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Dirichlet problem for a delayed diffusive hematopoiesis model

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A B S T R A C T

We study the dynamics of a delayed diffusive hematopoiesis model with two types of Dirichlet boundary conditions. For the model with a zero Dirichlet boundary condition, we establish global stability of the trivial equilibrium under certain conditions, and use the phase plane method to prove the existence and uniqueness of a positive spatially heterogeneous steady state. We further obtain delay-independent as well as delay-dependent conditions for the local stability of this steady state. For the model with a non-zero Dirichlet boundary condition, we show that the only positive steady state is a constant solution. Results for the local stability of the constant solution are also provided. By using the delay as a bifurcation parameter, we show that the model has infinite number of Hopf bifurcation values and the global Hopf branches bifurcated from these values are unbounded, which indicates the global existence of periodic solutions.

1. Introduction

In this paper, we are concerned with the Dirichlet problem for the delayed diffusive hematopoiesis model described by

$$\frac{\partial u(x,t)}{\partial t} = d\triangle u(x,t) - \delta u(x,t) + \frac{\alpha u(x,t-\tau)}{1+\beta u^k(x,t-\tau)}, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

where Ω is a connected bounded open domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $u(x,t) \in L^2(\bar{\Omega}, \mathbb{R})$ represents the density of mature cells in blood circulation, Δ is the Laplacian operator and $d > 0$ is the diffusion coefficient, τ is the time delay between the initiation of cellular production in the bone marrow and the release of mature cells into the blood. δ is the death rate, α is the intrinsic growth rate, β is a positive constant, and $k > 1$ is an integer.

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Note that when spatial effect is neglected (i.e., $d = 0$), Eq (1.1) reduces to the following Mackey–Glass equation proposed by Mackey and Glass [1] for studying the regulation of hematopoiesis

$$\frac{du}{dt} = -\delta u(x,t) + \frac{\alpha u(t-\tau)}{1+\beta u^k(t-\tau)}. \tag{1.2}$$

Together with the classical delayed logistic equation and the delayed Nicholson's equation, the Mackey–Glass equation has greatly promoted the development of theory of nonlinear functional differential equations; see [2–7] and references therein.

When $\alpha > \delta$, Model (1.1) admits a unique positive steady state

$$u_* = \left(\frac{1}{\beta}(\frac{\alpha}{\delta} - 1)\right)^{1/k} > 0. \tag{1.3}$$

In [8], the traveling wave solutions connecting the trivial equilibrium 0 and the unique positive equilibrium u_* were investigated. The Neumann problem of the delayed diffusive hematopoiesis model has been considered by Wang and Li [9] for its dynamics around these two equilibria. Neumann problems for other delayed diffusive systems have been extensively studied, see, for example, [10–12]. Compared to Neumann problems, it is generally more challenging to analyze Dirichlet problems and only limited results have been obtained; see [13–19].

In this work, we will consider Eq. (1.1) with a zero Dirichlet boundary condition and a non-zero Dirichlet boundary condition. The zero Dirichlet boundary condition indicates that the boundary of the domain is hostile to cells and the population vanishes on the boundary. For this case, we establish the existence, uniqueness and asymptotic stability of the spatially heterogeneous steady state. Unlike the Neumann boundary condition, zero Dirichlet boundary condition excludes the existence of positive homogeneous steady state. Thus, we impose a non-zero Dirichlet boundary condition for the diffusive model (1.1) in the sense that the environment on the boundary is hospitable to the cells and the population maintains at the positive equilibrium level. We then prove the existence, uniqueness and stability of the positive homogeneous steady state. Similar as in the ODE model, when the positive equilibrium is unstable, Hopf bifurcation occurs and time-periodic solutions dominate the model dynamics. We will use time delay as the bifurcation parameter to study the stability and direction of local Hopf branches. We also explore the continuation and structure of global Hopf branches and show that the global Hopf branches are all unbounded, which implies the global existence of periodic solutions.

We organize the rest of the paper as follows. In Section 2, we establish some preliminary results such as existence, uniqueness, positiveness and boundedness of the solution for the Dirichlet problem. Section 3 is devoted to the dynamics of Model (1.1) with zero Dirichlet boundary condition. In this section, a necessary and sufficient condition for the existence of heterogeneous positive steady state is given. Also, we investigate the stability of trivial equilibrium and positive steady state. In Section 4, we conduct stability and bifurcation analysis of the model with non-zero Dirichlet boundary condition. In Section 5, we present some simulation results to support our theoretical results and conclude the paper. An explicit algorithm for determining the direction and stability of Hopf bifurcation is given in the Appendix.

2. Preliminary results

We consider Eq. (1.1) with Dirichlet boundary condition and nonnegative initial condition:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\triangle u(x,t) - \delta u(x,t) + f(u(x,t-\tau)), & x \in \Omega, t > 0, \\ u(\cdot,t)\mid_{\partial\Omega} = u_b \in [0, f(c)/\delta], & t \geq 0, \\ u(x,\theta) = u_0(x,\theta) \geq 0, \quad x \in \Omega, & \theta \in [-\tau, 0], \end{cases} \tag{2.1}$$

where

$$f(u) = \frac{\alpha u}{1+\beta u^k}, \quad c = [\beta(k-1)]^{-1/k}. \tag{2.2}$$

Note that, given any integer $k > 1$, the function $f(u)$ attains the maximum value at $u = c$.

Our next result establishes the existence, uniqueness and boundedness of the solution to (2.1).

Theorem 2.1. *There exists a unique solution $u(x,t)$ of (2.1). Moreover, the solution $u(x,t)$ is nonnegative and eventually bounded by $f(c)/\delta$; namely, $u(x,t) \geq 0$ for all $(x,t) \in \bar{\Omega} \times [0,\infty)$, and*

$$\limsup_{t\to\infty} u(x,t) \leq f(c)/\delta.$$

If further, $u_0(x,\theta) \geq 0 (\not\equiv 0)$, then $u(x,t) > 0$ for all $(x,t) \in \Omega \times (\tau,\infty)$.

Proof. Let $\bar{u}_0 = \max\{u_b, \sup_{\bar{\Omega}\times[-\tau,0]} u_0(x,\theta)\}$, and $\bar{u}(t)$ be the unique solution to the ordinary differential equation:

$$\begin{cases} \frac{d\bar{u}(t)}{dt} = -\delta\bar{u}(t) + f(c), \\ \bar{u}(0) = \bar{u}_0. \end{cases} \tag{2.3}$$

It is readily seen that $\lim_{t\to\infty} \bar{u}(t) = f(c)/\delta$ and $\bar{u}(t)$ is monotone in t. Furthermore, since $u_b \leq f(c)/\delta$ and $u_b \leq \bar{u}_0$, we have $\bar{u}(t) \geq u_b$ for any $t \geq 0$.

Now, we introduce $\underline{u}(x,t) = 0$ and $\overline{u}(x,t) = \bar{u}(t)$, and claim that these two functions formulate a pair of lower-solution and upper-solution to (2.1). To see this, we note that

$$\underline{u}(x,0) = 0 \leq u_0(x,t) \leq \bar{u}_0 = \overline{u}(x,0),$$

and for any $\underline{u}(x,t) \leq h(x,t) \leq \overline{u}(x,t)$ in $\bar{\Omega} \times [-\tau,\infty)$,

$$\begin{aligned} &\tfrac{\partial \overline{u}(x,t)}{\partial t} - d\Delta\overline{u}(x,t) + \delta\overline{u}(x,t) = f(c) \geq f(h(x,t)), \\ &\tfrac{\partial \underline{u}(x,t)}{\partial t} - d\Delta\underline{u}(x,t) + \delta\underline{u}(x,t) = 0 \leq f(h(x,t)). \end{aligned}$$

Furthermore, for any $x \in \partial\Omega$ and $t \geq 0$, $\underline{u}(x,t) \leq u_b \leq \overline{u}(x,t)$. This proves our claim; see [20, Definition 8.1.2] or [21]. It then follows from [20, Theorem 8.3.3] or [21, Theorem 3.4] that (2.1) has a unique global solution $u(x,t)$ which satisfies

$$0 \leq u(x,t) \leq \bar{u}(t) \quad \text{for all} \quad (x,t) \in \bar{\Omega} \times (0,\infty).$$

Consequently, $\limsup_{t\to\infty} u(x,t) \leq \lim_{t\to\infty} \bar{u}(t) = f(c)/\delta$.

If further, $u_0(x,\theta) \geq 0(\not\equiv 0)$, we then have $u(x,t) \not\equiv 0$ on $\Omega \times [0,\tau]$. Otherwise, it follows from (2.1) that $u_0(x,\theta) \equiv 0$ on $\Omega \times [-\tau,0]$, which contradicts to the nontrivial initial condition. Choose $t_0 \in [0,\tau]$ such that $u(x,t_0) \geq 0(\not\equiv 0)$ for $x \in \Omega$. We obtain from (2.1) that

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - d\Delta u(x,t) + \delta u(x,t) \geq 0, & x \in \Omega, t > t_0, \\ u(\cdot,t)\mid_{\partial\Omega} \geq 0, & t \geq t_0, \\ u(x,t_0) = u_0(x,t_0) \geq 0(\not\equiv 0), & x \in \Omega. \end{cases}$$

The strong maximum principle implies that $u(x,t) > 0$ for all $(x,t) \in \Omega \times (t_0,\infty)$. This completes the proof. □

Since mature cells are circulating in thin blood tubes, we can treat the space Ω to be one dimensional, and in particular, we normalize it to be $\Omega = (0,\pi)$. Throughout the rest of this paper, we set $\Omega = (0,\pi)$.

3. Zero Dirichlet boundary condition

3.1. Existence and uniqueness of heterogeneous steady state

Throughout this subsection, we assume that $\alpha > \delta$. The steady state $u(x)$ of (2.1) satisfies the boundary value problem:

$$\begin{cases} du''(x) = \delta u(x) - f(u(x)), \\ u(0) = u(\pi) = 0. \end{cases} \tag{3.1}$$

Taking $v(x) = u'(x)$, we can rewrite the above second-order differential equation into the following system:

$$\begin{cases} u'(x) = v(x), \\ v'(x) = \frac{1}{d}\left(\delta u(x) - f(u(x))\right). \end{cases} \tag{3.2}$$

We treat x as a time variable and find the Hamiltonian function for System (3.2):

$$H(u,v) = \frac{v^2}{2} - \frac{\delta}{2d}u^2 + \frac{1}{d}\int_0^u f(\xi)d\xi. \tag{3.3}$$

Note that $H(u,v)$ remains a constant along the solution curve of System (3.2). It is easily observed that, System (3.2) has two equilibria $(0,0)$ and $(u_*,0)$ when k is odd, and three equilibria $(0,0)$ and $(\pm u_*,0)$ when k is even, where u_* is given in (1.3). The Jacobian matrix for the system (3.2) is given by

$$J(u,v) = \begin{pmatrix} 0 & 1 \\ \frac{\delta - f'(u)}{d} & 0 \end{pmatrix}.$$

Since $f'(0) > \delta > f'(u_*)$, $(0,0)$ is a center and $(u_*,0)$ is a saddle. If k is even, then $f'(-u_*) = f'(u_*) < \delta$, which implies that $(-u_*,0)$ is also a saddle.

By using the Hamiltonian function (3.3), we plot the phase portrait of (3.2) in Fig. A.1. A nonnegative solution of (3.2) with boundary conditions $u(0) = u(\pi) = 0$ corresponds to a trajectory starting from $(0,v_0)$ when $x = 0$, striking at $(\tilde{u},0)$ when $x = \pi/2$, and terminating at $(0,-v_0)$ when $x = \pi$, where $v_0 > 0$ and $0 < \tilde{u} < u_*$. To prove the existence or nonexistence of such a trajectory, we treat $\tilde{u} \in (0,u_*)$ as a variable and consider a family of trajectories striking at $(\tilde{u},0)$ on the positive u-axis. Since $H(u,-v) = H(u,v)$, we only need to study the traveling times of the trajectories in the first quadrant, which are denoted by $t(\tilde{u})$. A nonnegative solution of (3.2) with boundary conditions $u(0) = u(\pi) = 0$ exists if and only if $t(\tilde{u}) = \pi/2$ for some $\tilde{u} \in (0,u_*)$. We first obtain the following properties of $t(\tilde{u})$.

Proposition 3.1.

(i) $\lim_{\tilde{u}\to 0} t(\tilde{u}) = \frac{\pi}{2}\sqrt{\frac{d}{\alpha-\delta}}$.
(ii) The function $t(\tilde{u})$ *is strictly increasing for* $\tilde{u} \in (0,u_*)$.
(iii) The function $t(\tilde{u})$ *satisfies* $0 < t(\tilde{u}) < \infty$ *for any* $\tilde{u} \in [0,u_*)$. *Moreover,* $\lim_{\tilde{u}\to u_*} t(\tilde{u}) = \infty$.

Proof. (i) Since the trajectory goes through $(\tilde{u},0)$, the Hamiltonian function (3.3) has the expression

$$H(u,v) = H(\tilde{u},0) = -\frac{\delta}{2d}\tilde{u}^2 + \frac{1}{d}\int_0^{\tilde{u}} f(\xi)d\xi.$$

On the other hand, we obtain from (3.2) and (3.3) that

$$\frac{du}{dt} = v = \left[H(u,v) + \frac{\delta}{d}u^2 - \frac{2}{d}\int_0^u f(\xi)d\xi\right]^{1/2}.$$

Coupling the above two equations gives

$$\frac{dt}{du} = \left[\frac{\delta}{d}(u^2 - \tilde{u}^2) + \frac{2}{d}\int_u^{\tilde{u}} f(\xi)d\xi\right]^{-1/2}.$$

An integration of the above equation yields

$$t(\tilde{u}) = \int_0^{\tilde{u}} \left[\frac{\delta}{d}(u^2 - \tilde{u}^2) + \frac{2}{d}\int_u^{\tilde{u}}\int_0^{\xi} f'(\eta)d\eta d\xi\right]^{-1/2} du.$$

By a change of variable $u = \tilde{u}\sin\theta$ with $\theta \in [0, \pi/2]$, we have

$$t(\tilde{u}) = \int_0^{\pi/2} \left[-\frac{\delta}{d} + \frac{2}{d\tilde{u}^2\cos^2\theta}\int_{\tilde{u}\sin\theta}^{\tilde{u}}\int_0^{\xi} f'(\eta)d\eta d\xi\right]^{-1/2} d\theta.$$

We further set $\xi = \tilde{u}\tilde{\xi}$ and $\eta = \tilde{u}\tilde{\eta}$. It follows that

$$t(\tilde{u}) = \int_0^{\pi/2} \left[-\frac{\delta}{d} + \frac{2}{d\cos^2\theta}\int_{\sin\theta}^{1}\int_0^{\tilde{\xi}} f'(\tilde{u}\tilde{\eta})d\tilde{\eta}d\tilde{\xi}\right]^{-1/2} d\theta. \tag{3.4}$$

Since $f'(0) = \alpha$, we obtain

$$\lim_{\tilde{u}\to 0} t(\tilde{u}) = \int_0^{\pi/2} \left[-\frac{\delta}{d} + \frac{2}{d\cos^2\theta}\int_{\sin\theta}^{1}\int_0^{\tilde{\xi}} f'(0)d\tilde{\eta}d\tilde{\xi}\right]^{-1/2} d\theta = \frac{\pi}{2}\sqrt{\frac{d}{\alpha - \delta}}.$$

(ii) In view of (3.4), to show $t(\tilde{u})$ is a strictly increasing function in $\tilde{u}$, it suffices to show that

$$g(\sin\theta) = \int_{\sin\theta}^{1}\int_0^{\tilde{\xi}} \tilde{\eta}f''(\tilde{u}\tilde{\eta})d\tilde{\eta}d\tilde{\xi} < 0$$

for any $\theta \in [0, \pi/2]$. Let $w = \sin\theta \in [0, 1]$. It can be shown that $g(1) = 0$ and

$$g'(w) = -\int_0^{w} \tilde{\eta}f''(\tilde{u}\tilde{\eta})d\tilde{\eta} = -\frac{1}{\tilde{u}^2}\int_0^{\tilde{u}w} \eta f''(\eta)d\eta = \frac{f(\tilde{u}w) - \tilde{u}wf'(\tilde{u}w)}{\tilde{u}^2}.$$

Since $f(u) > uf'(u)$ for any $u > 0$, we obtain $g'(w) > 0$ for $0 < w < 1$, which together with $g(1) = 0$ implies that $g(w) < 0$ for all $0 < w < 1$. Thus, $t(\tilde{u})$ is strictly increasing for $\tilde{u} \in (0, u_*)$.

(iii) For any $\tilde{u} \in [0, u_*)$, we have $f(\tilde{u}) > \delta\tilde{u}$, and thus

$$\int_{\sin\theta}^{1}\int_0^{\tilde{\xi}} f'(\tilde{u}\tilde{\eta})d\tilde{\eta}d\tilde{\xi} = \int_{\sin\theta}^{1} \frac{f(\tilde{u}\tilde{\xi})}{\tilde{u}}d\tilde{\xi} > \int_{\sin\theta}^{1} \delta\tilde{\xi}d\tilde{\xi} = \frac{\delta\cos^2\theta}{2}.$$

This implies that the integral on the right-hand side of (3.4) is well defined; namely, $0 < t(\tilde{u}) < \infty$ for all $\tilde{u} \in [0, u_*)$. If $\lim_{\tilde{u}\to u_*} t(\tilde{u})$ exists, then from the monotonicity of $t(\tilde{u})$ and the integral representation in (3.4), we have

$$t(u_*) = \int_0^{\pi/2} \left[-\frac{\delta}{d} + \frac{2}{d\cos^2\theta}\int_{\sin\theta}^{1} \frac{f(u_*\tilde{\xi})}{u_*}d\tilde{\xi}\right]^{-1/2} d\theta < \infty.$$

As $\theta \to \pi/2$, the integrand has the following asymptotic formula:

$$\left[-\frac{\delta}{d} + \frac{2}{d\cos^2\theta}\int_{\sin\theta}^{1} \frac{f(u_*\tilde{\xi})}{u_*}d\tilde{\xi}\right]^{-1/2} \sim \left[-\frac{\delta}{d} + \frac{2\delta(1-\sin\theta)}{d\cos^2\theta}\right]^{-1/2} \sim \frac{\sqrt{d/\delta}}{\pi/2 - \theta},$$

which implies that the integrand has a simple pole at $\theta = \pi/2$, a contradiction. Thus, we conclude that $t(\tilde{u}) \to \infty$ as $\tilde{u} \to u_*$. This ends the proof. □

By Proposition 3.1, there exists a unique heterogeneous positive solution, denoted by $\phi(x)$, of (3.1) if and only if $\lim_{\tilde{u}\to 0} t(\tilde{u}) < \pi/2$; i.e., $\alpha > d+\delta$. Let x^* be a maximum point of $\phi(x)$ in $[0,\pi]$. We claim $\phi(x^*) \le u_*$. Otherwise, $d\phi''(x^*) = \delta\phi(x^*) - f(\phi(x^*)) > 0$. But $\phi''(x^*) \le 0$ because x^* is a maximum point, a contradiction.

Summarizing the above analysis, we have the following steady state bifurcation theorem for (2.1).

Theorem 3.2. *If $\alpha \in (\delta, d+\delta]$, then (2.1) has no positive steady state. If $\alpha > d+\delta$, then (2.1) has a unique heterogeneous positive steady state $\phi(x)$. Moreover, $\phi(x) \le u_*$ for all $x \in [0,\pi]$, where u_* is defined in (1.3).*

3.2. Stability of the trivial equilibrium

In this subsection, we carry out the stability analysis of the trivial equilibrium for (2.1). Denote by $X = L^2(\Omega)$ the Hilbert space of integrable functions with usual inner product $<\cdot,\cdot>$, and $\mathcal{C} := C([-\tau,0],X)$ the Banach space of continuous maps from $[-\tau,0]$ to X with the sup norm. Given a continuous function $u(x,t)$ on $\Omega \times [-\tau,\infty)$, we define $u_t \in \mathcal{C}$ as $u_t(\theta) = u(\cdot, t+\theta)$ for $\theta \in [-\tau,0]$.

Linearizing Eq. (2.1) about $u(x,t) = 0$, we obtain

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - \delta u(x,t) + \alpha u(x,t-\tau). \tag{3.5}$$

Let $L : \mathcal{C} \to X$ be a bounded linear operator defined as

$$L(\phi) = \alpha\phi(-\tau) - \delta\phi(0) \quad \text{for} \quad \phi \in \mathcal{C}. \tag{3.6}$$

It then follows from [22] that the characteristic equation for the above linearized equation is

$$\lambda z - d\Delta z - L(e^{\lambda\cdot} z) = 0, \quad \text{for some } z \in \mathcal{D}\backslash\{0\}, \tag{3.7}$$

where

$$\mathcal{D} = \left\{u : u \in C^2(0,\pi) \cap C[0,\pi], u(0) = u(\pi) = 0\right\}.$$

We also use $\mathcal{D}^+$ to denote the subset of all nonnegative functions in $\mathcal{D}$. Note that the eigenvalue problem

$$-\psi''(x) = \lambda\psi(x),\ x \in (0,\pi),\ \psi(0) = \psi(\pi) = 0$$

has eigenvalues $\{n^2\}_{n=1}^{\infty}$ with eigenfunctions $\psi_n(x) = \sin(nx)$. Substituting $z = \sum_{n=0}^{\infty} z_n \sin(nx)$ into (3.7) gives

$$F_n(\lambda) := \lambda + dn^2 + \delta - \alpha e^{-\lambda\tau} = 0, \quad (n = 1,2,3,\ldots). \tag{3.8}$$

We then analyze the distribution of the zeros of $F_n(\lambda)$. From Lemma 6 in [23], we have (i) if $\alpha < d+\delta$, which implies that $\alpha < dn^2+\delta$ for any $n \ge 1$, then all zeros of $F_n(\lambda)$ have negative real parts for any $n = 1,2,\ldots$; (ii) if $\alpha = d+\delta$, then 0 is the only real zero of $F_1(\lambda)$, all other zeros of $F_n(\lambda)$ have negative real parts for any $n = 1,2,\ldots$; (iii) if $\alpha > d+\delta$, then $F_1(\lambda)$ admits one positive real zero and all other zeros of $F_1(\lambda)$ are complex numbers with negative real parts. To summarize, we obtain the following results on stability of the trivial equilibrium.

Theorem 3.3. *The trivial equilibrium 0 of (2.1) is locally asymptotically stable if $\alpha < d+\delta$, and unstable if $\alpha > d+\delta$.*

Theorem 3.4. *If $\alpha = d+\delta$, then the trivial equilibrium 0 of (2.1) is unstable if k is odd, and stable if k is even.*

Proof. If $\alpha = d+\delta$, then 0 is the only real eigenvalue for $n = 1$, and all other eigenvalues have negative real parts. We investigate the stability of the trivial equilibrium by using the normal forms for partial functional differential equations introduced by Faria in [24]. Let

$$\Lambda = \{\lambda \in \mathbb{C}, \lambda \text{ is an eigenvalue of (3.8) with } \mathrm{Re}\lambda = 0\}.$$

It is easily obtained that $\Lambda = \{0\}$ if $\alpha = d + \delta$, and (2.1) satisfies the nonresonance condition relative to Λ. Thus, there exists a one-dimensional ODE which governs the dynamics of (2.1) near the trivial equilibrium.

Eq. (2.1) can be written in abstract form in the phase space $\mathcal{C} = C([-\tau, 0], X)$ as

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(U_t) + H(u_t),$$

where $X = \{v \in L^2(0, \pi) : \ v = 0 \text{ at } x = 0, \pi\}$, L is defined in (3.6), and $H(\phi) = -\alpha\beta\phi^{k+1}(-\tau)+O(\|\phi\|^{2k+1})$ for any $\phi \in \mathcal{C}$. Define

$$\chi_0 = \chi_0(\theta) = \begin{cases} 0, & -\tau \le \theta < 0, \\ 1, & \theta = 0, \end{cases}$$

and

$$\chi_{-\tau} = \chi_{-\tau}(\theta) = \begin{cases} 0, & -\tau < \theta \le 0, \\ 1, & \theta = -\tau. \end{cases}$$

By choosing

$$\eta(\theta) = -(dn^2 + \delta)\chi_0 - \alpha\chi_{-\tau},$$

we obtain $-dn^2\phi(0) + L(\phi) = \int_{-\tau}^0 \phi(\theta)d\eta(\theta)$ for any $\phi \in \mathcal{C}$. Let P be the center space for the linear system (3.5), and P^* the adjoint space of P. The adjoint bilinear form $(\cdot,\cdot)$ on $C^* \times C$ is defined as

$$(\psi(s), \phi(\theta)) = \psi(0)\phi(0) - \int_{-\tau}^0 \int_0^\theta \psi(\xi-\theta)\phi(\xi)d\xi d\eta(\theta),$$

where $C = C([-\tau, 0], \mathbb{R})$ and $C^* = C([0, \tau], \mathbb{R})$. We use the adjoint theory to decompose $\mathcal{C}$ by Λ as

$$\mathcal{C} = P \oplus Q, \quad \text{where } P = \text{span}\{\Phi(\theta)\}, \ \Phi'(\theta) = 0.$$

Then $\Phi(\theta) = 1$. We further choose a basis $\Psi(s) = \psi(s)$ for P^* such that $(\Psi, \Phi) = 1$. Therefore, $\psi(s) = \frac{1}{1+\alpha\tau}$. We take the following Banach space as phase space

$$B\mathcal{C} = \{\phi : [-\tau, 0] \to X : \phi \text{ is continuous on } [-\tau, 0), \ \exists \lim_{\theta\to 0^-} \phi(\theta) \in X\}.$$

Then (2.1) can be rewritten as the following abstract differential equation in $B\mathcal{C}$:

$$\frac{du_t}{dt} = Au_t + \chi_0 H(u_t),$$

where $A\phi = \dot\phi(\theta) + \chi_0[d\Delta\phi(0) + L(\phi) - \dot\phi(0)]$. Define the projection $\pi : B\mathcal{C} \to P$ by

$$\pi(\phi + \chi_0\xi) = \Phi\left[(\Psi, <\dot\phi(\cdot), \varphi_1>) + \Psi(0) <\xi, \varphi_1>\right]\varphi_1,$$

where $\varphi_1 = \sqrt{\frac{2}{\pi}}\sin x$. The projection π leads to the topological decomposition $B\mathcal{C} = P \oplus \ker\pi$. By using the decomposition, we have

$$u_t = \Phi z(t)\varphi_1 + y(t),$$

where $z(t) = (\Psi, <u_t(\cdot), \varphi_1>) \in \mathbb{R}$, $y(t) \in \mathcal{C}_0^1 \cap \ker\pi := \mathcal{Q}^1$ and $\mathcal{C}_0^1 = \{\phi \in \mathcal{C} : \dot\phi \in \mathcal{C}, \phi(0) \in \text{dom}(\Delta)\}$. Then we see that in $B\mathcal{C}$ system (2.1) is equivalent to the system

$$\begin{cases} \dot z = Bz + \Psi(0) < H(\Phi z\varphi_1 + y), \varphi_1 >, \\ \frac{dy}{dt} = A_1 y + (I-\pi)\chi_0 H(\Phi z\varphi_1 + y), \end{cases}$$

where $B=0$, and A_1 is defined by $A_1 : \mathcal{Q}^1 \to \ker \pi, A_1\phi = A\phi$ for $\phi \in \mathcal{Q}^1$. Thus, the flow on the center manifold is given by the following one-dimensional ODE

$$\dot{z} = \frac{-\alpha\beta \int_0^\pi \varphi_1^{k+2}(x)dx}{1+\alpha\tau}(z^{k+1} + O(\|z\|^{2k+1})).$$

Therefore, the trivial equilibrium of (2.1) is unstable if k is odd, and stable if k is even. □

Next, we use energy method to prove the global stability of the trivial equilibrium if $\alpha < d+\delta$.

Theorem 3.5. *If $\alpha < d+\delta$, then the solution of (2.1) satisfies $\lim_{t\to\infty} u(x,t) = 0$ uniformly in x.*

Proof. Define

$$E(t) = \int_0^\pi u^2(x,t)dx.$$

We multiply the first equation of (2.1) by $u(x,t)$, integrate it over $[0,\pi]$, make use of integration by parts, zero Dirichlet boundary condition, and the fact that $f(u) \le \alpha u$ for all $u \ge 0$. It follows that

$$\frac{1}{2}E'(t) \le -d\int_0^\pi |\nabla u(x,t)|^2 dx - \delta E(t) + \alpha \int_0^\pi u(x,t-\tau)u(x,t)dx.$$

Poincaré inequality gives

$$\int_0^\pi |\nabla u(x,t)|^2 dx \ge \int_0^\pi |u(x,t)|^2 dx = E(t).$$

Cauchy inequality yields

$$\int_0^\pi u(x,t-\tau)u(x,t)dx \le \int_0^\pi \frac{|u(x,t-\tau)|^2 + |u(x,t)|^2}{2}dx = \frac{E(t-\tau)+E(t)}{2}.$$

Thus, we have

$$E'(t) \le \alpha E(t-\tau) - (2d+2\delta-\alpha)E(t).$$

Let $y(t)$ be a solution of the differential equation

$$y'(t) = \alpha y(t-\tau) - (2d+2\delta-\alpha)y(t)$$

with initial condition $y(\theta) = E(\theta)$ for $\theta \in [-\tau, 0]$. By comparison principle, we have $y(t) \ge E(t)$ for all $t \ge 0$. If $\alpha < d+\delta$, it follows from [25, Example 5.1] that $\lim_{t\to\infty} y(t) = 0$. Consequently, $\lim_{t\to\infty} E(t) = 0$; that is, the trivial equilibrium is globally attractive. This, together with the local stability established in Theorem 3.3, yields the global stability of the trivial equilibrium. This completes the proof. □

Remark 3.6. Theorem 3.5 implies that 0 is the unique nonnegative steady state of (2.1) if $\alpha < d+\delta$.

3.3. Stability of the unique heterogeneous positive steady state

If $\alpha > d+\delta$, it follows from Theorem 3.2 that (2.1) has a unique heterogeneous positive steady state $\phi(x)$. In this subsection, we investigate the stability of $\phi(x)$. The linearized equation of (2.1) at $u(x,t) = \phi(x)$ is given by

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\triangle u(x,t) - \delta u(x,t) + f'(\phi(x))u(x,t-\tau), & x \in (0,\pi), t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \ge 0, \end{cases}$$

where

$$f'(u) = \frac{\alpha}{1+\beta u^k} - \frac{\alpha k \beta u^k}{(1+\beta u^k)^2}.$$

The corresponding eigenvalue problem is

$$\begin{cases} -d\triangle\psi(x) + (\delta+\lambda)\psi(x) - f'(\phi(x))e^{-\lambda\tau}\psi(x) = 0, & x \in (0,\pi), \\ \psi(0) = \psi(\pi) = 0. \end{cases} \tag{3.9}$$

Denote

$$\widetilde{\Delta}_{\phi}^{\lambda} := d\triangle - \delta - \lambda + f'(\phi(x))e^{-\lambda\tau}. \tag{3.10}$$

Eq. (3.9) can be written as $\widetilde{\Delta}_{\phi}^{\lambda}\psi = 0$, and (λ,ψ) with $\psi \neq 0$ formulate a pair of eigenvalue and eigenfunction. We also introduce a differential operator $\Delta_\phi := d\Delta - \delta + f(\phi(x))/\phi(x)$. Clearly, $\Delta_\phi\phi = 0$. It is readily seen that

$$\Delta_\phi - \widetilde{\Delta}_{\phi}^{\lambda} = \lambda - f'(\phi(x))e^{-\lambda\tau} + \frac{f(\phi(x))}{\phi(x)}.$$

To establish local stability of $\phi(x)$, we shall make use of the following lemma.

Lemma 3.7. *For any $P \in C[0,\pi]$ such that $P(x)$ is positive in $(0,\pi)$, the differential operator $-\Delta_\phi + P(x)$ has a trivial kernel in $\mathcal{D}$; namely, $\varphi(x) \equiv 0$ is the unique solution to the boundary value problem*

$$\begin{cases} -\Delta_\phi\varphi(x) + P(x)\varphi(x) = 0, & x \in (0,\pi), \\ \varphi(0) = \varphi(\pi) = 0. \end{cases}$$

Proof. Assume $\varphi(x_0) < 0$ for some $x_0 \in [0,\pi]$. Let (x_1,x_2) be the largest interval containing x_0 where φ remains negative. Thus, $\varphi'(x_1) \le 0$, $\varphi'(x_2) \ge 0$, and $\varphi(x_1) = \varphi(x_2) = 0$. Especially,

$$[\varphi'(x)\phi(x) - \phi'(x)\varphi(x)]\Big|_{x_1}^{x_2} \ge 0.$$

We rewrite the differential equations for φ and ϕ below:

$$\begin{aligned} -d\varphi''(x) + \left(\delta - \frac{f(\phi(x))}{\phi(x)} + P(x)\right)\varphi(x) = 0, \\ -d\phi''(x) + \left(\delta - \frac{f(\phi(x))}{\phi(x)}\right)\phi(x) = 0. \end{aligned}$$

Multiplying the first equation by $\phi(x)$ and the second equation by $\varphi(x)$, then subtracting and integrating over $[x_1,x_2]$, we obtain

$$\begin{aligned} \int_{x_1}^{x_2} P(x)\varphi(x)\phi(x)dx &= d\int_{x_1}^{x_2}[\varphi''(x)\phi(x) - \phi''(x)\varphi(x)]dx \\ &= [\varphi'(x)\phi(x) - \phi'(x)\varphi(x)]\Big|_{x_1}^{x_2} \ge 0. \end{aligned}$$

However, $P(x) > 0$, $\varphi(x) < 0$ and $\phi(x) > 0$ for $x \in (x_1,x_2)$, which implies that the integral on the left-hand side of the above inequality is negative, a contradiction. Thus, φ cannot be negative in $(0,\pi)$. A similar argument shows that φ is non-positive in $(0,\pi)$. Hence, $\varphi(x) \equiv 0$ in $[0,\pi]$. The proof is complete. □

A direct application of the above lemma is that the differential operator $\widetilde{\Delta}_{\phi}^{0}$ has only trivial kernel in $\mathcal{D}$. This is because $\Delta_\phi - \widetilde{\Delta}_{\phi}^{0} = -f'(\phi(x)) + \frac{f(\phi(x))}{\phi(x)}$ and $-f'(u) + f(u)/u > 0$ for any $u > 0$. Another application of the above lemma is that the eigenvalues of $-\Delta_\phi$ are all nonnegative. To see this, we first note that $-\Delta_\phi$ is a self-adjoint operator and thus has only real eigenvalues. If it has a negative eigenvalue $-\mu < 0$ with eigenfunction $\varphi \in \mathcal{D}$, then $(-\Delta_\phi + \mu)\varphi = 0$, which contradicts to the above lemma that $-\Delta_\phi + \mu$ has only trivial kernel in $\mathcal{D}$.

For any $u \geq 0$, it follows from (2.2) that

$$\frac{uf'(u)}{f(u)} = 1 - \frac{k\beta u^k}{1+\beta u^k} < 1.$$

Moreover, $uf'(u)/f(u) \geq -1$ if and only if $\beta u^k \leq 2/(k-2)$. Recall that u_* is the unique positive solution of $f(u) = \delta u$; namely, $\beta u_*^k = \alpha/\delta - 1$. We conclude that $|f'(u)| \leq f(u)/u$ for all $u \in [0, u_*]$ if and only if $\alpha \leq \delta k/(k-2)$.

Our next result gives delay-independent conditions for the local stability of the heterogeneous positive steady state $\phi(x)$.

Theorem 3.8. *Assume $d + \delta < \alpha \leq \delta k/(k-2)$. The unique heterogeneous positive steady state $\phi(x)$ of (2.1) is locally asymptotically stable for all $\tau \geq 0$.*

Proof. Let λ and $\psi \in \mathcal{D}$ be a pair of eigenvalue and eigenfunction for the eigenvalue problem $\widetilde{\Delta}_\phi^\lambda \psi = 0$. We have

$$\begin{aligned} 0 =< -\widetilde{\Delta}_\phi^\lambda \psi, \psi >=& < -\Delta_\phi \psi, \psi > \\ &+ \int_0^\pi \left(\lambda - f'(\phi(x))e^{-\lambda\tau} + \frac{f(\phi(x))}{\phi(x)}\right) |\psi(x)|^2 dx. \end{aligned}$$

Since all eigenvalues of $-\Delta_\phi$ are real and nonnegative, the inner product $< -\Delta_\phi \psi, \psi >$ is nonnegative. Separating the real and imaginary parts, the above equation can be rewritten as

$$\int_0^\pi \left(\mathrm{Im}\lambda + f'(\phi(x))e^{-\mathrm{Re}\lambda\tau} \sin(\mathrm{Im}\lambda\tau)\right) |\psi(x)|^2 dx = 0, \tag{3.11}$$

and

$$< \Delta_\phi \psi, \psi >= \int_0^\pi \left(\mathrm{Re}\lambda - f'(\phi(x))e^{-\mathrm{Re}\lambda\tau} \cos(\mathrm{Im}\lambda\tau) + \frac{f(\phi(x))}{\phi(x)}\right) |\psi(x)|^2 dx.$$

Since $\alpha \leq \delta k/(k-2)$, we have $|f'(u)| \leq f(u)/u$ for any $u > 0$. It then follows from the above equation and non-positiveness of $< \Delta_\phi \psi, \psi >$ that

$$0 \geq \int_0^\pi \left(\mathrm{Re}\lambda + (1 - e^{-\mathrm{Re}\lambda\tau} |\cos(\mathrm{Im}\lambda\tau)|) \frac{f(\phi(x))}{\phi(x)}\right) |\psi(x)|^2 dx. \tag{3.12}$$

Clearly, $\mathrm{Re}\lambda \leq 0$; otherwise, the quantity

$$\mathrm{Re}\lambda + (1 - e^{-\mathrm{Re}\lambda\tau} |\cos(\mathrm{Im}\lambda\tau)|) \frac{f(\phi(x))}{\phi(x)}$$

is positive in $(0, \pi)$, and thus the integral on the right-hand side of (3.12) is positive, a contradiction. If $\mathrm{Re}\lambda = 0$ and $|\cos(\mathrm{Im}\lambda\tau)| < 1$, we still have positiveness of the above quantity in $(0, \pi)$, which again contradicts the inequality (3.12). If $\mathrm{Re}\lambda = 0$ and $|\cos(\mathrm{Im}\lambda\tau)| = 1$, then $\sin(\mathrm{Im}\lambda\tau) = 0$. It follows from (3.11) that $\mathrm{Im}\lambda = 0$. Hence, $\lambda = 0$. However, by Lemma 3.7 and the fact $f(u)/u > f'(u)$ for any $u > 0$, the equation $\widetilde{\Delta}_\phi^0 \psi = 0$ has only trivial solution $\psi(x) \equiv 0$, which again leads to a contraction. Therefore, we obtain $\mathrm{Re}\lambda < 0$, and consequently, the heterogeneous positive steady state $\phi(x)$ of (2.1) is locally stable. This ends the proof. □

Remark 3.9. If $k = 2$, then $\phi(x)$ is locally stable whenever it exists; i.e., $\alpha > d + \delta$.

Theorem 3.8 implies that if $d + \delta < \alpha \leq \delta k/(k-2)$, the positive steady state $\phi(x)$ is locally stable for any value of τ. In the following, we consider the case with $\alpha > \max\{\delta k/(k-2), d + \delta\}$, and prove that a small time delay is still harmless to the stability of $\phi(x)$.

Theorem 3.10. *Assume* $\alpha > \max\{\delta k/(k-2), d+\delta\}$. *The positive steady state* $\phi(x)$ *is locally stable for all* $\tau \in [0, \hat{\tau}]$, *where*

$$\hat{\tau} = \frac{\arccos(\frac{-1}{k-1-k\delta/\alpha})}{\alpha\sqrt{(k-1-k\delta/\alpha)^2-1}}. \tag{3.13}$$

Proof. We will prove the contraposition that if $\phi(x)$ is unstable then $\tau > \hat{\tau}$. To see this, we assume $\phi(x)$ is unstable, then the eigenvalue problem $\widetilde{\Delta}_\phi^\lambda \psi = 0$ has a complex eigenvalue, denoted by λ, with nonnegative real part. Since the conjugate $\bar{\lambda}$ is also an eigenvalue, we may assume without loss of generality that $\mathrm{Im}\lambda \geq 0$. Also, we normalize the eigenfunction $\psi(x)$ such that $<\psi, \psi>= 1$. Thus, we have

$$0 =< -\Delta_\phi \psi, \psi > +\lambda + e^{-\lambda\tau} I_1 + I_2,$$

where

$$I_1 = -\int_0^\pi f'(\phi(x))|\psi(x)|^2 dx, \quad I_2 = \int_0^\pi \frac{f(\phi(x))}{\phi(x)}|\psi(x)|^2 dx.$$

Since $f(u)/u > 0$ and $f(u)/u > f'(u)$ for all $u > 0$, we have $I_2 > 0$ and $I_2 > -I_1$. Similar as in the proof of Theorem 3.8, we separate the real and imaginary parts of the characteristic equation. It follows that

$$-e^{-\mathrm{Re}\lambda\tau} \cos(\mathrm{Im}\lambda\tau) I_1 =< -\Delta_\phi \psi, \psi > +\mathrm{Re}\lambda + I_2, \tag{3.14}$$

$$\mathrm{Im}\lambda = e^{-\mathrm{Re}\lambda\tau} \sin(\mathrm{Im}\lambda\tau) I_1. \tag{3.15}$$

Since $\mathrm{Re}\lambda \geq 0$ and $< -\Delta_\phi \psi, \psi >\geq 0$, we have from (3.14)

$$-\cos(\mathrm{Im}\lambda\tau) I_1 \geq I_2. \tag{3.16}$$

In view of $I_2 > -I_1$, the above inequality gives $I_1(1 - \cos(\mathrm{Im}\lambda\tau)) > 0$. Thus, $I_1 > 0$ and $\cos(\mathrm{Im}\lambda\tau) < 1$. Especially, $\mathrm{Im}\lambda\tau > 0$ (recall that we have assumed without loss of generality $\mathrm{Im}\lambda\tau \geq 0$), which, together with $I_1 > 0$ and (3.15), implies that $\sin(\mathrm{Im}\lambda\tau) > 0$. Furthermore, it follows from $I_1 > 0$ and (3.16) that $\cos(\mathrm{Im}\lambda\tau) < 0$. Thus, $\mathrm{Im}\lambda\tau \in (\pi/2 + 2j\pi, \pi + 2j\pi)$ for some $j \in \mathbb{N}_0$. Since $\mathrm{Re}\lambda\tau \geq 0$, we obtain from (3.15) that

$$\frac{\mathrm{Im}\lambda}{\sin(\mathrm{Im}\lambda\tau)} \leq I_1. \tag{3.17}$$

To obtain a lower bound of τ, we shall derive a lower bound for I_2/I_1 and an upper bound for I_1. For any $u \in (0, u_*]$, we have from (1.3) and (2.2)

$$\frac{f(u)}{u} = \frac{\alpha}{1+\beta u^k} < \alpha,$$

$$\frac{-uf'(u)}{f(u)} = \frac{k\beta u^k}{1+\beta u^k} - 1 \leq \frac{k\beta u_*^k}{1+\beta u_*^k} - 1 = k - 1 - \frac{k\delta}{\alpha}.$$

Consequently,

$$I_2 < \int_0^\pi \alpha|\psi(x)|^2 dx = \alpha,$$

$$I_1 \leq \int_0^\pi (k-1-k\delta/\alpha)\frac{f(\phi(x))}{\phi(x)}|\psi(x)|^2 dx = (k-1-k\delta/\alpha) I_2.$$

Especially, $I_1 < \alpha(k-1-k\delta/\alpha)$. Substituting the above inequalities into (3.16) and (3.17) yields

$$-\cos(\mathrm{Im}\lambda\tau) \geq (k-1-k\delta/\alpha)^{-1}$$

and

$$\frac{\mathrm{Im}\lambda\tau}{\sin(\mathrm{Im}\lambda\tau)} < \alpha(k-1-k\delta/\alpha)\tau.$$

Let $\mathrm{Im}\lambda\tau = \xi + 2j\pi$ for some $\xi \in (\pi/2, \pi)$ and $j \in \mathbb{N}_0$. The first inequality implies $\xi \geq \arccos(\frac{-1}{k-1-k\delta/\alpha})$. Thus, the second inequality gives

$$\alpha(k-1-k\delta/\alpha)\tau > \frac{\xi}{\sin\xi} \geq \frac{\arccos(\frac{-1}{k-1-k\delta/\alpha})}{\sqrt{1-(\frac{1}{k-1-k\delta/\alpha})^2}}.$$

Here, we have used the fact that the function $\xi/\sin\xi$ is increasing for $\xi \in (\pi/2, \pi)$. Recall the definition of $\hat{\tau}$. The above inequality is the same as $\tau > \hat{\tau}$. The proof is complete. $\square$

4. Non-zero Dirichlet boundary problem

We consider the non-zero Dirichlet problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\triangle u(x,t) - \delta u(x,t) + f(u(x,t-\tau)), & x \in (0,\pi), t > 0, \\ u(0,t) = u(\pi,t) = u_*, & t \geq 0, \\ u(x,\theta) = u_0(x,\theta) \geq 0, \quad x \in [0,\pi], & \theta \in [-\tau, 0], \end{cases} \tag{4.1}$$

where u_* is defined in (1.3). Throughout this section, we assume $\alpha > \delta$, which guarantees that $u_* > 0$. First, we exclude the existence of heterogeneous steady state. Then, we carry out stability analysis of the unique positive steady state u_* and establish the existence conditions for Hopf bifurcation of (4.1). Next, we analyze the properties of Hopf bifurcations and prove the existence of both spatial-dependent and independent periodic solutions to (4.1). Finally, we study the global continuation of periodic solutions bifurcating from u_*.

4.1. Nonexistence of heterogeneous steady states

Since $u_* = f(u_*)/\delta \leq f(c)/\delta$, where f and c are defined as in (2.2), it follows from Theorem 2.1 that the solution of (4.1) is nonnegative for $t \geq 0$, positive for $t > \tau$, and eventually bounded above by $f(c)/\delta$. Furthermore, we have the following result regarding nonexistence of heterogeneous positive steady state of (4.1).

Theorem 4.1. *If $\alpha > \delta$, then the constant solution $u(x) \equiv u_*$ is the only positive steady state solution of* (4.1).

Proof. The steady state of (4.1) satisfies the following boundary value problem:

$$\begin{cases} du''(x) = \delta u(x) - f(u(x)), \\ u(0) = u(\pi) = u_*. \end{cases}$$

A steady state solution is equivalent to a trajectory of (3.2) that starts on the vertical line $u = u_*$ at $x = 0$ and comes back on the same line when $x = \pi$.

First, we consider a special trajectory, denoted by $\mathcal{X}$, passing through the point $(u_*, 0)$. By (3.3), the trajectory consists of all points (u, v) satisfying the equation:

$$v^2 = \frac{\delta}{d}(u^2 - u_*^2) + \frac{2}{d}\int_u^{u_*} f(\xi)d\xi.$$

It is readily seen that the trajectory in the right-half plane ($u \geq 0$) formulates an X-shape as illustrated by the green curves in Fig. A.1. We are ready to prove the nonexistence of heterogeneous positive steady state of (4.1). Assume to the contrary that there exists a trajectory $\mathcal{S} = (u(x), v(x))$ which starts on the line $u = u_*$ at $x = 0$ and comes back on the same line at $x = \pi$. Obviously, $v(0)$ and $v(\pi)$ cannot have different signs; otherwise, the two trajectories $\mathcal{S}$ and $\mathcal{X}$ would intersect with each other, which contradicts with the uniqueness of solution to (3.2). On the other hand, if both $v(0)$ and $v(\pi)$ are positive, the trajectory $\mathcal{S}$ should always stay in the first quadrant, which implies that $u(x)$ is increasing and thus, the trajectory cannot move back onto the line $u = u_*$, a contradiction. Similarly, if both $v(0)$ and $v(\pi)$ are negative, then $u(x)$ is decreasing, which again, leads to a contradiction. Therefore, we have excluded the existence of heterogeneous positive steady state of (4.1). This ends the proof. □

4.2. Stability analysis and existence of Hopf bifurcation

To translate the non-zero Dirichlet boundary condition to zero Dirichlet boundary condition, we replace $u(x,t)$ by $u(x,t) + u_*$ in (4.1). The resulting equation is

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\triangle u(x,t) - \delta(u(x,t) + u_*) + \frac{\alpha(u(x,t-\tau)+u_*)}{1+\beta(u(x,t-\tau)+u_*)^k}, & x \in (0,\pi), t > 0, \\ u(0,t) = u(\pi,t) = 0, & t \geq 0, \\ u(x,\theta) = u_0(x,\theta) - u_* \geq -u_* \ (\not\equiv -u_*), \qquad x \in [0,\pi], & \theta \in [-\tau, 0]. \end{cases} \tag{4.2}$$

Linearizing the above equation at the trivial steady state, we obtain

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) - \delta u(x,t) + f'(u_*)u(x,t-\tau),$$

where

$$f'(u_*) = \delta - \frac{k\delta(\alpha - \delta)}{\alpha} < \delta.$$

The corresponding characteristic equation is

$$\lambda + dn^2 + \delta - f'(u_*)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}. \tag{4.3}$$

We use the delay $\tau > 0$ as a bifurcation parameter and investigate the stability changes at u_* and the existence of Hopf bifurcation. Note that when $\tau = 0$, the eigenvalues are

$$\lambda = -dn^2 - \delta + f'(u_*) < 0 \ \text{ for all } \ n \in \mathbb{N}.$$

Thus a stability change at u_* can only happen when one or more eigenvalues cross the imaginary axis to the right. Since $f'(u_*) < \delta$, 0 cannot be an eigenvalue. We only need to look for a pair of purely imaginary eigenvalues $\lambda = \pm i\omega$ with $\omega > 0$ for some $\tau > 0$. Substituting $\lambda = i\omega$ into (4.3) and separating the real and imaginary parts, we have

$$dn^2 + \delta = f'(u_*)\cos(\omega\tau), \quad -\omega = f'(u_*)\sin(\omega\tau). \tag{4.4}$$

Squaring and adding both equations of (4.4) lead to

$$\omega^2 = (-f'(u_*) + dn^2 + \delta)(-f'(u_*) - dn^2 - \delta). \tag{4.5}$$

Note that $f'(u_*) < \delta$. The above equation has a unique positive root

$$\omega_n = \sqrt{(-f'(u_*) + dn^2 + \delta)(-f'(u_*) - dn^2 - \delta)} \ \text{ for } \ n \in \mathbb{N} \tag{4.6}$$

if and only if $-f'(u_*) - dn^2 - \delta > 0$, or equivalently, $\frac{\delta}{\alpha} < 1 - \frac{dn^2+2\delta}{k\delta}$. If this condition is not satisfied for all $n \in \mathbb{N}$; i.e., $\frac{\delta}{\alpha} \geq 1 - \frac{d+2\delta}{k\delta}$, then all eigenvalues of (4.3) stay in the open left-half complex plane. Consequently, we have the following result on the stability of u_*.

Theorem 4.2. *Assume that $1-\frac{d+2\delta}{k\delta} \le \frac{\delta}{\alpha} < 1$. The positive steady state u_* of (4.1) is locally asymptotically stable for all $\tau \ge 0$.*

Corollary 4.3. *If $k=2$ and $\alpha > \delta$, then u_* of (4.1) is locally asymptotically stable for all $\tau \ge 0$.*

In the sequel, we assume that $\frac{\delta}{\alpha} < 1-\frac{d+2\delta}{k\delta}$; namely, $f'(u_*) < -d-\delta$. Let n_{max} be the maximum value of n such that $\frac{\delta}{\alpha} < 1-\frac{dn+2\delta}{k\delta}$ is satisfied; namely,

$$n_{max} = [\sqrt{\frac{(1-\delta/\alpha)k\delta - 2\delta}{d}}],$$

where $[\cdot]$ denotes the greatest integer function. For any $n \in \{1,\dots,n_{max}\}$, we choose $\omega = \omega_n > 0$ to be defined as in (4.6). Since $f'(u_*) < 0$, it follows from (4.4) that $\sin(\omega\tau) > 0$ and $\cos(\omega\tau) < 0$, which implies that $\omega\tau$ lies in the second quadrant; i.e., $\omega\tau \in (\frac{\pi}{2}+2j\pi, \pi+2j\pi)$ for some $j \in \mathbb{N}_0$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$ denotes the set of all nonnegative integers. Consequently, (4.4) gives infinitely many solutions:

$$\tau = \tau_n^j = \frac{1}{w_n}(\arccos\frac{dn^2+\delta}{f'(u_*)} + 2j\pi) \tag{4.7}$$

for all $j \in \mathbb{N}_0$. For convenience, we introduce the index set

$$\mathcal{P} = \{1,\dots,n_{max}\} \times \mathbb{N}_0.$$

For any pair $(n,j) \in \mathcal{P}$, it is obvious that $\tau_n^{j+1} > \tau_n^j$. Furthermore, since ω_n is strictly decreasing and $\arccos\frac{dn^2+\delta}{f'(u_*)}$ is strictly increasing with respect to n, we conclude that τ_n^j is strictly increasing with respect to n. Thus, we arrive at the following lemma.

Lemma 4.4. *Assume that $\frac{\delta}{\alpha} < 1-\frac{d+2\delta}{k\delta}$. Then τ_n^j with $(n,j) \in \mathcal{P}$ is strictly increasing with respect to n and j. Especially, $\tau_1^0 = \min_{(n,j)\in\mathcal{P}} \tau_n^j$.*

Now, we verify the transversality condition for the occurrence of Hopf bifurcation at the bifurcation point τ_n^j. For any τ in a complex neighborhood of τ_n^j, we denote $\lambda_n(\tau) = \gamma_n(\tau) + i\omega_n(\tau)$ to be the root of (4.3) such that $\gamma_n(\tau_n^j) = 0$ and $\omega_n(\tau_n^j) = \omega_n$. Note that the conjugate $\bar{\lambda}_n(\tau) = \gamma_n(\tau) - i\omega_n(\tau)$ is also a root of (4.3), but these two roots have the same real part. We have the following result.

Lemma 4.5. *Assume that $\frac{\delta}{\alpha} < 1-\frac{d+2\delta}{k\delta}$. Then $\frac{dRe\lambda_n(\tau)}{d\tau}|_{\tau=\tau_n^j} > 0$ for any $(n,j) \in \mathcal{P}$.*

Proof. By implicit differentiation, we obtain from (4.3) that

$$\left.\frac{d\lambda_n(\tau)}{d\tau}\right|_{\tau=\tau_n^j} = \frac{-i\omega_n(i\omega_n+dn^2+\delta)}{1+\tau_n^j(i\omega_n+dn^2+\delta)}.$$

Taking the real part, we have

$$\left.\frac{d\mathrm{Re}\lambda_n(\tau)}{d\tau}\right|_{\tau=\tau_n^j} = \frac{\omega_n^2}{(1+\tau_n^j dn^2+\tau_n^j\delta)^2+(\omega_n\tau_n^j)^2} > 0.$$

This completes the proof. □

Now, we consider the collection of all τ_n^j with $(n,j) \in \mathcal{P}$. If a value appears more than once in the collection, then there are at least two pairs of purely imaginary roots and thus the condition of Hopf bifurcation is violated. For this reason, we only keep the values which appear exactly once in the collection and rearrange them in increasing order. Denote the new set by

$$\Sigma = \{\tau_0, \tau_1, \ldots, \tau_n, \ldots\}. \tag{4.8}$$

Obviously, $\tau_0 = \tau_1^0$, $\tau_i < \tau_{i+1}$ and Hopf bifurcation occurs when $\tau = \tau_i$ for each $i = 0, 1, 2\cdots$. Applying Lemmas 4.4 and 4.5, we can draw the conclusion on the distribution of the roots of (4.3).

Lemma 4.6. *Assume that $\frac{\delta}{\alpha} < 1 - \frac{d+2\delta}{k\delta}$. If $\tau \in [0, \tau_0)$, then all eigenvalues have negative real parts; if $\tau = \tau_0$, then all eigenvalues except $\pm i\omega_1$ have negative real parts; if $\tau > \tau_0$, at least two eigenvalues have positive real parts. Moreover, when τ increases through some τ_i for $i = 0, 1, 2\cdots$, the sum of the multiplicities of the eigenvalues with positive real parts will increase by two.*

The following results give the stability and existence of Hopf bifurcations.

Theorem 4.7. *Assume that $\frac{\delta}{\alpha} < 1 - \frac{d+2\delta}{k\delta}$. Then the positive steady state u_* of (4.1) is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Moreover, System (4.1) undergoes a Hopf bifurcation at u_* when $\tau = \tau_i \in \Sigma$ for $i = 0, 1, 2\cdots$, and the bifurcating periodic solutions are spatially non-homogeneous.*

In what follows, by using the center manifold and normal form method for delay partial differential equations developed by Hassard [26], Faria [24] and Wu [22,27], we study the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions from u_*, with the calculation of $\mathrm{Re}(c_1(\tau_i))$ postponed to the Appendix.

Theorem 4.8. *Given any $i \in \mathbb{N}_0$, if $Re(c_1(\tau_i))$ is negative (resp. positive), then the direction of the Hopf bifurcation is $\tau > \tau_i$ (resp. $\tau < \tau_i$). Moreover, if $i \geq 1$, the bifurcating periodic solutions are unstable; if $i = 0$, the bifurcating periodic solutions are unstable when $Re(c_1(\tau_0)) > 0$ and orbitally asymptotically stable when $Re(c_1(\tau_0)) < 0$.*

4.3. Global Hopf bifurcation analysis

Assume $\frac{\delta}{\alpha} < 1 - \frac{d+2\delta}{k\delta}$. Theorem 4.7 states that periodic solutions can bifurcate from u_* when τ is near the local Hopf bifurcation values $\tau_i \in \Sigma$. In this section, we study the global continuation of these local bifurcating periodic solutions via the global Hopf bifurcation theorem [22,28].

Let $z(t) = u(\cdot, \tau t) - u_*$. Eq. (4.1) can be rewritten as a semilinear functional differential equation

$$z'(t) = A_T z(t) + g(z_t), \tag{4.9}$$

where $z_t \in C([-1, 0], X)$ with $z_t(\theta) = z(t+\theta)$ for $\theta \in [-1, 0]$, $A_T = \tau d\Delta - \tau\delta$ is the shifted Laplace operator, and

$$g(z_t) = \frac{\tau\alpha(z_t(-1) + u_*)}{1 + \beta(z_t(-1) + u_*)^k} - \tau\delta u_*$$

is the nonlinear reaction function. Let $\{T(t)\}_{t\geq 0}$ be the semigroup generated by the linear differential operator A_T with zero Dirichlet boundary condition on the interval $(0, \pi)$. Since the principal eigenvalue of A_T is negative, we have $T(t) \to 0$ as $t \to \infty$. Moreover, the solution of (4.9) satisfies the following integral equation:

$$z(t) = T(t)z(0) + \int_0^t T(t-s)g(z_s)ds. \tag{4.10}$$

If $z(t)$ is periodic, then it also satisfies the integral equation:

$$z(t) = \int_{-\infty}^{t} T(t-s)g(z_s)ds. \tag{4.11}$$

To see this, we denote the period of $u(t)$ by ω and obtain from (4.10)

$$\begin{aligned} z(0) = z(\omega) &= T(\omega)z(0) + \int_0^{\omega} T(\omega - s)g(z_s)ds \\ &= T(\omega)z(0) + \int_{-\omega}^{0} T(-s)g(z_s)ds. \end{aligned}$$

By using the above equation repeatedly, we have

$$z(0) = T(n\omega)z(0) + \int_{-n\omega}^{0} T(-s)g(z_s)ds.$$

Letting $n \to \infty$, since $T(n\omega)z(0) \to 0$, the above equation gives (4.11). On the other hand, it is easily seen that a periodic solution of (4.11) is also a periodic solution of (4.10). We obtain from [22, Chapter 6.5] that the integral operator on the right-hand side of (4.11) is differentiable, completely continuous, and G-equivariant.

If $\frac{\delta}{\alpha} < 1 - \frac{d+2\delta}{k\delta}$, Theorem 4.1 shows that u_* is the unique positive steady state solution of (4.1). Note that 0 cannot be an eigenvalue of (4.3) for any $\tau \geq 0$ since $f'(u_*) < \delta$. Hence, the assumption (H1) in [22, Chapter 6.5] holds. It follows from Lemma 4.6 that when $\tau = \tau_i \in \Sigma$, there exists a unique pair $(r, j) \in \mathcal{P}$ such that $\tau = \tau_r^j$, and the characteristic equation (4.3) has exactly one pair of purely imaginary eigenvalues $\pm i\omega_r$. Thus, the assumption (H2) in [22, Chapter 6.5] holds. Following the definitions in [22, Chapter 6.5], we introduce the local steady state manifold

$$M = \{(u_*, \tau, T) : |\tau - \tau_i| < \epsilon_1, |T - 2\pi/(\omega_r \tau)| < \epsilon_2\} \subset E^{S^1} \times \mathbb{R}_+^2$$

for sufficiently small $\epsilon_1, \epsilon_2 > 0$, where $E = C(S^1, X)$ is a real isometric Banach representation of the group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and $E^{S^1} = \{x \in E : \ gx = x \text{ for all } g \in S^1\}$. Then $E^{S^1} = X$, and $E = E^{S^1} \bigoplus_{k=1}^{\infty}\{e^{ikt}x : x \in X\}$. From Lemma 4.6, we obtain that for $(\tau, \omega) \in [\tau_i - \epsilon_1, \tau_i + \epsilon_1] \times [\omega_r - \epsilon_2, \omega_r + \epsilon_2]$, $\pm i\omega_r$ is an eigenvalue of (4.3) if and only if $\tau = \tau_i$ and $\omega = \omega_r$. Thus, the assumption (H3) in [22, Chapter 6.5] is satisfied, and we further conclude that $(u_*, \tau_i, 2\pi/(\omega_r \tau_i))$ is an isolated singular point in M.

Next, we define a closed subset Γ of $X \times \mathbb{R}_+^2$ by

$$\Gamma = Cl\{(z, \tau, T) \in X \times \mathbb{R}_+^2 : z \text{ is a nontrivial } T\text{-periodic soltuion of (4.9)}\}.$$

Denote by $\mathcal{C}_i(u_*, \tau_i, T_i)$ the connected component of (u_*, τ_i, T_i) in Γ. It follows from Theorem 4.7 that $\mathcal{C}_i(u_*, \tau_i, T_i)$ is a nonempty subset in Γ. Define the complete steady state smooth manifold:

$$M^* = \{(u_*, \tau) : \tau \in \mathbb{R}_+\} \subset X \times \mathbb{R}_+.$$

Then $M = M^* \times (0, \infty)$. Applying the global Hopf bifurcation theorem in [22, Theorem 6.5.5], we obtain the following result.

Theorem 4.9. *Assume that $\frac{\delta}{\alpha} < 1 - \frac{d+2\delta}{k\delta}$. For each $i \in \mathbb{N}_0$, the connected component $\mathcal{C}_i(u_*, \tau_i, T_i)$ is unbounded, i.e.,*

$$\sup\left\{\max_{t\in\mathbb{R}} |z(t)| + |\tau| + T + T^{-1} : (z, \tau, T) \in \mathcal{C}_i(u_*, \tau_i, T_i)\right\} = \infty.$$

Proof. By Theorem 6.5.5 in [22], at least one of the following holds:

(a) $\mathcal{C}_i(u_*, \tau_i, T_i)$ is unbounded in $X \times \mathbb{R}_+^2$, or
(b) $\mathcal{C}_i(u_*, \tau_i, T_i) \cap M$ is finite, and $\sum_{(u_*,\tau,T)\in\mathcal{C}_i\cap M} \zeta_m(u_*, \tau, T) = 0$ for all integers $m \geq 1$, where $\zeta_m(u_*, \tau, T)$ is the mth generalized crossing number.

It follows from Lemmas 4.5 and 4.6 that $\zeta_m(u_*, \tau, T) = 0$ for all $m > 1$, and

$$\zeta_1(\mu_*, \tau_i, T_i) = -\mathrm{sgn}\left(\left. \frac{d(\mathrm{Re}\lambda_r(\tau))}{d\tau} \right|_{\tau=\tau_i} \right) = -1.$$

We only need to exclude the possibility of case (b). If case (b) is true, then $\sum \zeta_1(u_*, \tau, T) = -k < 0$, where k is the number of elements in $\mathcal{C}_i \cap M$, which leads to a contradiction. This completes the proof. $\square$

To find the interval of τ in which periodic solutions exist, we shall further investigate the properties of periodic solutions of (4.1).

Lemma 4.10. *If $\alpha > \delta$, then all nonnegative periodic solutions of (4.1) are uniformly bounded; namely, there exist two constants $B_1 > 0$ and $B_2 > 0$ such that for any nonnegative periodic solution $u(x, t)$, we have $B_1 \leq u(x, t) \leq B_2$ for all $(x, t) \in \bar{\Omega} \times \mathbb{R}_+$.*

Proof. Note that $u_* = f(u_*)/\delta \leq f(c)/\delta$, where f and c are defined in (2.2). By Theorem 2.1, we have

$$\limsup_{t\to\infty} u(x, t) \leq f(c)/\delta$$

for all $x \in [0, \pi]$. Denote $B_2 = f(c)/\delta > 0$. We claim $u(x, t) \leq B_2$ for all $(x, t) \in \bar{\Omega} \times \mathbb{R}_+$. Otherwise, if $u(x_1, t_1) > B_2$ for some $(x_1, t_1) \in \bar{\Omega} \times \mathbb{R}_+$, then

$$\lim_{n\to\infty} u(x_1, t_1 + nT) = u(x_1, t_1) > B_2,$$

where T is the period of $u(x, t)$. This contradicts the fact that $u(x_1, t)$ is eventually bounded above by B_2 as $t \to \infty$. Thus, B_2 is indeed the upper bound of $u(x, t)$. To find a uniform lower bound for $u(x, t)$, we first note from Theorem 2.1 that $u(x, t)$ should be strictly positive, and thus possesses a positive minimum at a point (x_2, t_2). Since $\frac{\partial}{\partial t} u(x_2, t_2) = 0$ and $\Delta u(x_2, t_2) \geq 0$, we obtain from the differential equation (4.1) that

$$\delta u(x_2, t_2) \geq f(u(x_2, t_2 - 1)).$$

If $0 < u(x_2, t_2 - 1) < u_*$, then

$$f(u(x_2, t_2 - 1)) > \delta u(x_2, t_2 - 1) \geq \delta u(x_2, t_2),$$

a contradiction. Hence, we have $u_* \leq u(x_2, t_2 - 1) \leq B_2$. Consequently,

$$\delta u(x_2, t_2) \geq f(u(x_2, t_2 - 1)) \geq \frac{\alpha u_*}{1 + \beta B_2^k}.$$

Denote $B_1 = \alpha u_*/[\delta(1 + \beta B_2^k)]$. It follows that $u(x, t) \geq u(x_2, t_2) \geq B_1$ for all (x, t) in $\bar{\Omega} \times \mathbb{R}_+$. Since B_1 and B_2 depend only on the model parameters, we conclude that all periodic solutions of (4.1) are uniformly bounded in $[B_1, B_2]$. This completes the proof. $\square$

Lemma 4.11. *Assume that $\alpha > \delta$. Then System (4.1) has no nontrivial periodic solution of period τ.*

Proof. Assume to the contrary, $v(x,t)$ is a nontrivial periodic solution of (4.1) with period τ. Then it satisfies the following system

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = d\triangle v(x,t) - \delta v(x,t) + f(v(x,t)), & x \in (0,\pi), t > 0, \\ v(0,t) = v(\pi,t) = u_*, & t \geq 0, \\ v(x,0) = u_0(x,0) \geq 0, & x \in [0,\pi]. \end{cases} \tag{4.12}$$

We claim that

$$\lim_{t\to\infty} v(x,t) = u_*,$$

which precludes the existence of nontrivial heterogeneous periodic solution of system (4.12), and thus leads to a contradiction. To prove the claim, we consider the non-diffusive ordinary differential equation corresponding to (4.12):

$$z'(t) = -\delta z(t) + \frac{\alpha z(t)}{1+\beta z^k(t)}, \quad z(0) = z_0 > 0 \tag{4.13}$$

Note that the right-hand side of the above equation, as a function of z, has two equilibria $z = 0$ and $z = u_* > 0$. Moreover, this function is positive for $z \in (0, u_*)$ and negative when $z > u_*$. It is readily seen that the solution of Eq. (4.13) with positive initial condition converges to the unique positive equilibrium u_*, that is, $\lim_{t\to\infty} z(t, z_0) = u_*$.

Denote $M_0 = \max\{u_*, \max_{[0,\pi]} u_0(x,0)\}$, then $\overline{v}(x,t) = z(t, M_0)$ and $\underline{v}(x,t) = 0$ are upper-solution and lower-solution of (4.12), respectively. Hence, System (4.12) has a unique solution $v(x,t)$, which satisfies $0 \leq v(x,t) \leq z(t, M_0)$. Furthermore, by strong maximum principle, we have $v(x,t) > 0$ for any $t > 0$ and $x \in [0,\pi]$. Fix any $t_1 > 0$, we have $v(x,t_1) > 0$ for $x \in [0,\pi]$. Denote

$$m_1 := \min\{u_*, \min_{x\in[0,\pi]} v(x,t_1)\} > 0, \quad M_1 := \max\{u_*, \max_{x\in[0,\pi]} v(x,t_1)\} > 0,$$

and $w(x,t) = v(x, t+t_1)$. It is clear that $w(x,t)$ satisfies the following system

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = d\triangle w(x,t) - \delta w(x,t) + f(w(x,t)), & x \in (0,\pi), t > 0, \\ w(0,t) = w(\pi,t) = u_*, & t \geq 0, \\ w(x,0) = v(x,t_1), & x \in [0,\pi]. \end{cases}$$

Moreover, $z(t, M_1)$ and $z(t, m_1)$ are upper-solution and lower-solution of the above equation, respectively. Thus, we have

$$z(t, m_1) \leq w(x,t) = v(x, t+t_1) \leq z(t, M_1)$$

Note that $\lim_{t\to\infty} z(t, m_1) = \lim_{t\to\infty} z(t, M_1) = u_*$. This, together with the above inequality, yields $\lim_{t\to\infty} v(x,t) = u_*$. Therefore, System (4.1) has no nontrivial periodic solution of period τ. □

We are now ready to analyze the structure of $\mathcal{C}_i$ and prove global existence of periodic solutions of (4.1). For simplicity, let $\mathcal{P}_\Sigma$ be the collection of all index pairs $(n,j) \in \mathcal{P}$ such that $\tau_n^j \in \Sigma$; namely, τ_n^j only appears once in the set $\mathcal{P}$.

Theorem 4.12. *Assume that $\frac{\delta}{\alpha} < 1 - \frac{d+2\delta}{k\delta}$. For each $(n,j) \in \mathcal{P}_\Sigma$, denote by $\mathcal{C}_n^j = \mathcal{C}_n^j(u_*, \tau_n^j, T_n^j)$ the connected component of (u_*, τ_n^j, T_n^j) in Γ. Then we have the following results.*

(i) If $i \neq j$, then $\mathcal{C}_n^j \cap \mathcal{C}_m^i = \emptyset$ for any (m,i) and (n,j) in $\mathcal{P}_\Sigma$.
(ii) For each $(n,j) \in \mathcal{P}_\Sigma$ with $j \geq 1$, the global Hopf branch $\mathcal{C}_n^j$ is unbounded with unbounded τ-component, bounded period component in $(1/(j+1), 1/j)$, and bounded solution component in $[B_1, B_2]$.
(iii) Let $\hat{\tau} = \min\{\tau_n^j : (n,j) \in \mathcal{P}_\Sigma$ and $j \geq 1\}$. For any $\tau > \hat{\tau}$, there exists at least one periodic solution for System (4.1).

Proof. By Lemma 4.11, System (4.1) has no nontrivial periodic solution of period τ. Therefore, System (4.9) does not have nontrivial periodic solutions of period 1, or nontrivial periodic solutions of period $1/j$ for any $j \in \mathbb{N}$. When τ is close to the bifurcation point τ_n^j, we obtain from local Hopf bifurcation theorem that

$$w_n\tau \in (2j\pi + \pi/2, 2j\pi + \pi) \subset (2j\pi, 2j\pi + 2\pi).$$

Let $T = 2\pi/(w_n\tau)$ be the period. It follows that $1/(j+1) < T < 1/j$ for $j \in \mathbb{N}$, and $T > 1$ for $j = 0$. If $j \in \mathbb{N}$, since System (4.9) has no nontrivial periodic solution of period $1/j$ or $1/(j+1)$, by continuity of Hopf bifurcation branch, the periods on $\mathcal{C}_n^j$ are bounded by $1/j$ and $1/(j+1)$. If $j = 0$, a similar argument shows that the periods on $\mathcal{C}_n^0$ are always greater than 1. Therefore, any two global Hopf branches $\mathcal{C}_n^j$ and $\mathcal{C}_m^i$ with $i \neq j$ do not intersect. This proves (i).

For each $(n,j) \in \mathcal{P}_\Sigma$, we recall from Lemma 4.10 that the periodic solutions on $\mathcal{C}_n^j$ are bounded in the interval $[B_1, B_2]$. Moreover, as proven in the proof of (i), the periods are bounded in the interval $(1/(j+1), 1/j)$. It then follows from Theorem 4.9 and the unboundedness of $\mathcal{C}_n^j$ that the τ-component should be unbounded. This proves (ii).

Finally, (iii) follows immediately from the unboundedness of τ on the global Hopf branch bifurcating from the bifurcation point $\hat{\tau}$. $\square$

For simplicity, we consider the case when

$$\frac{\delta}{\alpha} \in \left(1 - \frac{4d+2\delta}{k\delta}, 1 - \frac{d+2\delta}{k\delta}\right); \tag{4.14}$$

namely, $n_{max} = 1$. It follows that $\Sigma = \{\tau_1^0, \tau_1^1, \ldots\}$ with $\tau_j = \tau_1^j$. We have the following results.

Corollary 4.13. *Assume (4.14) is satisfied, that is, $n_{max} = 1$. If $j \geq 1$ and $\tau > \tau_j$, System (4.1) has at least j periodic solutions. If $j = 0$, the global Hopf branch $\mathcal{C}_1^0$ has either unbounded period or unbounded τ-component.*

Proof. By Theorem 4.12, any two global Hopf branches $\mathcal{C}_1^i$ and $\mathcal{C}_1^j$ with $i \neq j$ do not intersect. Moreover, the τ-component of $\mathcal{C}_1^i$ with $1 \leq i \leq j$ is unbounded and thus has a periodic solution for any $\tau > \tau_j$. Therefore, there exist at least j periodic solutions of (4.1). The second statement follows from Theorem 4.9. $\square$

Note that if System (4.1) has no periodic solution of period 4τ, then we can further prove that the periods on the global Hopf branch $\mathcal{C}_1^0$ are bounded in the interval $[1,4]$, which together with Corollary 4.13, implies that the τ-component of this branch is unbounded. Consequently, for any $\tau > \tau_j$ with $j \geq 0$, there exist at least $j+1$ periodic solutions for (4.1). We leave the proof of nonexistence of 4τ-periodic solution of (4.1) as an open problem.

5. Numerical simulations and discussion

In this section, we present some numerical simulations to demonstrate our theoretical results. First, we simulate the system (2.1) with zero-Dirichlet boundary condition. The parameter values are chosen as

$$d = 0.5,\ \ \delta = 1, k = 3, \beta = 8. \tag{5.1}$$

If $\alpha < 1.5$, by Theorems 3.2 and 3.5, the zero equilibrium is globally asymptotically stable for any $\tau \geq 0$. If $\alpha \in (1.5, 3)$, by Theorems 3.2 and 3.8, there exists a unique heterogeneous positive steady state $\phi(x)$ which is locally asymptotically stable for any $\tau \geq 0$. We set $\tau = 1$ and choose α to be 1.25 and 2.8, respectively. The simulation results are illustrated in Fig. A.2.

If $\alpha > 3$, the dynamics of System (2.1) may depend on the value of τ. In Fig. A.3, we fix $\alpha = 6$ and choose τ to be 1 and 4, respectively. It is noted that when $\tau = 1$, the unique heterogeneous positive steady state is still locally asymptotically stable. However, when $\tau = 4$, the unique heterogeneous positive steady state becomes unstable and a periodic solution is bifurcated from the unique heterogeneous positive steady state. This suggests that if the condition in Theorem 3.8 is violated, then the delay may destabilize the positive steady state and induce a bifurcation.

Next, we consider System (4.1) with non-zero Dirichlet boundary condition. We choose the same parameter values as in (5.1). It follows from Theorem 4.2 that the unique positive steady state $u_* = \sqrt[3]{\alpha - 1}/2$ is locally asymptotically stable for all $\tau \geq 0$ if $1 < \alpha \leq 6$; see Fig. A.4, where we set $\alpha = 2, \tau = 1$, and the initial condition $u_0(x, \theta) = 0.5 + 0.015 \sin x, \theta \in [-\tau, 0]$.

If $\alpha > 6$, by Theorem 4.7, the positive steady state u_* is locally asymptotically stable when $\tau < \tau_1^0$ and unstable for $\tau > \tau_1^0$. Furthermore, System (4.1) undergoes a Hopf bifurcation at u_* when $\tau = \tau_i \in \Sigma$ with $i \in \mathbb{N}_0$; see Fig. A.5, where we set $\alpha = 9$, and the initial condition $u_0(x, \theta) = 1 + 0.1 \sin x, \theta \in [-\tau, 0]$. A simple calculation gives $u_* = 1$, $n_{max} = 1$, and $\omega_1 \approx 0.726483$. Moreover, the local Hopf bifurcation values are

$$\tau_1^0 \approx 3.70355 < \tau_1^1 \approx 12.3523 < \cdots < \tau_1^j \approx 3.70355 + 8.64877j < \cdots .$$

By using the formula given in the Appendix, we compute $c_1(\tau_1^0) \approx -0.854693 - 0.158908i$. Especially, $\mathrm{Re}(c_1(\tau_1^0)) < 0$. By Theorem 4.8, a forward Hopf bifurcation occurs at τ_1^0 and the bifurcating spatially non-homogeneous periodic solutions are stable; see Fig. A.5. Further simulations (not shown here) demonstrate that spatial non-homogeneous periodic solutions exist for any large τ, which coincides with Corollary 4.13.

We conclude this paper by a discussion on two open problems.

1. For zero Dirichlet problem, we obtained global stability of trivial equilibrium, and established existence, uniqueness and local stability of nontrivial steady state. It would be interesting, though challenging, to analyze the periodic solutions bifurcated from the spatially heterogeneous steady state solution.
2. For non-zero Dirichlet problem, we provided existence, uniqueness and local stability of steady state solution, studied properties of local Hopf branch and structures of global Hopf branches, and proved global existence of periodic solutions. There is one open problem on the structure of the unbounded Hopf branch which bifurcates from the first bifurcating point. We demonstrated that this branch is unbounded in the sense that either the period component or the delay component is unbounded. However, if one could exclude the existence of 4-periodic solution for the scaled problem, then the period component is bounded and the delay component is unbounded. The nonexistence conjecture has not been proved even for the spatial homogeneous problem.

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Appendix. Direction and stability of the bifurcating periodic solutions

Theorem 4.7 gives some sufficient conditions for the occurrence of Hopf bifurcation from the steady state solution u_* of System (4.1) at $\tau = \tau_n^j$ with $(n, j) \in \mathcal{P}_\Sigma$. In this appendix, we determine the direction

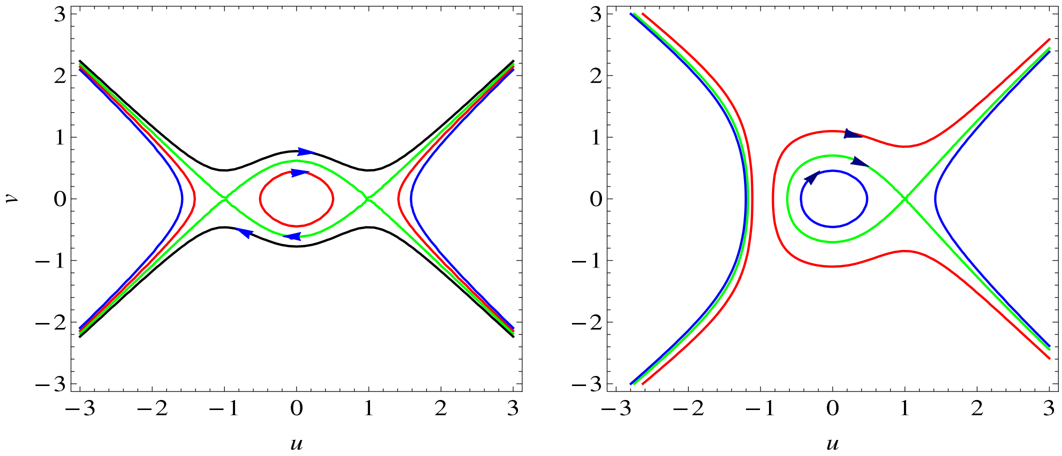

Fig. A.1. The phase portrait of (3.2). Left: k is even. Right: k is odd.

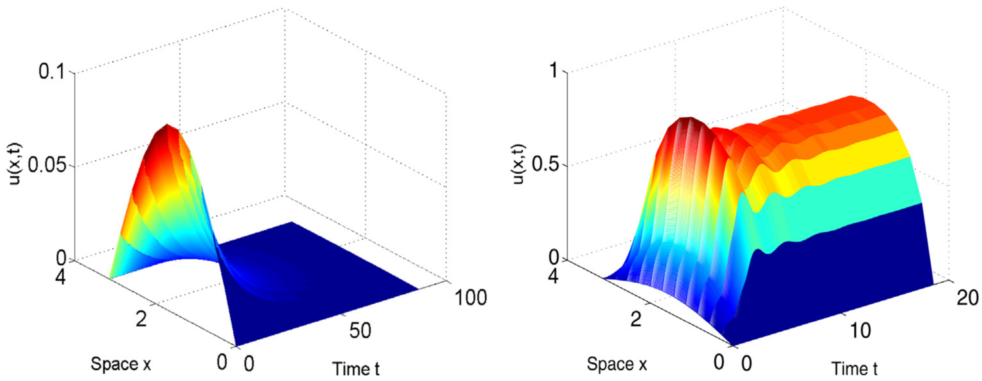

Fig. A.2. Left: $\alpha = 1.25$, the zero equilibrium is globally asymptotically stable. Right: $\alpha = 2.8$, the unique heterogeneous positive steady state is locally asymptotically stable.

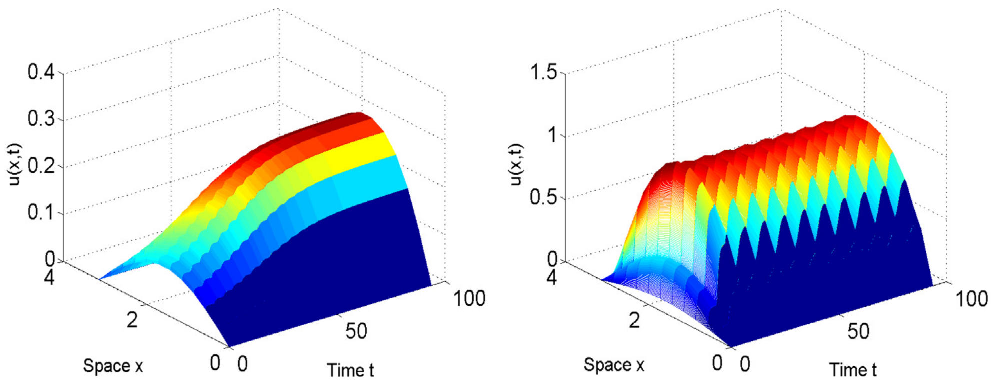

Fig. A.3. Left: $\tau = 1$, the unique heterogeneous positive steady state is locally asymptotically stable. Right: $\tau = 4$, a periodic solution is bifurcated from the unstable steady state.

and stability of the bifurcating spatially non-homogeneous periodic solutions established, and thus prove Theorem 4.8.

Set $\tau = \tau_n^j + \epsilon$, then $\epsilon = 0$ is the Hopf bifurcation value of System (4.1). We translate u_* to the origin and rescale the time to normalize the delay; namely, let $U(t) = u(\cdot, t\tau) - u_*$. System (4.1) can be written in

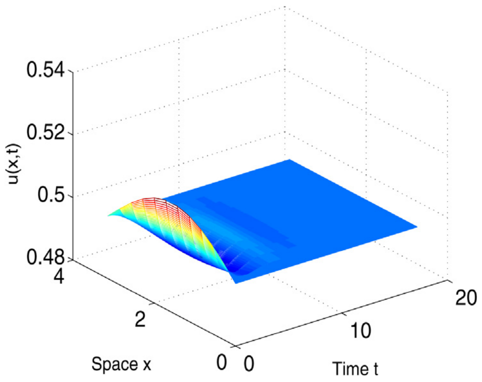

Fig. A.4. The solution of (4.1) tends to the unique positive steady state u_* for all $\tau \geq 0$.

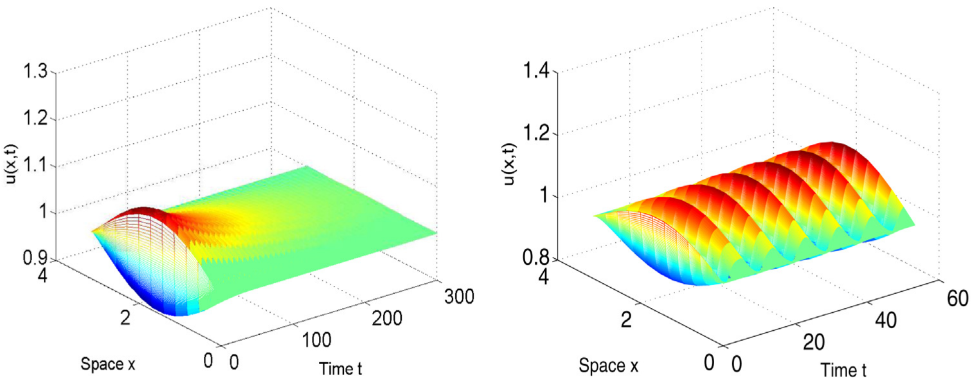

Fig. A.5. Left: $\tau = 3$, the unique positive steady state u_* is locally asymptotically stable for all $\tau < \tau_1^0$. Right: $\tau = 4$, a bifurcating spatially non-homogeneous periodic solution exists when $\tau > \tau_1^0$.

the following abstract form:

$$U'(t) = \tau_n^j L_0 U_t + F_\epsilon(U_t), \tag{A.1}$$

where $U_t \in C([-1,0],X)$, and L_0 is a linear functional and F_ϵ a nonlinear functional on $C([-1,0],X)$ such that

$$\begin{aligned} L_0\varphi &= (d\triangle - \delta)\varphi(0) + f'(u_*)\varphi(-1), \\ F_\epsilon(\varphi) &= \epsilon L_0\varphi + (\tau_n^j + \epsilon)\left(\tfrac{f''(u_*)}{2}\varphi^2(-1) + \tfrac{f'''(u_*)}{6}\varphi^3(-1) + \mathcal{O}(4)\right), \end{aligned}$$

for any $\varphi \in C([-1,0],X)$. Denote by $\mathcal{A}_{\tau_n^j}$ the infinitesimal generator of the semigroup induced by the solutions of the linearized equation $U'(t) = \tau_n^j L_0 U_t$. Then we have

$$\mathcal{A}_{\tau_n^j}(\varphi)(\theta) = \begin{cases} \dot\varphi(\theta), & \theta \in [-1,0), \\ \tau_n^j L_0\varphi, & \theta = 0, \end{cases}$$

for $\varphi \in C([-1,0],X)$ such that $\dot\varphi \in C([-1,0],X)$, $\varphi(0) \in \mathcal{D}$, and $\dot\varphi(0) = \tau_n^j L_0\varphi$. Thus, (A.1) can be rewritten as the abstract ODE:

$$U_t' = \mathcal{A}_{\tau_n^j} U_t + \chi_0 F_\epsilon(U_t).$$

By Riesz representation theorem, there exists a bounded variation function

$$\eta(\theta) = -\tau_n^j(dn^2 + \delta)\chi_0 - \tau_n^j f'(u_*)\chi_{-1}$$

such that

$$\tau_n^j L_0\varphi = \int_{-1}^0 \varphi(\theta)d\eta(\theta)$$

for $\varphi \in C([-1,0],\mathbb{R})$. Define an operator

$$\mathcal{A}^*(\psi)(s) = \begin{cases} -\dot{\psi}(s), & s \in (0,1], \\ \int_{-1}^{0} \psi(-s)d\eta(s), & s = 0. \end{cases}$$

It is easily seen that $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}_{\tau_n^j}$ under the bilinear form

$$(\psi,\varphi) = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \bar{\psi}(\xi-\theta)\varphi(\xi)d\xi d\eta(\theta)$$

for $\psi \in C([0,1],\mathbb{R}^*)$ and $\varphi \in C([-1,0],\mathbb{R})$; see [26,29]. By Lemma 4.6, $\mathcal{A}_{\tau_n^j}$ has exactly one pair of purely imaginary simple eigenvalues $\pm i\omega_n\tau_n^j$. Clearly, $q(\theta) = e^{i\omega_n\tau_n^j\theta}$ with $\theta \in [-1,0]$ is an eigenvector of $\mathcal{A}_{\tau_n^j}$ corresponding to the eigenvalue $i\omega_n\tau_n^j$, and $p(s) = D_n e^{i\omega_n\tau_n^j s}$ with $s \in [0,1]$ is an eigenvector of $\mathcal{A}^*$ corresponding to the eigenvalue $-i\omega_n\tau_n^j$. Here we choose $\overline{D}_n = (1+\tau_n^j f'(u_*)e^{-i\omega_n\tau_n^j})^{-1}$ such that $(p,q)=1$. We also note that $(p,\bar{q}) = 0$. To study the properties of Hopf bifurcation, we follow the computation algorithm as in [22,26] and calculate the following parameter values:

$$\begin{aligned}
g_{20} &= \overline{D}_n\tau_n^j e^{-2i\omega_n\tau_n^j} f''(u_*)\int_0^{\pi}\sin^3(nx)dx, \\
g_{11} &= \overline{D}_n\tau_n^j f''(u_*)\int_0^{\pi}\sin^3(nx)dx, \\
g_{02} &= \overline{D}_n\tau_n^j e^{2i\omega_n\tau_n^j} f''(u_*)\int_0^{\pi}\sin^3(nx)dx, \\
g_{21} &= \overline{D}_n\tau_n^j f''(u_*)\Big(e^{i\omega_n\tau_n^j}\int_0^{\pi} W_{20}(-1)\sin^2(nx)dx \\
&\quad + 2e^{-i\omega_n\tau_n^j}\int_0^{\pi} W_{11}(-1)\sin^2(nx)dx\Big) + \overline{D}_n\tau_n^j f'''(u_*)e^{-i\omega_n\tau_n^j}\int_0^{\pi}\sin^4(nx)dx,
\end{aligned}$$

where

$$\begin{aligned}
W_{20}(\theta) &= \left(\frac{ig_{20}}{\omega_n\tau_n^j}e^{i\omega_n\tau_n^j\theta} + \frac{i\overline{g}_{02}}{3\omega_n\tau_n^j}e^{-i\omega_n\tau_n^j\theta}\right)\sin(nx) + E_1 e^{2i\omega_n\tau_n^j\theta}, \\
W_{11}(\theta) &= \left(-\frac{ig_{11}}{\omega_n\tau_n^j}e^{i\omega_n\tau_n^j\theta} + \frac{i\overline{g}_{11}}{\omega_n\tau_n^j}e^{-i\omega_n\tau_n^j\theta}\right)\sin(nx) + E_2,
\end{aligned}$$

for $\theta \in [-1,0)$, and

$$\begin{aligned}
E_1 &= \sum_{m=1}^{\infty}\frac{\tau_n^j f''(u_*)e^{-2i\omega_n\tau_n^j}\int_0^{\pi}\sin^2(nx)\sin(mx)dx}{2i\omega_n\tau_n^j + dm^2 + \delta - f'(u_*)e^{-2i\omega_n\tau_n^j}}\sin(mx), \\
E_2 &= \sum_{m=1}^{\infty}\frac{\tau_n^j f''(u_*)\int_0^{\pi}\sin^2(nx)\sin(mx)dx}{dm^2 + \delta - f'(u_*)}\sin(mx).
\end{aligned}$$

Consequently, we can compute the following quantities:

$$c_1(\tau_n^j) = \frac{i}{2\omega_n\tau_n^j}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{\mathrm{Re}(c_1(\tau_n^j))}{\mathrm{Re}(\lambda_n'(\tau_n^j))}.$$

It follows from Hopf bifurcation theory [22,24,26] that the sign of μ_2 gives the direction of the Hopf bifurcation, and the sign of $\mathrm{Re}(c_1(\tau_n^j))$ determines the stability of the bifurcating spatially non-homogeneous periodic solutions. To be more specific, if $\mu_2 > 0$ (resp. $\mu_2 < 0$), then there exist bifurcating periodic solutions when τ moves to the right (resp. left) of bifurcation point τ_n^j; if $\mathrm{Re}(c_1(\tau_n^j))$ is negative (resp. positive), then the bifurcating periodic solutions are orbitally asymptotically stable (resp. unstable) on the center manifold. Since $\mathrm{Re}(\lambda_n'(\tau_n^j)) > 0$ by Lemma 4.5, μ_2 and $\mathrm{Re}(c_1(\tau_n^j))$ have different signs. Theorem 4.8 then follows from the above argument.

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