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Research article

Traveling waves of diffusive disease models with time delay and degeneracy

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Abstract: In this paper, we propose a diffusive epidemic model with a standard incidence rate and distributed delays in disease transmission. We also consider the degenerate case when one of the diffusion coefficients vanishes. By establishing existence theory of traveling wave solutions and providing sharp lower bound for the wave speeds, we prove linear determinacy of the proposed model system. Sensitivity analysis suggests that disease propagation is slowed down by transmission delay but fastened by spatial diffusion. The existence proof is based on the construction of a suitable convex set which is invariant under the integral map of traveling wave equations. An innovative argument is formulated to study the boundary value problems of nonlinear elliptic equations satisfied by the traveling wave solutions, which enables us to prove that there does not exist a positive traveling wave connecting two nontrivial equilibria.

Keywords: traveling waves; diffusive disease models; time delay; degeneracy

1. Introduction

The study of deterministic epidemic model can be dated back in 1927 when Kermack and McKendrik [1, Eq. (29)] proposed a simple ordinary differential system with three compartments: susceptible individuals (S), infected individuals (I) and removed/recovered individuals (R). The corresponding three-dimensional system can be decoupled into two subsystems:

$$S'(t) = -\varphi(S(t), I(t)), \tag{1.1}$$

$$I'(t) = \varphi(S(t), I(t)) - \gamma I(t), \tag{1.2}$$

and $R'(t) = \gamma I(t)$. Here, the incidence rate is chosen as the mass-action function:

$$\varphi(S(t), I(t)) = \beta S(t)I(t),$$

The two parameters $\beta > 0$ and $\gamma > 0$ in the model denote the disease transmission rate and the removal/recovery rate, respectively. Given initial values $\{S_0, I_0, R_0\}$ at $t = 0$, one can integrate the R-equation to obtain

$$R(t) = R_0 + \int_0^t \gamma I(s)ds,$$

where $I(t)$ can be solved from the subsystem for S and I:

$$I(t) = S_0 + I_0 - S(t) + \frac{\gamma}{\beta} \ln \frac{S(t)}{S_0}.$$

Substituting this into (1.1) gives a separable equation for $S(t)$, whose solution can be expressed as the inverse of an integral:

$$\int_{S_0}^{S(t)} \left[u(S_0 + I_0 - u + \frac{\gamma}{\beta} \ln \frac{u}{S_0}) \right]^{-1} du = -\beta t.$$

As noted by Kermack and McKendrik, there is no explicit formula, in terms of elementary functions, instead of the integral representation as given above, for the solution of their SIR model.

In real application, however, it is more desirable to have a closed formula for the functions of interest so as to fit the epidemic wave data collected during a disease outbreak. The Richards empirical model [2], which is also called the theta-logistic model [3, 4], has been used to estimate the key parameters in recent outbreaks of SARS [5], dengue fever [6] and pandemic influenza H1N1 [7]. This model has the advantage that it can be solved in terms of elementary functions. However, the Richards model was originally introduced in ecological studies, and it did not have any epidemiological justification. To make a connection between Richards empirical model with Kermack-Mckendrick compartmental model, it was proposed in [8] that the following standard incidence rate shall be used:

$$\varphi(S(t), I(t)) = \frac{\beta S(t)I(t)}{S(t) + I(t)}.$$

Here, the removed individuals are assumed to be isolated from the community so that the denominator of the incidence function is $S(t) + I(t)$, instead of $S(t) + I(t) + R(t)$. This assumption ensures that the R-equation can be decoupled from the model system. The above system can be integrated with solutions explicitly given by elementary functions. Actually, if we add (1.1) to (1.2) and then divide both sides of the resulting equation by both sides of (1.1) respectively, we arrive at a separable equation

$$\frac{d(S + I)}{dS} = \frac{\gamma(S + I)}{\beta S}.$$

Solving the above equation gives $S(t) + I(t) = cS(t)^{\gamma/\beta}$, where $c > 0$ is a constant of integration which depends on initial conditions. Now, we use this algebraic equation to eliminate $I(t)$ in (1.1) and obtain a Richards-type equation:

$$S'(t) = -\beta S(t)[1 - S(t)^{1-\gamma/\beta}/c],$$

whose solution has a closed form.

Moreover, an intrinsic relation between the parameters in Kermack-Mckendrick model and Richards model was obtained in [8], which provided a satisfactory interpretation of Richards model in epidemiology.

It is worth to remark that "standard incidence is not realistic for small population sizes" [9]. This is because "for small population, the contact rate increases with population size" and mass action is more appropriate [10]. However, when population increases, the contact rate cannot increase linearly but it should be saturated as population size goes to infinity [11]. So, our model is biologically relevant when the population size is large.

If the transmission delay is taken into consideration [12], one should change the equation (1.2) to

$$I'(t) = \int_0^\infty \eta(\tau)q(\tau)\varphi(S(t-\tau), I(t-\tau))d\tau - \gamma I(t), \tag{1.3}$$

where $q(\tau)$ is a probability density function which characterizes the possibility of a susceptible individual having a latency period of τ after being infected and $\eta(\tau) < 1$ is the survival probability. Note that the time delay does not appear in the S-equation because the susceptible individual should be removed from this group once being infected. We refer to the equations (1.1) and (1.3) as a *type-I delayed disease model*. For convenience, we define $\theta = \int_0^\infty \eta(\tau)q(\tau)d\tau < 1$ and $p(\tau) = \eta(\tau)q(\tau)/\theta$. It is noted that $p(\tau)$ is still a probability density function and θ can be regarded as average survival rate during latency stage. We then adopt the convolution symbol $*$ as

$$(p * g)(t) := \int_0^\infty p(\tau)g(t-\tau)d\tau,$$

and rewrite (1.3) as $I'(t) = \theta[p * \varphi(S, I)](t) - \gamma I(t)$. When $p(\tau) = \delta_r(\tau)$ is a Dirac delta function, then $(p * g)(t) = g(t-r)$ and the equation (1.3) has only one discrete delay:

$$I'(t) = \theta\varphi(S(t-r), I(t-r)) - \gamma I(t).$$

Let $i(t) = I(t+r)$. One may further obtain a closed system for S and i:

$$\begin{aligned} S'(t) &= -\varphi(S(t), i(t-r)), \\ i'(t) &= \theta\varphi(S(t), i(t-r)) - \gamma i(t). \end{aligned}$$

In some literature, the term $i(t-r)$ in both of the above two equations is replaced by a general functional with distributed delays $(p*i)(t)$. We refer to such generalized delay system as a *type-II delayed disease model*. Note that in the case of single discrete delay, type-I and type-II delayed disease models are equivalent with the relation $i(t) = I(t+r)$; see [13]. However, if more general distributed delays are taken into consideration, these two types of delayed disease models are different.

To study the spatial spread of infectious diseases, we assume random movement of each individual and propose the following delayed diffusive epidemic model:

$$\partial_t S(x,t) = d_1\partial_{xx}S(x,t) - \varphi(S(x,t), I(x,t)), \tag{1.4}$$

$$\partial_t I(x,t) = d_2\partial_{xx}I(x,t) + \theta\int_0^\infty p(\tau)\varphi(S(x,t-\tau), I(x,t-\tau))d\tau - \gamma I(x,t), \tag{1.5}$$

where $d_1 \geq 0$ and $d_2 \geq 0$ are the diffusion rates of the susceptible and infective individuals, respectively. If both d_1 and d_2 vanish, the above system reduces to the delay differential system with spatial homogeneity. Throughout this paper, we assume that at least one diffusion rate is positive. However, it is biologically relevant that one of the diffusion rates may vanish. For instance, in the study of spatial spread of rabies among foxes [14], the uninfected foxes $S(x,t)$ always stay in their territories, while the rabid foxes $I(x,t)$ may loss their sense of direction and wonder randomly. Thus, we shall consider the degenerate case $d_1 = 0$. Here, we also assume for simplicity that the individuals do not diffuse during the incubation period. This is reasonable because, in the rabies model, the fox moves rapidly and randomly only when it becomes infective. Note that for the special case when $p(\tau) = \sigma e^{-\sigma\tau}$, the system (1.4)-(1.5) reduces to an SEIR diffusive model without delay but with degeneracy in the equation for the exposed class.

Our objective is to provide existence theory for the traveling wave solutions of the proposed model system with time delay and degeneracy. Especially, we will calculate the minimal traveling speed and analyze the effect of disease transmission delay and spatial movement in the propagation of an infectious disease. We shall show that the delayed system with degeneracy is linearly determinant in the sense that the minimal traveling speed can be obtained by considering the linearized system at a disease-free equilibrium. The traveling wave solutions of (1.4)-(1.5) with speed $c \geq 0$ take the special forms $S(x,t) = S(\xi)$ and $I(x,t) = I(\xi)$, where $\xi = x + ct \in \mathbb{R}$. The solution with zero wave speed is actually the steady state of the diffusive system. So, we always assume $c > 0$. A simple application of chain rule gives the following nonlocal differential system

$$cS'(\xi) = d_1 S''(\xi) - \varphi(S(\xi), I(\xi)), \tag{1.6}$$

$$cI'(\xi) = d_2 I''(\xi) + \theta \int_0^\infty p(\tau)\varphi(S(\xi - c\tau), I(\xi - c\tau))d\tau - \gamma I(\xi). \tag{1.7}$$

For simplicity, we denote $p_c(\tau) := p(\tau/c)/c$. Then the integral on the right-hand side of (1.7) has the simple convolution expression: $p_c * \varphi(S,I)$. We also note that p_c is still a probability density function whose total integral on the positive real line equals to one.

It follows from maximum principle that any non-constant and non-negative solutions of (1.6)-(1.7) should be positive at any finite point. In general, the traveling wave connects two equilibria at infinities. Note that the system (1.4)-(1.5) has infinitely many equilibrium points $(S^*, 0)$, where S^* is any nonnegative number. However, as we will demonstrate later, it is impossible to find a traveling wave solution connecting two different non-trivial equilibria. In other words, if a positive traveling wave solution of (1.6)-(1.7) satisfies the boundary conditions:

$$S(-\infty) = S_0 > 0, \quad I(-\infty) = 0, \quad S'(-\infty) = I'(-\infty) = 0 \tag{1.8}$$

then we should also have $S(\infty) = I(\infty) = 0$.

Unlike the rich theory and general tools developed for monotone systems [15, 16, 17, 18, 19], the study of non-monotone systems such as the diffusive disease models is far from complete and unified. Especially, the results on traveling wave solutions in epidemic models are obtained on a case-by-case basis. Most of the earlier studies only consider the non-delay case. When the incidence rate is a bilinear function in S and I, Källén [20] investigated the degenerate case $d_1 = 0$, while Hosono and Ilyas [21] considered the degenerate case $d_2 = 0$. Hosono and Ilyas [22] further studied the non-degenerate case by using a geometric shooting method introduced by Dunbar [23, 24]. This technique was recently developed by Huang [25] in a work on non-monotone diffusive systems with general reaction functions. The applications of geometric shooting method are limited to the non-delay models. For the diffusive disease models with time delay, a commonly used method is the Schauder fixed point theorem; see [26] and [27] for the study of two delayed epidemic models with standard incidence rate and mass-action, respectively. In [28], a delayed diffusive SIR model with external supplies is considered. The disease models all have type-II delays. A type-I delayed disease model with mass-action and recruitment was proposed in [29], where a critical wave speed was calculated and numerical simulation was conducted to suggest that this critical value should be the minimal wave speed. This observation was recently proved in [30] with the aid of Schauder fixed point theorem. To the best of our knowledge, there is no results concerning the traveling wave solutions of type-I delayed diffusive models with standard incidence rate. In this paper, we will fill in this gap and establish existence theory for the traveling wave equations (1.6)-(1.7). Moreover, we will extend our results to the degenerate cases when one of the diffusion coefficients vanishes.

The rest of the paper is organized as follows. In Section 2, we obtain a critical wave speed by linearizing the traveling wave equations at a non-trivial equilibrium. Sensitivity analysis is conducted to study the dependence of this critical value on diffusion coefficient and time delay. In Section 3, we introduce some preliminary results about shifted Laplacian operators and their inverses. In Section 4, we provide an existence theorem for the positive traveling wave solutions. In Section 5, we investigate some properties of the positive traveling wave solutions. In Section 6, we conclude this paper with some discussions on the main results and an open problem.

2. Linear determinacy and critical wave speed

It is well known that the minimal wave speed for monotone systems can be determined from the corresponding linearized system at a certain equilibrium point [31, 32, 33]. However, for non-monotone systems, especially predator-prey type systems in disease models, it is still an open problem to find the conditions for linear determinacy [34, 35]. In this section, we will calculate a critical wave speed c^* by linearization. As we shall see later, this critical value is exactly the minimal wave speed, which proves linear determinacy of our model system.

By linearizing the I-equation (1.7) at the nontrivial equilibrium $(S_0, 0)$, we find the characteristic function:

$$f(\lambda, c) := -d_2\lambda^2 + c\lambda - \theta\beta \int_0^\infty p(\tau)e^{-c\tau\lambda}d\tau + \gamma, \tag{2.1}$$

where $\lambda \geq 0$ and $c \geq 0$. Recall that p is a probability density function such that $p(\tau) \geq 0$ for all $\tau \geq 0$ and the total integral on the real line is one. Throughout this paper, we also assume that $p(\tau)$ decays exponentially at infinity; namely, there exists $\sigma > 0$ such that $p(\tau)e^{\sigma\tau}$ is uniformly bounded for $\tau \in [0, \infty)$. We will need to use this assumption to prove continuity of the integral map associated with traveling wave equations. As a simple consequence of the assumption, all moments of p exist. In this section, we are only interested in the case when $\theta\beta > \gamma$; as we shall prove later, no positive traveling wave solution exists for the case $\theta\beta \leq \gamma$. First, we note that if $c = 0$, then $f(\lambda, 0) = -d_2\lambda^2 - \theta\beta + \gamma$ is always negative. Fix any $c > 0$, we have

$$\partial_{\lambda\lambda} f(\lambda, c) = -2d_2 - \theta\beta c^2 \int_0^\infty p(\tau)\tau^2 e^{-c\tau\lambda}d\tau < 0,$$

which implies that

$$\partial_\lambda f(\lambda, c) = -2d_2\lambda + c + \theta\beta c \int_0^\infty p(\tau)\tau e^{-c\tau\lambda}d\tau$$

is decreasing in λ.

We have the following lemma about the critical wave speed.

Lemma 1. *If $\theta\beta > \gamma$, then there exists $c^* \geq 0$ such that for any $c > c^*$, the equation $f(\lambda, c) = 0$ has a unique positive solution, denoted by $\lambda_1 = \lambda_1(c)$, such that $\partial_\lambda f(\lambda_1, c) > 0$, and for any $0 < c < c^*$, the equation $f(\lambda, c) = 0$ has no positive solution. In the later case, $f(\lambda, c) \to -\infty$ as $\lambda \to \infty$. Especially, $c^* = 0$ for the degenerate case $d_2 = 0$.*

Proof. We consider the non-degenerate case $d_2 > 0$ and degenerate case $d_2 = 0$, respectively. If $d_2 > 0$, then $\partial_\lambda f(0, c) > 0$ and $\partial_\lambda f(\lambda, c) < 0$ for sufficiently large $\lambda > 0$. There exists a unique $\bar{\lambda}(c) > 0$ such that $\partial_\lambda f(\bar{\lambda}(c), c) = 0$. Actually $\bar{\lambda}(c)$ is the global maximum point of $f(\lambda, c)$ for $\lambda \in [0, \infty)$. Moreover,

$$2d_2\bar{\lambda} = c + \theta\beta c \int_0^\infty p(\tau)\tau e^{-c\tau\bar{\lambda}} d\tau \in (c, c + \theta\beta c m_1),$$

where $m_1 = \int_0^\infty p(\tau)\tau d\tau$ is the first moment of the probability density function p. Biologically, the first moment m_1 is interpreted as the average delay of disease transmission. Since $f(0, c) = -\theta\beta + \gamma < 0$, the equation $f(\lambda, c) = 0$ has at least one positive solution if and only if $g(c) := f(\bar{\lambda}(c), c) \geq 0$. By chain rule, we have

$$g'(c) = \partial_\lambda f(\bar{\lambda}(c), c)\bar{\lambda}'(c) + \partial_c f(\bar{\lambda}(c), c) = \bar{\lambda} + \theta\beta\bar{\lambda} \int_0^\infty p(\tau)\tau e^{-c\tau\bar{\lambda}} d\tau > 0,$$

which implies that $g(c)$ is increasing in c. As $c \to 0^+$, we obtain $\bar{\lambda}_c \to 0^+$ and $g(c) \to f(0, 0) = -\theta\beta + \gamma < 0$. As $c \to \infty$, we have $f(1, c) \to \infty$. Thus, $g(c) > 0$ for sufficiently large c. The equation $g(c) = 0$ has a unique positive solution, denoted by c^*. For any $c > c^*$, we have $g(c) = f(\bar{\lambda}(c), c) > 0$, $f(0, c) < 0$, and $\partial_\lambda f(\lambda, c) > 0$ for $\lambda \in (0, \bar{\lambda}(c))$. There exists a unique $\lambda_1 = \lambda_1(c) \in (0, \bar{\lambda}(c))$ such that of $f(\lambda_1(c), c) = 0$ and $\partial_\lambda f(\lambda_1(c), c) > 0$. For any $c < c^*$, we have $g(c) = f(\bar{\lambda}(c), c) < 0$ and the equation $f(\lambda, c) = 0$ has no positive real solution. It is obvious that $f(\lambda, c) \to -\infty$ as $\lambda \to \infty$.

If $d_2 = 0$ and $c > 0$, then $\partial_\lambda f(\lambda, c) > 0$ for all $\lambda \geq 0$. Moreover, $f(0, c) < 0$ and $f(\lambda, c) > c\lambda - \theta\beta + \gamma > 0$ for sufficiently large $\lambda > 0$. The existence of λ_1 follows immediately. □

Remark 2. *From the proof of Lemma 1, we observe that in the non-degenerate case $d_2 > 0$, the equation $f(\lambda, c) = 0$ with $c > c^*$ has another positive solution $\lambda_2(c) > \bar{\lambda}(c)$. But, this solution is irrelevant because $\partial_\lambda f(\lambda_2(c), c) < 0$. We only need to use $\lambda_1(c)$ to construct the upper and lower solutions. Especially, we observe from $\partial_\lambda f(\lambda_1(c), c) > 0$ that $f(\lambda_1(c) + \varepsilon, c) > 0$ for any sufficiently small $\varepsilon > 0$.*

To investigate the dependence of critical wave speed on the time delay, we assume that $p(\tau) = \delta_r(\tau)$ is a Dirac delta function such that $(p * g)(t) = g(t - r)$. In this special case, we emphasize the dependence of f on r and write $f(\lambda, c, r) = -d_2\lambda^2 + c\lambda - \theta\beta e^{-\lambda cr} + \gamma$. From the proof of Lemma 1, we note that $c = c^*$ and $\lambda^* := \bar{\lambda}(c^*)$ are a pair of positive solutions to the equations $f(\lambda^*, c^*, r) = 0$ and $\partial_\lambda f(\lambda^*, c^*, r) = 0$. We treat r as an independent variable, while λ^* and c^* are dependent variables. By chain rule, we have

$$\partial_\lambda f(\lambda^*, c^*, r)\frac{d\lambda^*}{dr} + \partial_c f(\lambda^*, c^*, r)\frac{dc^*}{dr} + \partial_r f(\lambda^*, c^*, r) = 0.$$

On the other hand, since $\partial_\lambda f(\lambda^*, c^*, r) = 0$, we then have

$$\frac{dc^*}{dr} = \frac{-\partial_r f(\lambda^*, c^*, r)}{\partial_c f(\lambda^*, c^*, r)} < 0$$

because f is an increasing function in both c and r. This implies that the critical value c^* is a decreasing function of time delay r; namely, time delay in the transmission mechanism will reduce the spatial propagation of infectious diseases. Since f is decreasing in d_2, a similar argument shows that the critical wave speed c^* is increasing in d_2; that is, a larger spatial diffusion rate will lead to a faster traveling speed.

3. Differential operators and their inverses

Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be two large constants to be determined later. We introduce the shifted Laplacian differential operators:

$$\Delta_i h := -d_i h'' + ch' + \alpha_i h$$

for any function h which is second-order differentiable on the whole real line except at finite many points. If $d_i > 0$, then the inverse of Δ_i have the following integral representation:

$$(\Delta_i^{-1} h)(\xi) := \frac{1}{d_i(\mu_i^+ - \mu_i^-)}\Big[\int_{-\infty}^{\xi} e^{\mu_i^-(\xi-y)} h(y)dy + \int_{\xi}^{\infty} e^{\mu_i^+(\xi-y)} h(y)dy\Big],$$

where $\mu_i^\pm := (c \pm \sqrt{c^2 + 4d_i\alpha_i})/(2d_i)$ are the two characteristic roots for the differential operator Δ_i. The integrals on the right-hand side of the above formula are well-defined if the function $h(\xi)$ belongs to the Banach space $B_\mu(\mathbb{R})$ which consists of all continuous functions $h(\xi)$ whose weighted norm

$$|h|_\mu := \sup_{\xi\in\mathbb{R}} e^{-\mu|\xi|}|h(\xi)|$$

is finite. Here, μ is a positive number such that $\mu_i^- < -\mu < \mu < \mu_i^+$. Now, we consider the degenerate case by taking limits. It is readily seen that $\mu_i^- \to -\alpha_i/c$, $\mu_i^+ \to \infty$, and $d_i(\mu_i^+ - \mu_i^-) \to c$ as $d_i \to 0^+$. The integral operator Δ_i^{-1} becomes:

$$(\Delta_i^{-1} h)(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\alpha_i}{c}(\xi-y)} h(y)dy.$$

For all $h \in B_\mu(\mathbb{R})$, it follows from fundamental theorem of calculus that $\Delta_i(\Delta_i^{-1}h) = h$. However, if $h \in B_\mu(\mathbb{R})$ such that h' and h'' may have finite number of points with jump discontinuity, it is now always true that $\Delta_i^{-1}(\Delta_i h) = h$. For convenience, we use $\check{B}_\mu(\mathbb{R})$ to denote the collection of all piecewise continuous functions with possible jump discontinuity at finite many points and finite norm $|\cdot|_\mu$. Obviously, the integral operator Δ_i^{-1} is well defined on $\check{B}_\mu(\mathbb{R})$. It is noted that $B_\mu(\mathbb{R}) = \check{B}_\mu(\mathbb{R}) \cap C(\mathbb{R})$, but the normed space $\check{B}_\mu(\mathbb{R})$ is not complete. We introduce the jump function associated with $h \in \check{B}_\mu(\mathbb{R})$ as

$$[h](\xi) := h(\xi^+) - h(\xi^-) = \lim_{\varepsilon\to 0+} h(\xi+\varepsilon) - \lim_{\varepsilon\to 0-} h(\xi+\varepsilon).$$

Note that $[h]$ vanishes at all continuous points of h, and $[h] = 0$ if $h \in B_\mu(\mathbb{R})$. We have the following results.

Lemma 3. *If $d_i > 0$, then $\Delta_i^{-1}(\Delta_i h) \le h$ for $h \in B_\mu(\mathbb{R})$ such that $h', h'' \in \check{B}_\mu(\mathbb{R})$ and $[h'] \le 0$. If $d_i = 0$, then $\Delta_i^{-1}(\Delta_i h) \le h$ for $h \in \check{B}_\mu(\mathbb{R})$ such that $h' \in \check{B}_\mu(\mathbb{R})$ and $[h] \ge 0$.*

Proof. We first consider the non-degenerate case $d_i > 0$. For simplicity, we assume that h' and h'' have only one jump point at ξ_0. The argument can be easily extended to the case of multiple jump points. By shifting, we may further set $\xi_0 = 0$. For $\xi \le 0$, it follows from the definition that

$$d_i(\mu_i^+ - \mu_i^-)[\Delta_i^{-1}(h'')](\xi) = \int_{-\infty}^{\xi} e^{\mu_i^-(\xi-y)} h''(y)dy + \int_{\xi}^{0} e^{\mu_i^+(\xi-y)} h''(y)dy + \int_{0}^{\infty} e^{\mu_i^+(\xi-y)} h''(y)dy.$$

Using integration by parts, we have

$$d_i(\mu_i^+ - \mu_i^-)[\Delta_i^{-1}(h'')](\xi) = (\mu_i^- - \mu_i^+)h(\xi) - [h'](0)e^{\mu_i^+\xi} + \int_{-\infty}^{\xi} (\mu_i^-)^2 e^{\mu_i^-(\xi-y)} h(y)dy + \int_{\xi}^{\infty} (\mu_i^+)^2 e^{\mu_i^+(\xi-y)} h(y)dy.$$

Similarly, we obtain

$$d_i(\mu_i^+ - \mu_i^-)[\Delta_i^{-1}(h')](\xi) = \int_{-\infty}^{\xi} \mu_i^- e^{\mu_i^-(\xi-y)} h(y)dy + \int_{\xi}^{\infty} \mu_i^+ e^{\mu_i^+(\xi-y)} h(y)dy.$$

Since $\mu_i^\pm$ are characteristic roots for the differential operator Δ_i, we have

$$[\Delta_i^{-1}(\Delta_i h)](\xi) = h(\xi) + \frac{[h'](0)e^{\mu_i^+\xi}}{\mu_i^+ - \mu_i^-} \le h(\xi).$$

In a similar manner, we can show that for $\xi \ge 0$,

$$[\Delta_i^{-1}(\Delta_i h)](\xi) = h(\xi) + \frac{[h'](0)e^{\mu_i^-\xi}}{\mu_i^+ - \mu_i^-} \le h(\xi).$$

Next, we consider the degenerate case $d_i = 0$. For $\xi \le 0$, by definition and integration by parts, we have

$$c[\Delta_i^{-1}(h')](\xi) = \int_{-\infty}^{\xi} e^{-\frac{\alpha_i}{c}(\xi-y)} h'(y)dy = h(\xi) - \frac{\alpha_i}{c}\int_{-\infty}^{\xi} e^{-\frac{\alpha_i}{c}(\xi-y)} h(y)dy,$$

which, together with $\Delta_i h = ch' + \alpha_i h$, implies that

$$[\Delta_i^{-1}(\Delta_i h)](\xi) = h(\xi).$$

For $\xi \ge 0$, we have

$$c[\Delta_i^{-1}(h')](\xi) = \int_{-\infty}^{0} e^{-\frac{\alpha_i}{c}(\xi-y)} h'(y)dy + \int_{0}^{\xi} e^{-\frac{\alpha_i}{c}(\xi-y)} h'(y)dy = h(\xi) - e^{-\alpha_i\xi/c}[h](0) - \frac{\alpha_i}{c}\int_{-\infty}^{\xi} e^{-\frac{\alpha_i}{c}(\xi-y)} h(y)dy,$$

which implies that

$$[\Delta_i^{-1}(\Delta_i h)](\xi) = h(\xi) - e^{-\alpha_i\xi/c}[h](0) \le h(\xi).$$

This completes the proof. □

Remark 4. *By replacing h with $-h$, we also obtain $\Delta_i^{-1}(\Delta_i h) \geq h$ if $[h'] \geq 0$ in the non-degenerate case or $[h] \leq 0$ in the degenerate case. Especially, if h' is continuous in the non-degenerate case or h is continuous in the degenerate case, we have $\Delta_i^{-1}(\Delta_i h) = h$.*

4. Existence results

Throughout this section, we assume that $\theta\beta > \gamma$ and $c > c^*$. Making use of the integral representation for the inverse of shifted Laplacian operator Δ_i, we define the integral map: $F = (F_1, F_2)^T$, where

$$F_1(u, v) := \Delta_1^{-1}(\alpha_1 u - \varphi(u, v)), \tag{4.1}$$

$$F_2(u, v) := \Delta_2^{-1}(\alpha_2 v + \theta p_c * \varphi(u, v) - \gamma v), \tag{4.2}$$

for any $u, v \in B_\mu(\mathbb{R})$. Recall that $p_c(\tau) = p(\tau/c)/c$ and

$$(p_c * g)(\xi) = \int_0^\infty p(\tau) g(\xi - c\tau) d\tau.$$

It is readily seen that a traveling wave solution of (1.6)-(1.7) is the same as a fixed point of the integral map F. We then use the following upper and lower solutions to construct an invariant convex set. Let $f(\lambda) = f(\lambda, c)$ be given as in (2.1). By Lemma 1, there exists a positive $\lambda_1 = \lambda_1(c)$ such that $f(\lambda_1) = 0$ and $f(\lambda_1 + \varepsilon) > 0$ for any sufficiently small $\varepsilon > 0$. We define

$$\begin{aligned} S_+(\xi) &:= S_0, \\ S_-(\xi) &:= \max\{S_0[1 - e^{\varepsilon(\xi - \xi_1)}], 0\}, \\ I_+(\xi) &:= \min\{e^{\lambda_1 \xi}, S_0(\theta\beta/\gamma - 1)\}, \\ I_-(\xi) &:= \max\{e^{\lambda_1 \xi}[1 - e^{\varepsilon(\xi - \xi_2)}], 0\}, \end{aligned}$$

where $\varepsilon > 0, \xi_1 < 0, \xi_2 < 0$ are constants to be determined later. It is readily seen that $S_- \in B_\mu(\mathbb{R})$ and $S'_-, S''_- \in \check{B}_\mu(\mathbb{R})$ with $[S'_-] \geq 0$. By Lemma 3, $\Delta_1^{-1}(\Delta_1 S_-) \geq S_-$. Similarly, we note that $\Delta_1^{-1}(\Delta_1 S_+) = S_+$, $\Delta_2^{-1}(\Delta_2 I_-) \geq I_-$ and $\Delta_2^{-1}(\Delta_2 I_+) \leq I_+$.

The convex set Γ is chosen as the collection of all function pairs $(u, v) \in B_\mu(\mathbb{R}) \times B_\mu(\mathbb{R})$ such that $S_- \leq u \leq S_+$ and $I_- \leq v \leq I_+$. To show that Γ is invariant under the map F, we first choose $\alpha_1 \geq \beta$, $\alpha_2 \geq \gamma$, and $0 < \varepsilon < \min\{\lambda_1, c/d_1\}$ such that $f(\lambda_1 + \epsilon) > 0$. Then, we let $\xi_1 < 0$ be negatively large such that

$$\beta e^{\lambda_1 \xi_1} \leq \varepsilon(c - d_1 \varepsilon) S_0.$$

Finally, we choose $\xi_2 < \xi_1$ such that

$$f(\lambda_1 + \varepsilon) S_0 [1 - e^{\varepsilon(\xi_2 - \xi_1)}] \geq \theta\beta e^{\lambda_1 \xi_2}.$$

Lemma 5. *Assume $\theta\beta > \gamma$ and $c > c^*$. Let $\alpha_1, \alpha_2, \varepsilon, \xi_1, \xi_2$ be chosen as above. For any $(u, v) \in \Gamma$, we have $F(u, v) \in \Gamma$.*

Proof. We need to prove four inequalities $S_- \leq F_1(u, v) \leq S_+$ and $I_- \leq F_2(u, v) \leq I_+$, respectively. First, we note that

$$\alpha_1 u - \varphi(u, v) \leq \alpha_1 u \leq \alpha_1 S_+ = \Delta_1 S_+,$$

which, together with (4.1), implies that

$$F_1(u,v) \leq \Delta_1^{-1}(\Delta_1 S_+) = S_+.$$

Next, we choose $\alpha_1 \geq \beta$ such that $\alpha_1 u - \varphi(u,v)$ is increasing in u and decreasing in v. Thus,

$$\alpha_1 u - \varphi(u,v) \geq \alpha_1 S_- - \varphi(S_-, I_+).$$

On the other hand, we have

$$(\Delta_1 S_-)(\xi) = \alpha_1 S_-(\xi) - \varepsilon(c - d_1\varepsilon) S_0 e^{\varepsilon(\xi-\xi_1)}$$

for $\xi \leq \xi_1$, and $(\Delta_1 S_-)(\xi) = 0$ for $\xi \geq \xi_1$. We need to show $\alpha_1 u - \varphi(u,v) \geq \Delta_1 S_-$. Note that $\varphi(S_-, I_+) \leq \beta S_- \leq \alpha_1 S_-$ and $\varphi(S_-, I_+) \leq \beta I_+ \leq \beta e^{\lambda_1 \xi}$. It suffices to prove

$$\beta e^{\lambda_1 \xi} \leq \varepsilon(c - d_1\varepsilon) S_0 e^{\varepsilon(\xi-\xi_1)}$$

for all $\xi \leq \xi_1$. We choose $0 < \varepsilon < \min\{\lambda_1, c/d_1\}$. The above inequality is a consequence of the inequality

$$\beta e^{\lambda_1 \xi_1} \leq \varepsilon(c - d_1\varepsilon) S_0,$$

which can be achieved by letting $\xi_1 = \xi_1(\varepsilon) < 0$ be negatively large. Therefore, in view of (4.1), we obtain

$$F_1(u,v) \geq \Delta_1^{-1}(\Delta_1 S_-) \geq S_-.$$

Now, we observe from monotonicity of $\varphi(u,v)$ in both u and v that

$$\varphi(u(\xi), v(\xi)) \leq \varphi(S_+(\xi), I_+(\xi)) \leq \min\{\beta e^{\lambda_1 \xi}, S_0(\beta - \gamma/\theta)\}.$$

Consequently, if $\alpha_2 \geq \gamma$, then

$$\begin{aligned}
&\alpha_2 v(\xi) + \theta[p_c * \varphi(u,v)](\xi) - \gamma v(\xi)\\
\leq&(\alpha_2 - \gamma) I_+(\xi) + \min\{\theta\beta e^{\lambda_1 \xi} \int_0^\infty p(\tau) e^{-\lambda_1 c\tau} d\tau, S_0(\theta\beta - \gamma)\}.
\end{aligned}$$

Recall that λ_1 is the root of the characteristic equation $f(\lambda_1) = 0$. We have

$$(\Delta_2 I_+)(\xi) = (\alpha_2 - \gamma) I_+(\xi) + \theta\beta e^{\lambda_1 \xi} \int_0^\infty p(\tau) e^{-\lambda_1 c\tau} d\tau$$

for $\xi \leq \ln[S_0(\theta\beta/\gamma - 1)]/\lambda_1$, and

$$(\Delta_2 I_+)(\xi) = (\alpha_2 - \gamma) I_+(\xi) + S_0(\theta\beta - \gamma)$$

for $\xi \geq \ln[S_0(\theta\beta/\gamma - 1)]/\lambda_1$. In either case, we have

$$(\Delta_2 I_+)(\xi) \geq \alpha_2 v(\xi) + \theta[p_c * \varphi(u,v)](\xi) - \gamma v(\xi),$$

which, together with (4.2), yields

$$F_2(u,v) \leq \Delta_2^{-1}(\Delta_2(I_+)) \leq I_+.$$

Finally, we show that

$$\alpha_2 v(\xi) + \theta[p_c * \varphi(u, v)](\xi) - \gamma v(\xi) \geq (\Delta_2 I_-)(\xi).$$

Since $\alpha_2 \geq \gamma$ and $I_-(\xi) = 0$ for $\xi \geq \xi_2$, we only need to verify the above inequality for $\xi \leq \xi_2$. Note that

$$\alpha_2 v(\xi) + \theta[p_c * \varphi(u, v)](\xi) - \gamma v(\xi) \geq (\alpha_2 - \gamma) I_-(\xi) + \theta[p_c * \varphi(S_-, I_-)](\xi),$$

and

$$(\Delta_2 I_-)(\xi) = (\alpha_2 - \gamma) I_-(\xi) + \theta\beta[p_c * I_-](\xi) - f(\lambda_1 + \varepsilon) e^{\lambda_1 \xi + \varepsilon(\xi - \xi_2)}.$$

It suffices to prove

$$f(\lambda_1 + \varepsilon) e^{\lambda_1 \xi + \varepsilon(\xi - \xi_2)} \geq \theta[p_c * (\beta I_- - \varphi(S_-, I_-))](\xi)$$

for all $\xi \leq \xi_2$. Since an increasing exponential function is always greater than its convolution with p_c, the above inequality is satisfied if we can show that

$$f(\lambda_1 + \varepsilon) e^{\lambda_1 \xi + \varepsilon(\xi - \xi_2)} \geq \theta[\beta I_- - \varphi(S_-, I_-)](\xi) = \frac{\theta\beta [I_-(\xi)]^2}{S_-(\xi) + I_-(\xi)}$$

for all $\xi \leq \xi_2$. Assuming $\xi_2 < \xi_1$, we only need to prove

$$f(\lambda_1 + \varepsilon) e^{\lambda_1 \xi + \varepsilon(\xi - \xi_2)} S_0[1 - e^{\varepsilon(\xi - \xi_1)}] \geq \theta\beta e^{2\lambda_1 \xi}$$

for all $\xi \leq \xi_2$. This can be done by choosing $\xi_2 = \xi_2(\xi_1, \varepsilon) < 0$ negatively large such that

$$f(\lambda_1 + \varepsilon) S_0 (1 - e^{\varepsilon(\xi_2 - \xi_1)}] \geq \theta\beta e^{\lambda_1 \xi_2}.$$

Hence, on account of (4.2), we obtain

$$F_2(u, v) \geq \Delta_2^{-1}(\Delta_2(I_-)) \geq I_-.$$

This completes the proof. □

We shall use Schauder fixed point theorem to establish existence of traveling wave solution. Recall that Γ is invariant under the integral map. We have to prove that F is continuous and compact on Γ with respect to the induced norm in $B_\mu(\mathbb{R}) \times B_\mu(\mathbb{R})$ for some sufficiently small $\mu > 0$. Recall that the probability density function $p(\tau)$ decays exponentially as $\tau \to \infty$. Especially, there exists a $\sigma > 0$ such that $p(\tau)e^{\sigma\tau}$ is bounded as $\tau \to \infty$. We choose $\mu \in (0, \sigma/c)$ such that $p(\tau)e^{\mu c \tau}$ is integrable on the positive real line. Recall that we also require $\mu_i^- < -\mu < \mu < \mu_i^+$ with $i = 1, 2$.

Lemma 6. *For any $\mu > 0$ such that $\mu < \sigma/c$ and $\mu_i^- < -\mu < \mu < \mu_i^+$ with $i = 1, 2$, the integral map F is continuous and compact on Γ with respect to the induced norm in $B_\mu(\mathbb{R}) \times B_\mu(\mathbb{R})$.*

Proof. For any (u_1, v_1) and (u_2, v_2) in Γ, we then have

$$|[\varphi(u_1, v_1) - \varphi(u_2, v_2)](y - c\tau)| \leq \beta e^{\mu|y| + \mu c \tau}(|u_1 - u_2|_\mu + |v_1 - v_2|_\mu)$$

for any $y \in \mathbb{R}$ and $\tau \geq 0$. Consequently,

$$|[p_c * \varphi(u_1, v_1) - p_c * \varphi(u_2, v_2)](y)| \leq \beta C_0 e^{\mu|y|}(|u_1 - u_2|_\mu + |v_1 - v_2|_\mu)$$

for all $y \in \mathbb{R}$, where

$$C_0 := \int_0^\infty p(\tau)e^{\mu c\tau}d\tau < \infty.$$

Let $g(y) := e^{\mu|y|}$. It is readily seen that

$$|(\Delta_2^{-1}g)(\xi)| \le C_2 e^{\mu|\xi|},$$

where

$$C_2 := \frac{1}{d_2(\mu_2^+ - \mu_2^-)}(\frac{1}{\mu_2^+ - \mu} - \frac{1}{\mu_2^- + \mu}).$$

For the degenerate case $d_2 = 0$, the above formula is still valid by taking limit $d_2 \to 0^+$. Note that $d_2(\mu_2^+ - \mu_2^-) \to c$, $\mu_2^- \to -\alpha_2/c$ and $\mu_2^+ \to \infty$ as $d_2 \to 0^+$. It follows that $C_2 = 1/(\alpha_2 - \mu c)$ if $d_2 = 0$. We then have

$$|[F_2(u_1, v_1) - F_2(u_2, v_2)](\xi)| \le (\theta\beta C_0 + \alpha_2 - \gamma)C_2 e^{\mu|\xi|}(|u_1 - u_2|_\mu + |v_1 - v_2|_\mu)$$

for all $\xi \in \mathbb{R}$. Similarly, we can prove that

$$|[F_1(u_1, v_1) - F_1(u_2, v_2)](\xi)| \le (\alpha_1 + \beta)C_1 e^{\mu|\xi|}(|u_1 - u_2|_\mu + |v_1 - v_2|_\mu)$$

for all $\xi \in \mathbb{R}$, where

$$C_1 := \frac{1}{d_1(\mu_1^+ - \mu_1^-)}(\frac{1}{\mu_1^+ - \mu} - \frac{1}{\mu_1^- + \mu})$$

is the upper bound for the operator norm of Δ_1^{-1} in $B_\mu(\mathbb{R})$ with respect to the norm $|\cdot|_\mu$. For the degenerate case $d_1 = 0$, we write $C_1 = 1/(\alpha_1 - \mu c)$. Thus, we have prove continuity of the map $F = (F_1, F_2)$ on Γ. To prove compactness, we shall make use of Arzela-Ascolli theorem. Note that Γ is bounded, it suffices to show that the image $F(\Gamma)$ is pre-compact. First, we note that $F(\Gamma) \subset \Gamma$ is bounded by the upper and lower solutions, which are uniformly bounded functions on the whole real line. For any $\varepsilon > 0$, there exists a $M > 0$ such that

$$|[F_1(u_1, v_1) - F_1(u_2, v_2)](\xi)|e^{-\mu|\xi|} + |[F_2(u_1, v_1) - F_2(u_2, v_2)](\xi)|e^{-\mu|\xi|} < \varepsilon, \tag{4.3}$$

for any $(u_1, v_1) \in \Gamma$, $(u_2, v_2) \in \Gamma$ and $|\xi| \ge M$. On the other hand, the functions in $F(\Gamma)$ continuous and uniformly bounded on the compact interval $[-M, M]$. Moreover, they are equi-continuous because

$$|[F_1(u, v)]'(\xi)| \le \frac{2(\alpha_1 + \beta)S_0}{d_1(\mu_1^+ - \mu_1^-)},$$

$$|[F_2(u, v)]'(\xi)| \le \frac{2(\alpha_2 + \theta\beta - \gamma)S_0(\theta\beta/\gamma - 1)}{d_2(\mu_2^+ - \mu_2^-)},$$

for all $(u, v) \in \Gamma$ and $\xi \in \mathbb{R}$. By Arzela-Ascolli theorem, there exists a finite ε-net of $F(\Gamma)$ with respect to the supremum norm in $C[-M, M] \times C[-M, M]$. In view of (4.3), this net is also a finite ε-net of $F(\Gamma)$ with respect to the weighted norm in $B_\mu(\mathbb{R}) \times B_\mu(\mathbb{R})$. Thus, $F(\Gamma)$ is precompact, which implies that F is a compact map on Γ. □

By Schauder fixed point theorem, F possesses a fixed point, denoted by (S, I), in Γ. Since $S_{\pm}(-\infty) = S_0$ and $I_{\pm}(-\infty) = 0$, by squeeze theorem, we have $S(-\infty) = S_0$ and $I(-\infty) = 0$. Actually, $I(\xi)e^{-\lambda_1\xi} \to 1$ as $\xi \to -\infty$, which implies that I can not be a constant function. From the integral representation of the operator F and L'Hôpital's rule, we note that S and I are differentiable, and $S'(-\infty) = I'(-\infty) = 0$. Thus, (S, I) satisfies the boundary conditions (1.8). To sum up, we have the following existence result.

Theorem 7. *Assume $\theta\beta > \gamma$ and $c > c^*$. For any $S_0 > 0$, there exists a positive and uniformly bounded traveling wave solution of* (1.6)-(1.7) *with the boundary conditions* (1.8).

5. Properties of traveling wave solutions

In this section, we shall derive some general properties for a positive and uniformly bounded traveling wave solution of (1.6)-(1.7) with the boundary conditions (1.8). Especially, we will show that $\theta\beta > \gamma$ and $c \geq c^*$ are necessary conditions for the existence of traveling wave solutions.

We denote the traveling wave solution by (S, I). An integration of (1.6) gives

$$d_1 S'(\xi) = c[S(\xi) - S_0] + \int_{-\infty}^{\xi} \varphi(S(y), I(y))dy,$$

which, together with boundedness of S, implies that $\varphi(S, I)$ is integrable, S is differentiable, and S' is uniformly bounded on the real line. If $d_1 > 0$, we integrate (1.6) to obtain

$$S'(\xi) = -\int_{\xi}^{\infty} \frac{e^{c(\xi-y)/d_1}}{d_1} \varphi(S(y), I(y))dy < 0.$$

If $d_1 = 0$, then (1.6) gives $S'(\xi) = -\varphi(S(\xi), I(\xi))/c < 0$. In either case, S is strictly decreasing on $\mathbb{R}$. Moreover, it follows from above formulas that $S'(\infty) = 0$ and

$$c[S_0 - S(\infty)] = \int_{-\infty}^{\infty} \varphi(S(y), I(y))dy.$$

Now, we solve (1.7) by variation of parameters. For non-degenerate case $d_2 > 0$, we let $\mu^{\pm} = (c \pm \sqrt{c^2 + 4d_2\gamma})/(2d_2)$ be the two roots of the characteristic equation $-d_2\mu^2 + c\mu + \gamma = 0$. On account of uniform boundedness of I and $\varphi(S, I)$, we have the following integral equation

$$I(\xi) = \frac{\theta}{d_2(\mu^+ - \mu^-)} \left[\int_{-\infty}^{\xi} e^{\mu^-(\xi-y)}[p_c * \varphi(S, I)](y)dy + \int_{\xi}^{\infty} e^{\mu^+(\xi-y)}[p_c * \varphi(S, I)](y)dy \right].$$

By taking the limit $d_2 \to 0^+$, we obtain an integral equation for I in the degenerate case:

$$I(\xi) = \frac{\theta}{c} \int_{-\infty}^{\xi} e^{-\gamma(\xi-y)/c}[p_c * \varphi(S, I)](y)dy.$$

Since $\varphi(S, I)$ is integrable on the real line, by Tonelli-Fubini theorem, I is also integrable on the real line, and

$$\int_{-\infty}^{\infty} I(\xi)d\xi = \frac{\theta}{\gamma} \int_{-\infty}^{\infty} \varphi(S(\xi), I(\xi))d\xi.$$

Note that $\varphi(S(\xi), I(\xi)) < \beta I(\xi)$ for all $\xi \in \mathbb{R}$. It follows from the above equation that $\theta\beta > \gamma$. Since $\varphi(S, I)$ is uniformly bounded, we obtain from the integral representation of I that I' is also uniformly bounded, this together with integrability of I implies that $I(\infty) = 0$.

Recall that $S'(\xi) < 0$ and $S(\xi) > 0$ for all $\xi \in \mathbb{R}$. So, the limit $S(\infty)$ exists. We claim that $S(\infty) = 0$. If not, by monotonicity of φ in the first component, it follows from (1.7) that

$$-d_2 I''(\xi) + cI'(\xi) \geq \theta \int_0^{\infty} p_c(\tau)\varphi(S(\infty), I(\xi - \tau))d\tau - \gamma I(\xi), \tag{5.1}$$

where $p_c(\tau) = p(\tau/c)/c$ is the scaled probability density function. The above inequality contradicts to the fact $I(\infty) = 0$ by the following result.

Lemma 8. *If $\theta\beta > \gamma$ and $S(\infty) > 0$, then there does not exist a positive and uniformly bounded function $I(\xi)$ satisfying the inequality* (5.1) *and boundary condition $I(\infty) = 0$.*

Proof. We prove by contradiction. Assume such function $I(\xi)$ exists. We choose a sufficiently large $T > 0$ and a sufficiently small $\delta > 0$ such that

$$\frac{\theta\varphi(S(\infty), \delta)}{\delta} \int_0^T p_c(\tau)d\tau > \gamma(1 + \delta). \tag{5.2}$$

This could be done because, as $T \to \infty$ and $\delta \to 0^+$, the left-hand side tends to $\theta\beta$ and the right-hand side tends to γ. Since $I(\infty) = 0$, there exists ξ_0 such that $I(\xi) < \delta$ for all $\xi \geq \xi_0$. Define a function

$$I_T(\xi) := \min_{0 \leq \tau \leq T} I(\xi - \tau).$$

We claim $I_T(\xi) = I(\xi)$ for sufficiently large ξ. If not, there exist an infinite sequence ξ_k with $k = 1, 2, \cdots$, such that $\xi_k > \xi_{k-1} + T$, $I(\xi_k) < I(\xi_{k-1})$ and $I_T(\xi_k) < I(\xi_k)$ for all $k \geq 1$. For each $k \geq 1$, let η_k be the minimum point of $I(\xi)$ in the interval $[\xi_k, \xi_{k+1}]$. Since $I_T(\xi_{k+1}) < I(\xi_{k+1}) < I(\xi_k)$, η_k lies in the open interval (ξ_k, ξ_{k+1}). Moreover, $I'(\eta_k) = 0$ and $I''(\eta_k) \geq 0$. It follows from (5.1) and (5.2) that

$$0 \geq \theta \int_0^T p_c(\tau)\varphi(S(\infty), I_T(\eta_k))d\tau - \gamma I(\eta_k) > \gamma(1 + \delta)I_T(\eta_k) - \gamma I(\eta_k) > \gamma[I_T(\eta_k) - I(\eta_k)].$$

Thus, $I(\eta_k) > I_T(\eta_k)$ for all $k \geq 1$. For each $k \geq 2$, since $I(\eta_k) \leq I(\xi)$ for all $\xi \in (\xi_k, \xi_{k+1})$, $I_T(\eta_k) = \min_{0 \leq \tau \leq T} I(\eta_k - \tau)$ should be achieved at some point in (ξ_{k-1}, ξ_k). Especially, $I(\eta_k) > I_T(\eta_k) \geq I(\eta_{k-1})$ for all $k \geq 2$. Thus, we have found a sequence $\eta_k \to \infty$, such that $I(\eta_k)$ is increasing, which contradicts to the fact $I(\infty) = 0$. This prove our claim that $I_T(\xi) = I(\xi)$ for sufficiently large ξ. For simplicity, we shift ξ_0 such that $I_T(\xi) = I(\xi)$ for all $\xi \geq \xi_0$, which is the same as $I'(\xi) \leq 0$ for $\xi \geq \xi_0$. By (5.1) and (5.2), we have

$$-d_2 I''(\xi) + cI'(\xi) \geq \theta \int_0^T p_c(\tau)\varphi(S(\infty), I(\xi))d\tau - \gamma I(\xi) > \gamma\delta I(\xi),$$

for all $\xi \geq \xi_0$. Since $I(\xi) > 0$, we may define $w(\xi) := I'(\xi)/I(\xi)$. It is readily seen that $w(\xi) \leq 0$ and $w'(\xi) = I''(\xi)/I(\xi) - w^2(\xi)$. For the degenerate case $d_2 = 0$, the above inequality reads $w(\xi) > \gamma\delta/c > 0$, which contradicts to the non-positiveness of $w(\xi)$. For the non-degenerate case, the above inequality can be written as

$$-d_2 w'(\xi) > \gamma\delta - cw(\xi) + d_2 w^2(\xi) > d_2 w^2(\xi), \quad \xi \geq \xi_0.$$

The solution of above differential inequality with nonpositive initial value at $\xi = \xi_0$ will blow up at finite value of $\xi > \xi_0$. Actually, an integration of the above inequality gives

$$\frac{1}{w(\xi)} > \frac{1}{w(\xi_0)} + \xi - \xi_0,$$

where we have assumed without loss of generality that $w(\xi_0) < 0$. The right-hand side of the above inequality becomes positive for large ξ, but the left-hand side is always negative, a contradiction. This completes the proof. □

Remark 9. *A similar result was obtained in [26] for a type-II delayed disease model. Their proof is based on comparison principle and asymptotic stability results of quisi-monotone diffusive equations on the real line. It is noted that our proof has the potential to be generalized for more complicated nonlinear elliptic differential equations when the corresponding reaction-diffusion system is difficult to handle.*

Recall that $\theta\beta > \gamma$ is a necessary condition for the existence of positive traveling wave solutions. In what follows, we will show that c can not be smaller than c^*. Note that $c^* = 0$ for the degenerate case $d_2 = 0$. We only need to consider the non-degenerate case $d_2 > 0$. In view of the boundary conditions $S(-\infty) = S_0 > 0$ and $I(-\infty) = 0$, we can find a $\xi_1 \in \mathbb{R}$ such that

$$\varphi(S(\xi), I(\xi)) > (\theta\beta + \gamma)I(\xi)/2$$

for all $\xi \leq \xi_1$. It then follows from (1.7) that

$$-d_2 I''(\xi) + cI'(\xi) > \frac{\theta\beta + \gamma}{2}[(p_c * I)(\xi) - I(\xi)] + \frac{\theta\beta - \gamma}{2} I(\xi).$$

Let $K(\xi) = \int_{-\infty}^{\xi} I(y)dy$. Integrating the above inequality twice gives

$$-d_2 I(\xi) + cK(\xi) > \frac{\theta\beta + \gamma}{2} \int_{-\infty}^{\xi} [(p_c * K)(y) - K(y)]dy + \frac{\theta\beta - \gamma}{2} \int_{-\infty}^{\xi} K(y)dy.$$

Note that

$$\begin{aligned}\int_{-\infty}^{\xi} [(p_c * K)(y) - K(y)]dy &= -\int_{-\infty}^{\xi} \int_0^{\infty} \int_0^{\tau} p_c(\tau) I(y - s)dsd\tau dy \\ &= -\int_0^{\infty} \int_0^{\tau} p_c(\tau)K(\xi - s)dsd\tau > -cm_1 K(\xi),\end{aligned}$$

where $m_1 = \int_0^{\infty} \tau p(\tau)d\tau$ is the first moment (also called the average delay). We then have

$$\frac{c + cm_1(\theta\beta + \gamma)/2}{(\theta\beta - \gamma)/2} K(\xi) > \frac{d_2}{(\theta\beta - \gamma)/2} I(\xi) + \int_{-\infty}^{\xi} K(y)dy > sK(\xi - s)$$

for any $s \geq 0$ and $\xi \leq \xi_1$. Especially, by choosing $s = 4[c + cm_1(\theta\beta + \gamma)/2]/(\theta\beta - \gamma)$, we have $K(\xi - s) < K(\xi)/2$. By iterating this inequality, we obtain from monotonicity and uniform boundedness of K that the function $K(\xi)e^{-\mu_0 \xi}$ with $\mu_0 = \ln 2/s$ is uniformly bounded on the real line. Note from above inequality that for all $\xi \le \xi_1$, $I(\xi)$ is bounded by a constant multiplication of $K(\xi)$. Thus, $I(\xi)e^{-\mu_0\xi}$ is also uniformly bounded on the real line. Similarly, we have uniform boundedness of $I'(\xi)e^{-\mu_0\xi}$ and $I''(\xi)e^{-\mu_0\xi}$ on the real line. Furthermore, $\varphi(S(\xi), I(\xi))e^{-\mu_0\xi}$ is uniformly bounded on the real line. For each $\mu \in (0, \mu_0)$, we introduce two-sided Laplace transform on (1.7) to obtain

$$-f(\mu, c)\int_{-\infty}^{\infty} e^{-\mu\xi} I(\xi)d\xi = \int_{-\infty}^{\infty} e^{-\mu\xi}\frac{\tilde{\beta}I^2(\xi)}{S(\xi)+I(\xi)}d\xi,$$

where f is the characteristic function defined in (2.1) and $\tilde{\beta} = \theta\beta\int_0^{\infty} e^{-\mu c\tau}p(\tau)d\tau$. If $c < c^*$, then $f(\mu, c)$ has no zero on the positive real line. By analytic continuation [36, Theorem 5b, p.58], the two integrals on both sides of the above equation are well defined for all $\mu > 0$. However, by rewriting the above equation as

$$0 = \int_{-\infty}^{\infty} e^{-\mu\xi} I(\xi)\left[f(\mu, c) + \frac{\tilde{\beta}I(\xi)}{S(\xi)+I(\xi)}\right]d\xi,$$

we note from $f(\mu, c) \to -\infty$ as $\mu \to \infty$ that the integrand is always negative for large μ, a contradiction. Thus, we require $f(\mu, c)$ to have at least one positive zero such that the analytic continuation fails to extend beyond this zero. This means that $c \ge c^*$ is a necessary conditions for the existence of positive traveling wave solutions. We then state the properties of traveling wave solutions in the following theorem.

Theorem 10. *If* (S, I) *is a positive and uniformly bounded solution pair of* (1.6)-(1.7) *with boundary conditions* (1.8)*, then we have* $\theta\beta > \gamma$*,* $c \ge c^*$*, and* $S(\infty) = I(\infty) = 0$*. The following identities are satisfied:*

$$\frac{\gamma}{\theta}\int_{-\infty}^{\infty} I(\xi)d\xi = \int_{-\infty}^{\infty} \varphi(S(\xi), I(\xi))d\xi = cS_0.$$

Moreover, $S'(\xi) < 0$ *for all* $\xi \in \mathbb{R}$.

6. Conclusion

We classify the delayed epidemic models into two types. The first type impose delays only on the I-equation, while the second type assumes delayed infective terms in both equations for S and I. Though these two types are mathematically equivalent in the case of one discrete delay, they are different when general distributed delay is taken into consideration. We note that the Susceptible-Infected-Recovered model with type-I distributed delay can be reduced to the classical Susceptible-Exposed-Infected-Recovered model with no delay by assuming that the probability density function takes a special form $p(\tau) = \sigma e^{-\sigma\tau}$. It is thus reasonable to study the model systems with type-I delay. However, not much has been done even for the non-diffusive model. On the contrast, there is a rich literature of works on the type-II delayed non-diffusive epidemic models; see for example, [37, 13, 38]. To establish global stability results for the model systems with type-II delays, one should take advantage of the fact that the differential equation for $S + I$ is simple and has no delayed terms. However, this phenomenon disappears in the model system with type-I delay and thus poses a challenge in the global analysis of model dynamics [39].

In this paper, we consider a diffusive epidemic model with type-I distributed delay. It is noted that when the probability density function for the distributed delay takes the special form of exponential function, then a standard application of linear chain trick reduces our model system to a diffusive delay-free model with an exposed compartment which does not diffuse. We also consider the degenerate cases when the diffusion coefficient of either susceptible individuals or infected individuals vanishes. We prove linear determinacy of our model system by establishing existence theory of positive traveling wave solutions. To be more specific, we calculate a critical value from the linearized I-equation and verify that this value is the sharp lower bound for the speeds of traveling wave solutions. Sensitivity analysis indicates that the critical wave speed is increasing in the diffusion coefficient but decreasing in time delay. Biological interpretation of this result is that random movement of infected individuals will enforce disease propagation, while time delay during transmission mechanism will inhibit spatial spread of infectious diseases.

In Lemma 8, we also provide a novel and elementary proof of a conjecture proposed in [40] that a positive traveling wave can only connect a nontrivial equilibrium with a trivial equilibrium. We should mention that this conjecture was first proved in [26] by using comparison principle and global stability result for quasi-monotone reaction-diffusion equations on an unbounded domain. In comparison, our proof is simpler and more natural, and it provides a new idea of understanding nonlocal elliptic differential equations/inequalities.

An open problem for our model system, and for many other diffusive epidemic models, is the existence of a positive traveling wave solution for the critical wave speed $c = c^*$. Due to the lack of monotonicity, the traditional limiting argument for the monotone systems fails. It is exciting to see some recent achievements in [41, 42] for existence proofs of weak traveling wave solutions with critical speed in non-cooperative diffusive systems, and in [43] for an existence result of traveling waves in a nonlocal dispersal epidemic model with critical speed. We conjecture that, for our proposed epidemic model with type-I distributed delays, a positive traveling wave solution with critical speed should exist.

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Conflict of interest

The authors declare there is no conflict of interest.

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