of Theorem 2.4, we have the following uniqueness property of bivariate Lagrange basis polynomials for S_τ with $\tau = 0$ or $\tau = 1$.

Theorem 4.2. *Given any $x_0 > x_1 > \cdots > x_n$ and $y_0 > y_1 > \cdots > y_{n+\sigma}$ where n and σ are nonnegative integers. Let S_τ with either $\tau = 0$ or $\tau = 1$ be defined in* (3.1) *or* (3.2). *Set $\delta = \lfloor \sigma/2 \rfloor$. The set of bivariate Lagrange basis polynomials for S_τ defined in* (3.17) *is unique in the quotient space $\mathbf{P}_{n+\delta}(x,y)/Q$, where Q is defined in* (4.2)-(4.4) *depending on odd-even properties of σ and n.*

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