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ASYMPTOTIC ANALYSIS OF DIFFERENCE EQUATIONS WITH QUADRATIC COEFFICIENTS*

XIANG-SHENG WANG†

Abstract. In this paper, we study asymptotic solutions of second-order difference equations with quadratic coefficients. According to the parameter values, we classify the difference equations into three cases and derive Plancherel-Rotach type asymptotic formulas of the solutions respectively. As direct applications of our main results, we also provide asymptotic formulas of associated Meixner-Pollaczek polynomials, associated Meixner polynomials, and associated Laguerre polynomials, respectively.

Key words. Asymptotic analysis, difference equations, associated polynomials.

AMS subject classifications. 39A06, 41A60.

1. Introduction. Asymptotic analysis of orthogonal polynomials has been studied extensively in the literature. If there is an integral representation for the polynomials, one may apply the classical Laplace method or the steepest-descent method [35]; if the polynomials satisfy a second-order differential equation, the WKB method [25] will be useful. A modern Riemann-Hilbert technique introduced by Deift and Zhou [14] and further developed in [13] and [5] has become a powerful tool when the weight function for the orthogonal polynomials is known and has nice analyticity. Wilson [34] proposed a convexity argument to derive a complete asymptotic expansion for certain hypergeometric series including Wilson polynomials. Recently, a discrete analogue of Laplace method was introduced to derive asymptotic formulas of q-orthogonal polynomials [19, 28, 29]. However, the aforementioned techniques are not applicable for many orthogonal polynomials that are defined directly from a difference equation, for example, the polynomials arising from coherent states [2, 11] and the birth-death type polynomials [12]. Also, the orthogonal polynomials in the top hierarchies of Askey scheme always have very complicated integral representations and non-analytic weights with several singularities. For these polynomials, it seems better to conduct asymptotic analysis from their difference equations which are comparably much simpler. Moreover, difference equations have wide applications not only in orthogonal polynomials, but also in continued fractions, mathematical quantum field theory and other disciplines [3, 6, 24]. It is thus important to develop a difference equation technique as an parallel result to the treatment of Olver [25] on differential equations.

Some early works on asymptotic analysis of difference equations can be found in [1, 7, 8, 9]. However, their papers were too complicated to be understood even by other experts in asymptotics [36]. More than half a century later, Geronimo and his collaborators [15, 16, 26] studied difference equations with varying parameters and obtained asymptotic formulas in the outer region which is bounded away from the polynomial zeros. In the oscillatory region where the zeros are distributed, Wong and Li [38] derived two asymptotic solutions to the difference equations which are linearly independent with each other. The coefficients of these two solutions can be determined by a matching principle introduceed in [30]. In a series of work, Wang

*Received October 25, 2015; accepted for publication February 16, 2016.
†Department of Mathematics, University of Louisiana at Lafayette, Lafayette, Louisiana, 70503, USA (xswang@louisiana.edu).

and Wong [31, 32, 33] developed a turning-point theory for difference equations and studied uniform asymptotic solutions near the turning points. This theory was further completed by Cao and Li [10]. The difference equation technique has been used in the study of coherent state polynomials [11] and birth-death type orthogonal polynomials [12]. We refer to [37] for an overview of asymptotic theory on liner difference equations.

In a previous paper [27], we provided Plancherel-Rotach asymptotics of second-order difference equations with linear coefficients. The corresponding results were applied to investigate associated orthogonal polynomials in the lowest hierarchy of Askey scheme (i.e., associated Hermite polynomials and associated Charlier polynomials). It is a natural desire to find asymptotic formulas for associated orthogonal polynomials in higher hierarchies. Hence, we will study a class of monic polynomials satisfying the following second-order difference equation with quadratic coefficients [23]:

$$\begin{aligned} \pi_{n+1}(x) &= (x - dn - d_0)\pi_n(x) - (an^2 + bn + c)\pi_{n-1}(x), \\ \pi_0(x) &= 1, \quad \pi_1(x) = x - d_0. \end{aligned} \tag{1.1}$$

Here, the parameters a, b, c, d and d_0 are all real constants. Three typical examples in the Askey scheme of classical hypergeometric orthogonal polynomials [20] are given below.

1. Meixner-Pollaczek polynomials: $a = 1/(4\sin^2\phi)$, $b = (2\lambda - 1)/(4\sin^2\phi)$, $c = 0$, $d = -\cot\phi$, $d_0 = -\lambda\cot\phi$ with $\lambda > 0$ and $0 < \phi < \pi$;
2. Meixner polynomials: $a = \lambda/(1-\lambda)^2$, $b = (\beta - 1)\lambda/(1-\lambda)^2$, $c = 0$, $d = (1+\lambda)/(1-\lambda)$, $d_0 = \beta\lambda/(1-\lambda)$ with $\beta > 0$ and $0 < \lambda < 1$;
3. Laguerre polynomials: $a = 1$, $b = \alpha$, $c = 0$, $d = 2$, $d_0 = \alpha + 1$ with $\alpha > -1$.

Upon a shift on x, we may assume without loss of generality that $d_0 = 0$. Also, it suffices to consider the case when $d \geq 0$, because when $d < 0$, we could introduce a reflection $p_n(x) := (-1)^n \pi_n(-x)$ and study the polynomials $p_n(x)$ satisfying the difference equation (1.1) with d being replaced with $-d > 0$.

Throughout this paper, we will also assume that $a > 0$. By introducing the canonical scale $x = ny$, we obtain from [21, Section 4.5] that the Mhaskar-Rakhmanov-Saff (MRS) numbers are $d \pm 2\sqrt{a}$. These numbers are also called turning points or transition points [32]. Following [5], we refer to the interval between MRS numbers as a band: $[d - 2\sqrt{a}, d + 2\sqrt{a}]$. According to the location of the origin with respect to the band, we classify the difference equations into three cases:

i) the origin lies in the band, namely, $0 < d < 2\sqrt{a}$;
ii) the origin lies outside the band, namely, $0 < 2\sqrt{a} < d$;
iii) the origin coincides with one of the MRS numbers, namely, $d = 2\sqrt{a}$.

It is readily seen that the Meixner-Pollaczek polynomials, Meixner polynomials and Laguerre polynomials belong to the cases i), ii) and iii) respectively. We will study these three cases in Sections 2-4, respectively. In Section 5, we apply our theorems to find asymptotic formulas for associated polynomials. Finally, a brief discussion will be given in Section 6.

2. Case I: $0 < d < 2\sqrt{a}$**.** As mentioned before, we assume without loss of generality that $d_0 = 0$. The difference equation (1.1) is written as

$$\pi_{n+1}(x) = (x - dn)\pi_n(x) - (an^2 + bn + c)\pi_{n-1}(x)$$

with initial conditions $\pi_0(x) = 1$ and $\pi_1(x) = x$.

THEOREM 2.1. *Assume* $0 < d < 2\sqrt{a}$. *Let* $x = ny$ *with* $y \in \mathcal{C} \setminus [d-2\sqrt{a}, d+2\sqrt{a}]$. *We have as* $n \to \infty$,

$$\begin{aligned}\pi_n(ny) \sim \left(\frac{n}{e}\right)^n &\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2}\right]^n \left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2\sqrt{(y-d)^2-4a}}\right]^{1/2} \\ &\times \left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2y}\right]^{b/(2a)} \\ &\times \exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{4a-d^2}}\left[\arcsin\left(\frac{dy+4a-d^2}{2y\sqrt{a}}\right) - \arcsin\left(\frac{d}{2\sqrt{a}}\right)\right]\right\}.\end{aligned} \tag{2.1}$$

Proof. For $k \geq 1$ and x away from the oscillatory region such that $\pi_{k-1}(x) \neq 0$, we define

$$w_k(x) := \frac{\pi_k(x)}{\pi_{k-1}(x)}.$$

It follows that $w_1(x) = x$ and

$$w_{k+1}(x) = x - dk - \frac{ak^2+bk+c}{w_k(x)}. \tag{2.2}$$

Let $x = ny$ with $y \in \mathcal{C} \setminus [d-2\sqrt{a}, d+2\sqrt{a}]$. By successive approximation, we have as $n \to \infty$,

$$\begin{aligned}w_k(x) =& \frac{x-dk+\sqrt{(x-dk)^2-4(ak^2+bk+c)}}{2} \\ &\times \left\{1 + \frac{d}{2\sqrt{(x-dk)^2-4(ak^2+bk+c)}}\right. \\ &\left.+ \frac{dx-d^2k+4ak}{2[(x-dk)^2-4(ak^2+bk+c)]} + O\left(\frac{1}{n^2}\right)\right\}.\end{aligned} \tag{2.3}$$

Actually, the above asymptotic formula is uniform for all $k = 1, \cdots, n$; see the proof in appendix. Taking summation of $\ln w_k(x)$ from $k=1$ to $k=n$ yields

$$\begin{aligned}\ln \pi_n(x) \sim& \sum_{k=1}^n \ln \frac{x-dk+\sqrt{(x-dk)^2-4(ak^2+bk+c)}}{2} \\ &+ \sum_{k=1}^n \frac{d}{2\sqrt{(x-dk)^2-4(ak^2+bk+c)}} + \frac{dx-d^2k+4ak}{2[(x-dk)^2-4(ak^2+bk+c)]} \\ \sim& \frac{1}{2}\ln\frac{x-dn+\sqrt{(x-dn)^2-4(an^2+bn+c)}}{x+\sqrt{x^2-4c}} \\ &+ n\ln n + n\int_0^1 \ln\frac{y-dt+\sqrt{(y-dt)^2-4(at^2+bt/n+c/n^2)}}{2}dt \\ &+ \int_0^1 \frac{d}{2\sqrt{(y-dt)^2-4(at^2+bt/n+c/n^2)}} \\ &+ \frac{dy-d^2t+4at}{2[(y-dt)^2-4(at^2+bt/n+c/n^2)]}dt.\end{aligned}$$

Here we have made use of $\ln(1+\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0$ and the trapezoidal rule. Denote the right-hand side of the above formula as $I_1 + n \ln n + nI_2 + I_3$. It is readily seen that

$$I_1 \sim \frac{1}{2} \ln \frac{y - d + \sqrt{(y-d)^2 - 4a}}{2y},$$

and

$$\begin{aligned} I_2 \sim & \int_0^1 \ln \frac{y - dt + \sqrt{(y-dt)^2 - 4at^2}}{2} dt \\ & - \int_0^1 \frac{2bt}{n\sqrt{(y-dt)^2 - 4at^2}[y - dt + \sqrt{(y-dt)^2 - 4at^2}]} dt; \\ I_3 \sim & \int_0^1 \frac{d}{2\sqrt{(y-dt)^2 - 4at^2}} + \frac{dy - d^2 t + 4at}{2[(y-dt)^2 - 4at^2]} dt. \end{aligned}$$

By an integration by parts, we have

$$\begin{aligned} & \int_0^1 \ln[y - dt + \sqrt{(y-dt)^2 - 4at^2}] dt \\ = & \ln[y - d + \sqrt{(y-d)^2 - 4a}] \\ & + \frac{y}{\sqrt{4a - d^2}} \left[\arcsin \left(\frac{dy + 4a - d^2}{2y\sqrt{a}} \right) - \arcsin \left(\frac{d}{2\sqrt{a}} \right) \right] - 1. \end{aligned}$$

Next, we observe that

$$\begin{aligned} & \int_0^1 \frac{2bt}{n\sqrt{(y-dt)^2 - 4at^2}[y - dt + \sqrt{(y-dt)^2 - 4at^2}]} dt \\ = & \int_0^1 \frac{b[y - dt - \sqrt{(y-dt)^2 - 4at^2}]}{2atn\sqrt{(y-dt)^2 - 4at^2}} dt \\ = & \frac{-b}{2an} \left\{ \frac{d}{\sqrt{4a - d^2}} \left[\arcsin \left(\frac{dy + 4a - d^2}{2y\sqrt{a}} \right) - \arcsin \left(\frac{d}{2\sqrt{a}} \right) \right] \right. \\ & \left. + \ln \frac{y - d + \sqrt{(y-d)^2 - 4a}}{2y} \right\}. \end{aligned}$$

Finally, we have

$$\begin{aligned} & \int_0^1 \frac{d}{2\sqrt{(y-dt)^2 - 4at^2}} + \frac{dy - d^2 t + 4at}{2[(y-dt)^2 - 4at^2]} dt \\ = & \frac{d}{2\sqrt{4a - d^2}} \left[\arcsin \left(\frac{dy + 4a - d^2}{2y\sqrt{a}} \right) - \arcsin \left(\frac{d}{2\sqrt{a}} \right) \right] - \frac{1}{4} \ln \frac{(y-d)^2 - 4a}{y^2}. \end{aligned}$$

To sum up, we have the following asymptotic formula for $\pi_n(x)$ with $x = ny$ and $y \in \mathcal{C} \setminus [d - 2\sqrt{a}, d + 2\sqrt{a}]$.

$$
\begin{aligned}
&\pi_n(ny)\\
&\sim \left(\frac{n}{2e}\right)^n \left[y - d + \sqrt{(y-d)^2 - 4a}\right]^{n+1/2+b/(2a)} (2y)^{-1/2-b/(2a)} \left(\frac{y^2}{(y-d)^2 - 4a}\right)^{1/4}\\
&\quad \times \exp\left\{\frac{ny + d/2 + bd/(2a)}{\sqrt{4a - d^2}}\left[\arcsin\left(\frac{dy + 4a - d^2}{2y\sqrt{a}}\right) - \arcsin\left(\frac{d}{2\sqrt{a}}\right)\right]\right\}\\
&\sim \left(\frac{n}{e}\right)^n \left[\frac{y - d + \sqrt{(y-d)^2 - 4a}}{2}\right]^n \left[\frac{y - d + \sqrt{(y-d)^2 - 4a}}{2\sqrt{(y-d)^2 - 4a}}\right]^{1/2}\\
&\quad \times \left[\frac{y - d + \sqrt{(y-d)^2 - 4a}}{2y}\right]^{b/(2a)}\\
&\quad \times \exp\left\{\frac{ny + d/2 + bd/(2a)}{\sqrt{4a - d^2}}\left[\arcsin\left(\frac{dy + 4a - d^2}{2y\sqrt{a}}\right) - \arcsin\left(\frac{d}{2\sqrt{a}}\right)\right]\right\}.
\end{aligned}
$$

This ends our proof. □

To find the asymptotic behavior of $\pi_n(ny)$ near the oscillatory region, we need the following lemma.

LEMMA 2.2. *Let $\pi_n(x)$ be a class of polynomials satisfying the following second-order linear difference equation:*

$$\pi_{n+1}(x) = (x - A_n)\pi_n(x) - B_n\pi_{n-1}(x) \tag{2.4}$$

with initial conditions given by

$$\pi_0(x) = 1, \;\; \pi_1(x) = x - A_0. \tag{2.5}$$

Assume that for some scaling $x = s_n y$, we have asymptotic formula $\pi_n(s_n y) \sim \Phi(n, y)$ for $y \in \mathcal{C} \setminus \bar{\Gamma}$, where Γ is a finite union of some open smooth curves on the complex plane such that $\Phi(n, y)$ has singularities at the points of $\bar{\Gamma} \setminus \Gamma$. Furthermore, assume that $\Phi(n, y)$ is analytic for $y \in \mathcal{C} \setminus \bar{\Gamma}$ and it can be analytically continued from both sides of Γ, denoted by $\Phi^+(n, y)$ and $\Phi^-(n, y)$ respectively. If the ratio $\Phi^+(n, y)/\Phi^-(n, y)$ is exponentially large on one side and exponentially small on the other side of Γ as $n \to \infty$, we then have

$$\pi_n(s_n y) \sim \Phi^+(n, y) + \Phi^-(n, y)$$

for y in a complex neighborhood of any compact subset of Γ.

Proof. Note that $\Phi(n, y)$ is an asymptotic solution of the difference equation (2.4) with branch cut Γ. By analytic continuation, we obtain two linearly independent asymptotic solutions $\Phi^+(n, y)$ and $\Phi^-(n, y)$ satisfying the equation (2.4). These two solutions may not satisfy the initial conditions (2.5). But $\pi_n(s_n y)$ can be asymptotically expressed as a linear combination of $\Phi^\pm(n, y)$, namely,

$$\pi_n(s_n y) \sim C_1\Phi^+(n, y) + C_2\Phi^-(n, y)$$

for y in a complex neighborhood of any compact subset of Γ. Since $\pi_n(s_n y) \sim \Phi^+(n, y)$ on one side and $\pi_n(s_n y) \sim \Phi^-(n, y)$ on the other side of Γ, we conclude that $C_1 = C_2 = 1$. This ends the proof. □

Now, we are ready to prove the following asymptotic formulas of $\pi_n(ny)$ near the oscillatory region.

Theorem 2.3. *Assume $0 < d < 2\sqrt{a}$. For y in a complex neighborhood of a compact subset of $(0, d + 2\sqrt{a})$, we have as $n \to \infty$,*

$$\begin{aligned}\pi_n(ny) \sim &\left(\frac{n\sqrt{a}}{e}\right)^n \left[\frac{a}{4a-(y-d)^2}\right]^{1/4} \left(\frac{\sqrt{a}}{y}\right)^{b/(2a)} \\ &\times \exp\left[\frac{ny + d/2 + bd/(2a)}{\sqrt{4a-d^2}} \arccos\left(\frac{d}{2\sqrt{a}}\right)\right] \\ &\times 2\cos\left\{\left(n + 1/2 + \frac{b}{2a}\right)\arccos\left(\frac{y-d}{2\sqrt{a}}\right) - \frac{\pi}{4}\right. \\ &\left.+ \frac{ny + d/2 + bd/(2a)}{\sqrt{4a-d^2}} \ln \frac{-\sqrt{(4a-d^2)[4a-(y-d)^2]} + d(y-d) + 4a}{2y\sqrt{a}}\right\}.\end{aligned} \tag{2.6}$$

For y in a complex neighborhood of a compact subset of $(d - 2\sqrt{a}, 0)$, we have as $n \to \infty$,

$$\begin{aligned}\pi_n(ny) \sim &\left(\frac{n\sqrt{a}}{e}\right)^n \left[\frac{a}{4a-(y-d)^2}\right]^{1/4} \left(\frac{\sqrt{a}}{-y}\right)^{b/(2a)} \\ &\times \exp\left\{\frac{ny + d/2 + bd/(2a)}{\sqrt{4a-d^2}} \left[-\pi + \arccos\left(\frac{d}{2\sqrt{a}}\right)\right]\right\} \\ &\times 2\cos\left\{\left(n + 1/2 + \frac{b}{2a}\right)\arccos\left(\frac{y-d}{2\sqrt{a}}\right) - \frac{\pi}{4} - \frac{b\pi}{2a}\right. \\ &\left.+ \frac{ny + d/2 + bd/(2a)}{\sqrt{4a-d^2}} \ln \frac{-\sqrt{(4a-d^2)[4a-(y-d)^2]} + d(y-d) + 4a}{-2y\sqrt{a}}\right\}.\end{aligned} \tag{2.7}$$

Proof. Note that for $0 < d < 2\sqrt{a}$ and $y \in \mathcal{C} \setminus [d - 2\sqrt{a}, d + 2\sqrt{a}]$, we can write

$$\begin{aligned}&\arcsin\left(\frac{dy + 4a - d^2}{2y\sqrt{a}}\right) - \arcsin\left(\frac{d}{2\sqrt{a}}\right) \\ &= i \ln \frac{dy - (d^2 - 4a) + i\sqrt{4a-d^2}\sqrt{(y-d)^2 - 4a}}{(d + i\sqrt{4a-d^2})y}.\end{aligned}$$

It follows from (2.1) that $\pi_n(ny) \sim \Phi(n, y)$ for $y \in \mathcal{C} \setminus \bar{\Gamma}$, where $\Gamma := (d - 2\sqrt{a}, 0) \cup (0, d + 2\sqrt{a})$ and

$$\begin{aligned}&\phi(n, y) \\ &:= \left(\frac{n}{e}\right)^n \left[\frac{y - d + \sqrt{(y-d)^2 - 4a}}{2}\right]^n \left[\frac{y - d + \sqrt{(y-d)^2 - 4a}}{2\sqrt{(y-d)^2 - 4a}}\right]^{1/2} \\ &\quad\times \left[\frac{y - d + \sqrt{(y-d)^2 - 4a}}{2y}\right]^{b/(2a)} \\ &\quad\times \exp\left\{\frac{ny + d/2 + bd/(2a)}{\sqrt{4a-d^2}} \times i \ln \frac{dy - (d^2 - 4a) + i\sqrt{4a-d^2}\sqrt{(y-d)^2 - 4a}}{(d + i\sqrt{4a-d^2})y}\right\}.\end{aligned}$$

Taking one side limits on the branch cut Γ, we obtain

$$\begin{aligned}
&\phi^{\pm}(n,y)\\
&=\left(\frac{n}{e}\right)^n(\sqrt{a})^n\left[\frac{\sqrt{a}}{\sqrt{4a-(y-d)^2}}\right]^{1/2}\left(\frac{\sqrt{a}}{y}\right)^{b/(2a)}\\
&\quad\times\exp\left[\pm i\left(n+\frac{1}{2}+\frac{b}{2a}\right)\arccos\left(\frac{y-d}{2\sqrt{a}}\right)\mp\frac{i\pi}{4}\right]\\
&\quad\times\exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{4a-d^2}}\arccos\left(\frac{d}{2\sqrt{a}}\right)\right\}\\
&\quad\times\exp\left\{\pm i\frac{ny+d/2+bd/(2a)}{\sqrt{4a-d^2}}\ln\frac{dy-(d^2-4a)-\sqrt{4a-d^2}\sqrt{4a-(y-d)^2}}{2\sqrt{a}y}\right\}
\end{aligned}$$

for $y\in(0,d+2\sqrt{a})$. Thus, (2.6) follows from Lemma 2.2.
Similarly, for $y\in(d-2\sqrt{a},0)$, we have

$$\begin{aligned}
&\phi^{\pm}(n,y)\\
&=\left(\frac{n\sqrt{a}}{e}\right)^n\left[\frac{\sqrt{a}}{\sqrt{4a-(y-d)^2}}\right]^{1/2}\left(\frac{\sqrt{a}}{-y}\right)^{b/(2a)}\\
&\quad\times\exp\left[\pm i\left(n+\frac{1}{2}+\frac{b}{2a}\right)\arccos\left(\frac{y-d}{2\sqrt{a}}\right)\mp\frac{i\pi}{4}\mp\frac{\pi b}{2a}\right]\\
&\quad\times\exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{4a-d^2}}\left[-\pi+\arccos\left(\frac{d}{2\sqrt{a}}\right)\right]\right\}\\
&\quad\times\exp\left\{\pm i\frac{ny+d/2+bd/(2a)}{\sqrt{4a-d^2}}\ln\frac{dy-(d^2-4a)-\sqrt{4a-d^2}\sqrt{4a-(y-d)^2}}{-2\sqrt{a}y}\right\}.
\end{aligned}$$

A direct application of Lemma 2.2 yields (2.7). ☐

3. Case II: $0<2\sqrt{a}<d$.

THEOREM 3.1. *Assume $0<2\sqrt{a}<d$. Let $x=ny$ with $y\in\mathcal{C}\setminus[0,d+2\sqrt{a}]$. We have as $n\to\infty$,*

$$\begin{aligned}
&\pi_n(ny)\\
&\sim\left(\frac{n}{e}\right)^n\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2}\right]^n\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2\sqrt{(y-d)^2-4a}}\right]^{1/2} \qquad (3.1)\\
&\quad\times\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2y}\right]^{b/(2a)}\\
&\quad\times\exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}\ln\frac{(d+\sqrt{d^2-4a})y}{dy-(d^2-4a)+\sqrt{d^2-4a}\sqrt{(y-d)^2-4a}}\right\}.
\end{aligned}$$

Proof. Similar to the proof of Theorem 2.1, we have

$$
\begin{aligned}
\ln \pi_n(x) \sim & \sum_{k=1}^{n} \ln \frac{x-dk+\sqrt{(x-dk)^2-4(ak^2+bk+c)}}{2} \\
& + \sum_{k=1}^{n} \frac{d}{2\sqrt{(x-dk)^2-4(ak^2+bk+c)}} + \frac{dx-d^2k+4ak}{2[(x-dk)^2-4(ak^2+bk+c)]} \\
\sim & \frac{1}{2} \ln \frac{x-dn+\sqrt{(x-dn)^2-4(an^2+bn+c)}}{x+\sqrt{x^2-4c}} \\
& + n\ln n + n\int_0^1 \ln \frac{y-dt+\sqrt{(y-dt)^2-4(at^2+bt/n+c/n^2)}}{2} dt \\
& + \int_0^1 \frac{d}{2\sqrt{(y-dt)^2-4(at^2+bt/n+c/n^2)}} \\
& + \frac{dy-d^2t+4at}{2[(y-dt)^2-4(at^2+bt/n+c/n^2)]} dt.
\end{aligned}
$$

Denote the right-hand side as $I_1 + n\ln n + nI_2 + I_3$. It is readily seen that

$$
I_1 \sim \frac{1}{2} \ln \frac{y-d+\sqrt{(y-d)^2-4a}}{2y},
$$

and

$$
\begin{aligned}
I_2 \sim & \int_0^1 \ln \frac{y-dt+\sqrt{(y-dt)^2-4at^2}}{2} dt \\
& - \int_0^1 \frac{2bt}{n\sqrt{(y-dt)^2-4at^2}[y-dt+\sqrt{(y-dt)^2-4at^2}]} dt; \\
I_3 \sim & \int_0^1 \frac{d}{2\sqrt{(y-dt)^2-4at^2}} + \frac{dy-d^2t+4at}{2[(y-dt)^2-4at^2]} dt.
\end{aligned}
$$

We calculate the following three integrals.

$$
\begin{aligned}
& \int_0^1 \ln[y-dt+\sqrt{(y-dt)^2-4at^2}]dt \\
= & -1 + \ln[y-d+\sqrt{(y-d)^2-4a}] \\
& + \frac{y}{\sqrt{d^2-4a}} \ln \frac{dy-(d^2-4a)-\sqrt{d^2-4a}\sqrt{(y-d)^2-4a}}{(d-\sqrt{d^2-4a})y};
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 \frac{2bt}{n\sqrt{(y-dt)^2-4at^2}[y-dt+\sqrt{(y-dt)^2-4at^2}]} dt \\
= & \frac{-b}{2an} \left\{ \ln \frac{y-d+\sqrt{(y-d)^2-4a}}{2y} \right. \\
& \left. + \frac{d}{\sqrt{d^2-4a}} \ln \frac{dy-(d^2-4a)-\sqrt{d^2-4a}\sqrt{(y-d)^2-4a}}{(d-\sqrt{d^2-4a})y} \right\};
\end{aligned}
$$

and

$$\int_0^1 \frac{d}{2\sqrt{(y-dt)^2-4at^2}}+\frac{dy-d^2t+4at}{2[(y-dt)^2-4at^2]}dt$$
$$=\frac{d}{2\sqrt{d^2-4a}}\ln\frac{dy-(d^2-4a)-\sqrt{d^2-4a}\sqrt{(y-d)^2-4a}}{(d-\sqrt{d^2-4a})y}-\frac{1}{4}\ln\frac{(y-d)^2-4a}{y^2}.$$

To sum up, we have

$$\pi_n(ny)\sim\left(\frac{n}{e}\right)^n\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2}\right]^n\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2\sqrt{(y-d)^2-4a}}\right]^{1/2}$$
$$\times\left[\frac{y-d+\sqrt{(y-d)^2-4a}}{2y}\right]^{b/(2a)}$$
$$\times\exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}\ln\frac{(d+\sqrt{d^2-4a})y}{dy-(d^2-4a)+\sqrt{d^2-4a}\sqrt{(y-d)^2-4a}}\right\}.$$

This proves (3.1). □

THEOREM 3.2. *Assume $0<2\sqrt{a}<d$. For y in a complex neighborhood of a compact subset of $(d-2\sqrt{a},d+2\sqrt{a})$, we have as $n\to\infty$,*

$$\pi_n(ny)\sim\left(\frac{n\sqrt{a}}{e}\right)^n\left[\frac{a}{4a-(y-d)^2}\right]^{1/4}\left(\frac{\sqrt{a}}{y}\right)^{b/(2a)}$$
$$\times\exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}\ln\frac{2\sqrt{a}}{d-\sqrt{d^2-4a}}\right\}$$
$$\times 2\cos\left\{\left(n+1/2+\frac{b}{2a}\right)\arccos\left(\frac{y-d}{2\sqrt{a}}\right)-\frac{\pi}{4}\right.$$
$$\left.-\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}\arccos\left[\frac{dy-(d^2-4a)}{2\sqrt{a}y}\right]\right\}. \tag{3.2}$$

For y in a complex neighborhood of a compact subset of $(0,d-2\sqrt{a})$, we have as $n\to\infty$,

$$\pi_n(ny)$$
$$\sim\left(\frac{-n}{e}\right)^n\left[\frac{d-y+\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}{2}\right]^n$$
$$\times\left[\frac{d-y+\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}{2\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}\right]^{1/2} \tag{3.3}$$
$$\times\left[\frac{d-y+\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}{2y}\right]^{b/(2a)}2\cos\left[\pi\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}-\pi b/(2a)\right]$$
$$\times\exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}\ln\frac{(d+\sqrt{d^2-4a})y}{-dy+(d^2-4a)+\sqrt{d^2-4a}\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}\right\}.$$

Proof. Applying Lemma 2.2 to (3.1) gives (3.2). To obtain (3.3), we first rewrite (3.1) as

$$\begin{aligned}
&\pi_n(ny)\\
&\sim \left(\frac{-n}{e}\right)^n \left[\frac{d-y+\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}{2}\right]^n\\
&\quad\times \left[\frac{d-y+\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}{2\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}\right]^{1/2}\\
&\quad\times \left[\frac{d-y+\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}{-2y}\right]^{b/(2a)}\\
&\quad\times \exp\left\{\frac{ny+d/2+bd/(2a)}{\sqrt{d^2-4a}}\ln\frac{-(d+\sqrt{d^2-4a})y}{-dy+(d^2-4a)+\sqrt{d^2-4a}\sqrt{(2\sqrt{a}-y+d)(-2\sqrt{a}-y+d)}}\right\}.
\end{aligned}$$

Coupling the above formula with Lemma 2.2 yields (3.3). □

4. Case III: $d = 2\sqrt{a}$.

THEOREM 4.1. *Assume $d = 2\sqrt{a}$. Let $x = ny$ with $y \in \mathcal{C} \setminus [0, 2d]$. We have as $n \to \infty$,*

$$\begin{aligned}
&\pi_n(ny)\\
&\sim \left(\frac{n}{e}\right)^n \left(\frac{y-d+\sqrt{y^2-2yd}}{2}\right)^n \left(\frac{y-d+\sqrt{y^2-2yd}}{2y}\right)^{2b/d^2+1/2} \left(\frac{y-2d}{y}\right)^{-1/4}\\
&\quad\times \exp\left[\left(\frac{n}{d}+\frac{2b}{yd^2}+\frac{1}{2y}\right)\left(y-\sqrt{y^2-2yd}\right)\right]. \qquad (4.1)
\end{aligned}$$

Proof. Similar to the proof of Theorem 2.1, we have

$$\begin{aligned}
\ln\pi_n(x) \sim{}& \sum_{k=1}^{n}\ln\frac{x-dk+\sqrt{(x-dk)^2-4(ak^2+bk+c)}}{2}\\
&+\sum_{k=1}^{n}\frac{d}{2\sqrt{(x-dk)^2-4(ak^2+bk+c)}}+\frac{dx}{2[(x-dk)^2-4(ak^2+bk+c)]}\\
\sim{}&\frac{1}{2}\ln\frac{x-dn+\sqrt{(x-dn)^2-4(an^2+bn+c)}}{x+\sqrt{x^2-4c}}\\
&+n\ln n+n\int_0^1\ln\frac{y-dt+\sqrt{(y-dt)^2-4(at^2+bt/n+c/n^2)}}{2}dt\\
&+\int_0^1\frac{d}{2\sqrt{(y-dt)^2-4(at^2+bt/n+c/n^2)}}\\
&+\frac{dy}{2[(y-dt)^2-4(at^2+bt/n+c/n^2)]}dt.
\end{aligned}$$

Denote the right-hand side as $I_1 + n\ln n + nI_2 + I_3$. It is readily seen that

$$I_1 \sim \frac{1}{2}\ln\frac{y-d+\sqrt{y^2-2yd}}{2y},$$

and

$$I_2 \sim \int_0^1 \ln \frac{y-dt+\sqrt{y^2-2ydt}}{2}dt - \int_0^1 \frac{2bt}{n\sqrt{y^2-2ydt}[y-dt+\sqrt{y^2-2ydt}]}dt;$$

$$I_3 \sim \int_0^1 \frac{d}{2\sqrt{y^2-2ydt}} + \frac{d}{2[y-2dt]}dt = \frac{y-\sqrt{y^2-2yd}}{2y} - \frac{1}{4}\ln\frac{y-2d}{y}.$$

Note that

$$\int_0^1 \ln \frac{y-dt+\sqrt{y^2-2ydt}}{2}dt = \frac{y-d-\sqrt{y^2-2yd}}{d} + \ln\frac{y-d+\sqrt{y^2-2yd}}{2};$$

$$\begin{aligned}&\int_0^1 \frac{2bt}{n\sqrt{y^2-2ydt}[y-dt+\sqrt{y^2-2ydt}]}dt\\ =&\frac{2b}{nyd^2}\left[\sqrt{y^2-2yd}-y-y\ln\frac{y-d+\sqrt{y^2-2yd}}{2y}\right].\end{aligned}$$

We have

$$\begin{aligned}I_2 \sim &\frac{y-d-\sqrt{y^2-2yd}}{d} + \ln\frac{y-d+\sqrt{y^2-2yd}}{2}\\ &-\frac{2b}{nyd^2}\left[\sqrt{y^2-2yd}-y-y\ln\frac{y-d+\sqrt{y^2-2yd}}{2y}\right].\end{aligned}$$

Combining the asymptotic formulas for $I_1,\ I_2$ and I_3, we have

$$\begin{aligned}&\pi_n(ny)\\ \sim&\left(\frac{n}{e}\right)^n\left(\frac{y-d+\sqrt{y^2-2yd}}{2}\right)^n\left(\frac{y-d+\sqrt{y^2-2yd}}{2y}\right)^{2b/d^2+1/2}\left(\frac{y-2d}{y}\right)^{-1/4}\\ &\times\exp\left[\left(\frac{n}{d}+\frac{2b}{yd^2}+\frac{1}{2y}\right)\left(y-\sqrt{y^2-2yd}\right)\right].\end{aligned}$$

This gives (4.1). □

THEOREM 4.2. *Assume $d=2\sqrt{a}$. For y in a complex neighborhood of a compact subset of $(0,2d)$, we have as $n\to\infty$,*

$$\begin{aligned}&\pi_n(ny)\\ \sim&\left(\frac{n}{e}\right)^n\left(\frac{d}{2}\right)^n\left(\frac{d}{2y}\right)^{2b/d^2+1/2}\left(\frac{2d-y}{y}\right)^{-1/4}\exp\left[y\left(\frac{n}{d}+\frac{2b}{yd^2}+\frac{1}{2y}\right)\right]\\ &\times 2\cos\left[\left(n+\frac{2b}{d^2}+\frac{1}{2}\right)\arccos\frac{y-d}{d}-\frac{\pi}{4}-\left(\frac{n}{d}+\frac{2b}{yd^2}+\frac{1}{2y}\right)\sqrt{2yd-y^2}\right]. \quad (4.2)\end{aligned}$$

Proof. Coupling Lemma 2.2 with Theorem 4.1 yields (4.2). □

5. Associated polynomials. In this section, we will investigate asymptotic behaviors of associated polynomials when their degree tends to infinity.

5.1. Associated Laguerre polynomials. The (monic) associated Laguerre polynomials are defined by replacing n with $n+\gamma$ in the difference equation satisfied by Laguerre polynomials; see [4]. The resulting difference equation becomes

$$\begin{aligned}&\pi_{n+1}^{\gamma}(x) = (x-2n-2\gamma-\alpha-1)\pi_n^{\gamma}(x) - (n+\gamma)(n+\alpha+\gamma)\pi_{n-1}^{\gamma}(x);\\ &\pi_0^{\gamma}(x)=1, \quad \pi_1^{\gamma}(x) = x-2\gamma-\alpha-1.\end{aligned} \tag{5.1}$$

Noting that $a=1$, $b=\alpha+2\gamma$ and $d=2$, the associated Laguerre polynomials belong to the third case: $d=2\sqrt{a}$. A direct application of Theorems 4.1 and 4.2 yields the following results.

COROLLARY 5.1. *Let $x = ny+2\gamma+\alpha+1$. For any $y \in \mathcal{C}\setminus[0,4]$, we have as $n\to\infty$,*

$$\begin{aligned}\pi_n^{\gamma}(x) \sim \left(\frac{n}{e}\right)^n \left(\frac{y-2+\sqrt{y^2-4y}}{2}\right)^n \left(\frac{y-2+\sqrt{y^2-4y}}{2y}\right)^{(2\gamma+\alpha+1)/2} \left(\frac{y-4}{y}\right)^{-1/4}\\ \times \exp\left[\left(\frac{n}{2}+\frac{\alpha+2\gamma}{2y}+\frac{1}{2y}\right)\left(y-\sqrt{y^2-4y}\right)\right].\end{aligned} \tag{5.2}$$

For y in a complex neighborhood of a compact subset of $(0,4)$, we have as $n\to\infty$,

$$\begin{aligned}&\pi_n^{\gamma}(x)\\ &\sim \left(\frac{n}{e}\right)^n \left(\frac{1}{y}\right)^{(2\gamma+\alpha+1)/2} \left(\frac{y}{4-y}\right)^{1/4} \exp\left(\frac{ny+2\gamma+\alpha+1}{2}\right)\\ &\quad\times 2\cos\left(\frac{2n+2\gamma+\alpha+1}{2}\arccos\frac{y-2}{2} - \frac{\pi}{4} - \frac{ny+2\gamma+\alpha+1}{2y}\sqrt{4y-y^2}\right).\end{aligned} \tag{5.3}$$

5.2. Associated Meixner polynomials. The (monic) associated Meixner polynomials [17, 22] satisfy the following difference equation.

$$\begin{aligned}&\pi_{n+1}^{\gamma}(x) = \left[x - \frac{(1+\lambda)(n+\gamma)+\beta\lambda}{1-\lambda}\right]\pi_n^{\gamma}(x) - \frac{\lambda(n+\gamma)(n+\gamma+\beta-1)}{(1-\lambda)^2}\pi_{n-1}^{\gamma}(x);\\ &\pi_0^{\gamma}(x) = 1, \quad \pi_1^{\gamma}(x) = x - \frac{(1+\lambda)\gamma+\beta\lambda}{1-\lambda}.\end{aligned} \tag{5.4}$$

Noting that $a=\lambda/(1-\lambda)^2$, $b=\lambda(2\gamma+\beta-1)/(1-\lambda)^2$ and $d=(1+\lambda)/(1-\lambda)$, the associated Meixner polynomials belong to the second case: $0<2\sqrt{a}<d$. For the sake of simplicity, we define

$$y_{\pm} := d \pm 2\sqrt{a} = \frac{(1\pm\sqrt{\lambda})^2}{1-\lambda} = \frac{1\pm\sqrt{\lambda}}{1\mp\sqrt{\lambda}}. \tag{5.5}$$

A direct application of Theorems 3.1 and 3.2 yields the following results.

COROLLARY 5.2. *Let $x = ny + [(1+\lambda)\gamma+\beta\lambda]/(1-\lambda)$. For any $y\in\mathcal{C}\setminus[0,y_+]$, we have as $n\to\infty$,*

$$\begin{aligned}\pi_n(x) \sim \left(\frac{n}{e}\right)^n \left(\frac{\sqrt{y-y_-}+\sqrt{y-y_+}}{2}\right)^{2n+2\gamma+\beta} \frac{y^{(1-\beta)/2-\gamma}}{(y-y_-)^{1/4}(y-y_+)^{1/4}}\\ \times \exp\left\{\left[ny+\frac{(1+\lambda)(\gamma+\beta/2)}{(1-\lambda)}\right]\ln\frac{(\sqrt{y}_-+\sqrt{y}_+)^2 y}{(\sqrt{y_-y-1}+\sqrt{y_+y-1})^2}\right\}.\end{aligned} \tag{5.6}$$

For y in a complex neighborhood of a compact subset of (y_-, y_+), we have as $n \to \infty$,

$$\begin{aligned}\pi_n(x) \sim \left(\frac{n}{e}\right)^n & \frac{\lambda^{n(1-y)/2-\lambda(\gamma+\beta/2)/(1-\lambda)} y^{(1-\beta)/2-\gamma}}{(1-\lambda)^{n+\gamma+\beta/2}(y_+ - y)^{1/4}(y-y_-)^{1/4}} \\ & \times 2\cos\left\{(n+\gamma+\beta/2)\arccos\left(\frac{y-d}{2\sqrt{a}}\right) - \frac{\pi}{4}\right. \\ & \left. - \left[ny + \frac{(1+\lambda)(\gamma+\beta/2)}{(1-\lambda)}\right]\arccos\left(\frac{dy-1}{2\sqrt{a}y}\right)\right\}, \end{aligned} \tag{5.7}$$

where $a = \lambda/(1-\lambda)^2$ and $d = (1+\lambda)/(1-\lambda)$. For y in a complex neighborhood of a compact subset of $(0, y_-)$, we have as $n \to \infty$,

$$\begin{aligned}\pi_n(x) \sim & \left(\frac{-n}{e}\right)^n \left(\frac{\sqrt{y_- - y} + \sqrt{y_+ - y}}{2}\right)^{2n+2\gamma+\beta} \frac{y^{(1-\beta)/2-\gamma}}{(y_+ - y)^{1/4}(y_- - y)^{1/4}} \\ & \times 2\cos\left[\pi ny + \frac{\pi\lambda(2\gamma+\beta)}{1-\lambda} + \frac{\pi}{2}\right] \\ & \times \exp\left\{\left[ny + \frac{(1+\lambda)(\gamma+\beta/2)}{(1-\lambda)}\right]\ln\frac{(\sqrt{y}_- + \sqrt{y}_+)^2 y}{(\sqrt{1-y_- y} + \sqrt{1-y_+ y})^2}\right\}. \end{aligned} \tag{5.8}$$

5.3. Associated Meixner-Pollaczek polynomials. The (monic) associated Meixner-Pollaczek polynomials satisfy the following difference equation.

$$\begin{aligned} &\pi_{n+1}^\gamma(x) = \left(x + \frac{n+\gamma+\lambda}{\tan\phi}\right)\pi_n^\gamma(x) - \frac{(n+\gamma)(n+\gamma+2\lambda-1)}{4\sin^2\phi}\pi_{n-1}^\gamma(x); \\ &\pi_0^\gamma(x) = 1, \quad \pi_1^\gamma(x) = x + \frac{\gamma+\lambda}{\tan\phi}. \end{aligned} \tag{5.9}$$

The associated Meixner-Pollaczek polynomials can be viewed as the special case of the associated Wilson polynomials introduced in [18]. Without loss of generality, we may assume $\pi/2 < \phi < \pi$ so that $d = -\cot\phi > 0$. Noting that $a = 1/(4\sin^2\phi)$ and $b = (2\gamma + 2\lambda - 1)/(4\sin^2\phi)$, the associated Meixner-Pollaczek polynomials belong to the first case: $0 < d < 2\sqrt{a}$. For the sake of simplicity, we define

$$y_\pm := d \pm 2\sqrt{a} = \frac{-\cos\phi \pm 1}{\sin\phi}. \tag{5.10}$$

A simple calculation gives $y_+ = \tan(\phi/2)$ and $y_- = -\cot(\phi/2)$. We apply Theorems 2.1 and 2.3 to (5.9) and obtain the following results.

COROLLARY 5.3. *Let $x = ny - (\gamma+\lambda)/\tan\phi$. For any $y \in \mathcal{C} \setminus [y_-, y_+]$, we have as $n \to \infty$,*

$$\begin{aligned}\pi_n(x) \sim & \left(\frac{n}{e}\right)^n \left[\frac{\sqrt{y-y_+} + \sqrt{y-y_-}}{2}\right]^{2n+2\gamma+2\lambda} \frac{y^{1/2-\gamma-\lambda}}{(y-y_-)^{1/4}(y-y_+)^{1/4}} \\ & \times \exp\left\{[ny - (\gamma+\lambda)\cot\phi]\left[\arcsin\left(\frac{\sin\phi - y\cos\phi}{y}\right) - \phi + \frac{\pi}{2}\right]\right\}. \end{aligned} \tag{5.11}$$

For y in a complex neighborhood of a compact subset of $(0, y_+)$, we have as $n \to \infty$,

$$\pi_n(x) \sim \left(\frac{n}{e}\right)^n \frac{y^{1/2-\gamma-\lambda}}{(y-y_-)^{1/4}(y_+-y)^{1/4}(2\sin\phi)^{n+\gamma+\lambda}} \exp\{[ny-(\gamma+\lambda)\cot\phi](\pi-\phi)\}$$
$$\times 2\cos\left\{(n+\gamma+\lambda)\arccos(y\sin\phi+\cos\phi)-\frac{\pi}{4}\right.$$
$$\left.+[ny-(\gamma+\lambda)\cot\phi]\ln\frac{-\sqrt{(y-y_-)(y_+-y)}-y\cot\phi+1}{y\csc\phi}\right\}. \tag{5.12}$$

For y in a complex neighborhood of a compact subset of $(y_-, 0)$, we have as $n \to \infty$,

$$\pi_n(x) \sim \left(\frac{n}{e}\right)^n \frac{(-y)^{1/2-\gamma-\lambda}}{(y-y_-)^{1/4}(y_+-y)^{1/4}(2\sin\phi)^{n+\gamma+\lambda}} \exp\{[ny-(\gamma+\lambda)\cot\phi](-\phi)\}$$
$$\times 2\cos\left\{(n+\gamma+\lambda)\arccos(y\sin\phi+\cos\phi)-\pi(\gamma+\lambda-\frac{1}{4})\right.$$
$$\left.+[ny-(\gamma+\lambda)\cot\phi]\ln\frac{-\sqrt{(y-y_-)(y_+-y)}-y\cot\phi+1}{-y\csc\phi}\right\}. \tag{5.13}$$

6. Discussions. We have studied a family of orthogonal polynomials satisfying a general difference equation with quadratic coefficients. By introducing the Plancherel-Rotach scale $x = ny$, we obtain asymptotic formulas of the orthogonal polynomials in the outer and oscillatory regions, respectively. Applications of our results are given to several associated orthogonal polynomials on the second level (Laguerre polynomials) and third level (Meixner polynomials and Meixner-Pollaczek polynomials) of Askey scheme. It is noted that the parameter c does not appear in the leading term approximation. The main reason is that in the quadratic expression $an^2 + bn + c$, the value of c is negligible when n is large. However, we expect this parameter to play an important role when the scaled variable y is close to the origin. In a forthcoming paper, we will study uniform asymptotic behavior of the orthogonal polynomials for y in a neighborhood of the origin.

Appendix: Proof of asymptotic formula (2.3)**.** For simplicity, we denote $A_k := x - dk$ and $B_k := ak^2 + bk + c$. The equation (2.2) is rewritten as

$$w_{k+1} = A_k - B_k/w_k.$$

Next, we introduce

$$u_k := \frac{A_k + \sqrt{A_k^2 - 4B_k}}{2}, \quad v_k := \frac{A_k - \sqrt{A_k^2 - 4B_k}}{2}, \quad \delta_k := \frac{u_k - u_{k+1}}{u_k - v_k}.$$

It is readily seen that $|\delta_k| \le L/n$ for some $L > 0$ and all $1 \le k \le n$. Let $w_k = u_k(1+\delta_k+\varepsilon_k)$. We want to show by induction that $\varepsilon_k = O(1/n^2)$. Since $B_k/u_k = v_k$, we have from the recurrence relation that

$$u_{k+1}(1+\delta_{k+1}+\varepsilon_{k+1}) = A_k - v_k(1+\delta_k+\varepsilon_k)^{-1}.$$

In view of $A_k - v_k = u_k$, the above equation can be written as

$$u_{k+1} + u_{k+1}\delta_{k+1} + u_{k+1}\varepsilon_{k+1} = u_k + v_k\delta_k + v_k\varepsilon_k + v_k[1-\delta_k-\varepsilon_k-(1+\delta_k+\varepsilon_k)^{-1}].$$

Since $u_k - u_{k+1} = u_k\delta_k - v_k\delta_k$, we have

$$u_{k+1}\varepsilon_{k+1} = v_k\varepsilon_k + (u_k\delta_k - u_{k+1}\delta_{k+1}) + v_k[1-\delta_k-\varepsilon_k-(1+\delta_k+\varepsilon_k)^{-1}].$$

It follows that

$$|\varepsilon_{k+1}| \le |\frac{v_k}{u_{k+1}}|\cdot|\varepsilon_k| + |\frac{u_k\delta_k-u_{k+1}\delta_{k+1}}{u_{k+1}}| + |\frac{v_k}{u_{k+1}}[1-\delta_k-\varepsilon_k-(1+\delta_k+\varepsilon_k)^{-1}]|.$$

Since $x = ny$ with $y \in \mathcal{C}\setminus[d-2\sqrt{a}, d+2\sqrt{a}]$, we have $|v_k/u_{k+1}| < r$ for some constant $r \in (0,1)$ and all $1 \le k \le n-1$. Furthermore, we choose a large $M > 0$ such that $|\varepsilon_1| \le M/n^2$ and

$$|\frac{u_k\delta_k-u_{k+1}\delta_{k+1}}{u_{k+1}}| + |\frac{v_k}{u_{k+1}}|\cdot \sup_{|t|\le L+1}|1-\frac{t}{n}-(1+\frac{t}{n})^{-1}| \le \frac{M(1-r)}{n^2}$$

for all large n. If $|\varepsilon_k| \le M/n^2$, we choose $n > M$ such that $|\delta_k+\varepsilon_k| \le (L+1)/n$. It follows that

$$|\varepsilon_{k+1}| \le \frac{rM}{n^2} + \frac{M(1-r)}{n^2} \le \frac{M}{n^2}.$$

Therefore, $\varepsilon_k = O(1/n^2)$ uniformly for $1 \le k \le n$. Next, we observe that

$$\begin{aligned} u_k - u_{k+1} &= \frac{A_k - A_{k+1} + \sqrt{A_k^2-4B_k} - \sqrt{A_{k+1}^2-4B_{k+1}}}{2} \\ &= \frac{A_k-A_{k+1}}{2} + \frac{A_k^2-A_{k+1}^2+4B_{k+1}-4B_k}{4\sqrt{A_k^2-4B_k}} + O(\frac{1}{n}) \\ &= \frac{d}{2} + \frac{dx-d^2k+4ak}{2\sqrt{A_k^2-4B_k}} + O(\frac{1}{n}). \end{aligned}$$

Thus, we obtain

$$\delta_k = \frac{d}{2\sqrt{A_k^2-4B_k}} + \frac{dx-d^2k+4ak}{2(A_k^2-4B_k)} + O(\frac{1}{n^2}).$$

This ends the proof of (2.3).

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