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The importance of quarantine: modelling the COVID-19 testing process

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Abstract
We incorporate the disease state and testing state into the formulation of a COVID-19 epidemic model. For this model, the basic reproduction number is identified and its dependence on model parameters related to the testing process and isolation efficacy is discussed. The relations between the basic reproduction number, the final epidemic and peak sizes, and the model parameters are further explored numerically. We find that fast test reporting does not always benefit the control of the COVID-19 epidemic if good quarantine while awaiting test results is implemented. Moreover, the final epidemic and peak sizes do not always increase along with the basic reproduction number. Under some circumstances, lowering the basic reproduction number increases the final epidemic and peak sizes. Our findings suggest that properly implementing isolation for individuals who are waiting for their testing results would lower the basic reproduction number as well as the final epidemic and peak sizes.

Keywords COVID-19 epidemic · Basic reproduction number · Test · Final epidemic size · Peak size

Mathematics Subject Classification 92D30 · 37N25 · 34A34

In Memory of Professor Fred Brauer.

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1 Introduction

The Coronavirus Disease 2019 (COVID-19) was declared a global pandemic by the World Health Organization (WHO) on March 11, 2020, and remains a global threat to public health and economics worldwide (WHO 2022). Since early 2020, tremendous efforts have been made to understand the spread patterns of COVID-19, and various prevention and control strategies from mask-wearing and social distancing to massive city lockdowns have been implemented in affected areas (Fanelli and Piazza 2020; He et al. 2020; Jung et al. 2020; Li et al. 2020; Tang et al. 2020; Wang et al. 2020; Wu et al. 2020). COVID-19 related data such as daily new cases, hospitalizations, and deaths become crucial for policymakers and public health authorities to make informed decisions on appropriate interventions and resource allocations (Farsalinos et al. 2021).

Lab tests such as the polymerase chain reaction (PCR) test and at-home rapid tests based either on molecular or on antigen technology have been used in many countries to detect infection cases and assist clinical diagnosis (Web 2022). Testing processes as well as prevention strategies and social behavior changes based on testing results have greatly affected the dynamics of the COVID-19 epidemic (Lyng et al. 2021), and hence, mathematical models of their effects and implications merit further study (Akman et al. 2022; Gharouni et al. 2022).

In their recent work, Gharouni et al. proposed an SIR type model combining testing and disease states to gain some insights into testing and isolation efficacy on the effectiveness of controlling COVID-19 infection (Gharouni et al. 2022). Their model used three disease states (susceptible, infectious, and recovered) and four testing states (untested, waiting-for-positive, waiting-for-negative, and confirmed positive) to group individuals. One assumption in this model is that the individuals who are waiting for their test results do not change their disease states during the waiting period. This assumption does not hold in reality because some individuals may change their disease states; namely, an individual who was susceptible at the time of testing may get infected and become infectious during the waiting period. Since the infection happens after the testing, this individual may end up with a negative test result. After receiving a negative result, this individual will not self-isolate, even though is infectious. As a result, this individual is more likely to infect other individuals.

Assigning compartment-specific relative testing weights, Gharouni et al. showed that under some circumstances, both increased testing intensity and faster test reporting can reduce the effectiveness of control in the sense that the basic reproduction number would be increased (Gharouni et al. 2022). Note that the basic reproduction number cannot fully characterize the disease dynamics (Cui et al. 2019, 2022; Shaw and Kennedy 2021). For epidemic models, besides the basic reproduction number, the peak size and the final epidemic size are also very important for controlling and assessing the disease (Arino et al. 2007). The final epidemic size and the basic reproduction number are in general positively related in the homogeneous models (Cui et al. 2022), but may be negatively correlated in some heterogeneous models (Cui et al. 2022, 2019).

In what follows, we will extend the model of Gharouni et al. by assuming that individuals may change their disease states during the waiting period and characterize

the properties of the basic reproduction number. We will also study how parameters related to disease states and testing states affect the final and peak sizes.

The rest of this paper is organized as follows. In Sect. 2, we formulate our model which is governed by a system of fifteen differential equations. Model analysis, including the well-posedness of the model and studies of the basic reproduction number and the final epidemic size relation, is presented in Sect. 3. In Sect. 4, we carry out some numerical simulations to explore the effects of parameters on the basic reproduction number, the final epidemic size, and the peak size. We summarize our findings in Sect. 5.

2 Model formulation

As in the model of Gharouni et al., we consider three disease states (Susceptible, Infectious and Recovered). However we use a slightly different model of testing: we set the testing states as untested, waiting for results, received negative results, or received positive results. Let X_T denote the population of each subcompartment with $X \in \{S, I, R\}$ indicating disease state and $T \in \{u, w_s, w_i, w_r, n, p\}$ indicating testing state. Testing states are u, untested; w_x, waiting with disease state x at time of test and $x \in \{s, i, r\}$; n, recently received negative results; and p, recently received positive results. Specifically, individuals waiting for test results are subdivided into six compartments:

S_{w_s}, susceptible individuals who were tested and are waiting for testing results;
I_{w_i}, infectious individuals who were tested and are waiting for testing results;
I_{w_s}, infectious individuals who were infected during their waiting period (the period after the tests and before receiving testing results);
R_{w_r}, individuals who were recovered at their time of testing and are waiting for testing results;
R_{w_i}, individuals who were infectious at their time of testing and recovered during their waiting period;
R_{w_s}, individuals who were susceptible at their time of testing, became infectious and recovered during their waiting period.

We further make the following assumptions:

(i) Infectivity is multiplied by factors $\eta_w \in [0, 1]$, $\eta_p \in [0, 1]$, respectively, in the classes I_{w_i}, I_{w_s}, I_p due to quarantine and isolation, that is, the force of infection Λ is given by

$$\Lambda = \frac{\beta \left[I_u + \eta_w (I_{w_i} + I_{w_s}) + I_n + \eta_p I_p \right]}{N}, \tag{2.1}$$

where η_w measures quarantine efficacy during the waiting period and η_p measures isolation efficacy for testing positive individuals, and N is the total population;

(ii) Infection-induced death is negligible;

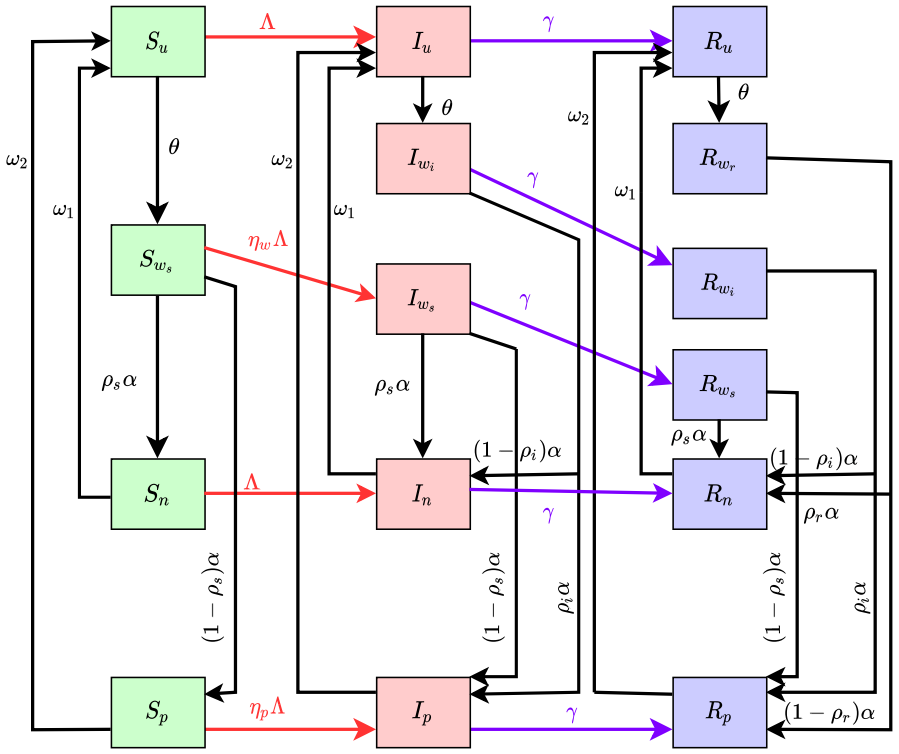


Fig. 1 Flow diagram for the transmission and testing of COVID-19

(iii) Random testing (Web 2022) is conducted across the untested population, namely, each untested individual is equally likely to get tested.

Note that $\eta_w = 0$ or $\eta_p = 0$ means the quarantine or isolation is perfect, $\eta_w = 1$ or $\eta_p = 1$ means no quarantine or isolation at all; $\rho_s = 1$ or $\rho_r = 1$ means a susceptible (recovered) individual receives a negative testing result; $\rho_i = 1$ means an infectious individual receives a positive testing result (The test is 100% accurate).

Our model consists of the following system of differential equations. See Fig. 1 for an illustration of the progression of individuals through testing and disease states.

$$\begin{aligned}
\frac{dS_u}{dt} &= -\Lambda S_u - \theta S_u + \omega_1 S_n + \omega_2 S_p,\\
\frac{dS_{w_s}}{dt} &= -\eta_w \Lambda S_{w_s} + \theta S_u - \alpha S_{w_s},\\
\frac{dS_n}{dt} &= -\Lambda S_n + \rho_s \alpha S_{w_s} - \omega_1 S_n,\\
\frac{dS_p}{dt} &= -\eta_p \Lambda S_p + (1-\rho_s)\alpha S_{w_s} - \omega_2 S_p,\\
\frac{dI_u}{dt} &= \Lambda S_u - \theta I_u + \omega_1 I_n + \omega_2 I_p - \gamma I_u,
\end{aligned}$$

$$
\begin{aligned}
\frac{dI_{w_i}}{dt} &= \theta I_u - \alpha I_{w_i} - \gamma I_{w_i},\\
\frac{dI_{w_s}}{dt} &= \eta_w \Lambda S_{w_s} - \alpha I_{w_s} - \gamma I_{w_s},\\
\frac{dI_n}{dt} &= \Lambda S_n + (1-\rho_i)\alpha I_{w_i} + \rho_s \alpha I_{w_s} - \omega_1 I_n - \gamma I_n,\\
\frac{dI_p}{dt} &= \eta_p \Lambda S_p + \rho_i \alpha I_{w_i} + (1-\rho_s)\alpha I_{w_s} - \omega_2 I_p - \gamma I_p,\\
\frac{dR_u}{dt} &= \gamma I_u - \theta R_u + \omega_1 R_n + \omega_2 R_p,\\
\frac{dR_{w_r}}{dt} &= \theta R_u - \alpha R_{w_r},\\
\frac{dR_{w_i}}{dt} &= \gamma I_{w_i} - \alpha R_{w_i},\\
\frac{dR_{w_s}}{dt} &= \gamma I_{w_s} - \alpha R_{w_s},\\
\frac{dR_n}{dt} &= \gamma I_n + \rho_r \alpha R_{w_r} + (1-\rho_i)\alpha R_{w_i} + \rho_s \alpha R_{w_s} - \omega_1 R_n,\\
\frac{dR_p}{dt} &= \gamma I_p + (1-\rho_r)\alpha R_{w_r} + \rho_i \alpha R_{w_i} + (1-\rho_s)\alpha R_{w_s} - \omega_2 R_p,
\end{aligned} \tag{2.2}
$$

with the nonnegative initial conditions

$$
\begin{aligned}
&S_u(0) = N - I_{u0},\ S_{w_s}(0) = 0,\ S_n(0) = 0,\ S_p(0) = 0,\\
&I_u(0) = I_{u0},\ I_{w_i}(0) = 0,\ I_{w_s}(0) = 0,\ I_n(0) = 0,\ I_p(0) = 0,\\
&R_u(0) = 0,\ R_{w_r}(0) = 0,\ R_{w_i}(0) = 0,\ R_{w_s}(0) = 0,\ R_n(0) = 0,\ R_p(0) = 0.
\end{aligned}
$$

Since the total population is conserved, we can assume without loss of generality $N = 1$. The parameters appearing in System (2.2) are presented in Table 1.

3 Model analysis

For convenience, we set

$$Q_1 = \{u, w_s, n, p\},\ Q_2 = \{u, w_i, w_s, n, p\} \text{ and } Q_3 = \{u, w_r, w_i, w_s, n, p\}.$$

First, we show that our model (2.2) is well posed and the disease eventually dies out.

Theorem 3.1 *For any nonnegative initial condition, System* (2.2) *possesses a unique solution, which remains nonnegative and is bounded for* $t \geq 0$. *Moreover,* $\lim_{t\to+\infty} I_j(t) = 0$, $j \in Q_2$, $\lim_{t\to+\infty} R_{w_i}(t) = 0$ *and* $\lim_{t\to+\infty} R_{w_s}(t) = 0$.

Proof Using the proof similar to (Ji et al. 2022, Theorem 3.1), one can show that System (2.2) possesses a unique solution, which remains nonnegative for any given

Table 1 Model parameters

Symbol	Description	Value/Source
β	Transmission rate	[0.5, 1.5] day^{-1} Eikenberry et al. (2020)
γ	Recovery rate	[0.071, 0.33] day^{-1} Eikenberry et al. (2020)
θ	Testing rate	[0.2, 0.5] day^{-1} [Assumed]
ω_1	Rate back to untested group for individuals with negative results	0.25 day^{-1} [Assumed]
ω_2	Rate back to untested group for individuals with positive results	0.071 day^{-1} WHO (2022)
α	Rate of test return	[0.33, 1] day^{-1} Web (2022)
ρ_i	Testing accuracy for infectious individuals	[0.7, 0.9] Homza et al. (2021)
ρ_s	Testing accuracy for susceptible individuals	[0.7, 0.9] Homza et al. (2021)
ρ_r	Testing accuracy for recovered individuals	[0.7, 0.9] Homza et al. (2021)
η_w	Quarantine efficacy during the waiting period	$[\eta_p, 1]$ [Assumed]
η_p	Isolation efficacy for testing positive individuals	$[0, \eta_w]$ [Assumed]

nonnegative initial condition. In addition, this solution is bounded since the total population is conserved. Adding all S-component and I-component equations of (2.2), we obtain

$$\frac{d\left(\sum_{i\in Q_1} S_i(t) + \sum_{j\in Q_2} I_j(t)\right)}{dt} = -\gamma\left(\sum_{j\in Q_2} I_j(t)\right) \le 0. \tag{3.1}$$

Therefore, $\sum_{i\in Q_1} S_i(t) + \sum_{j\in Q_2} I_j(t)$ is decreasing. This, together with the nonnegativity, shows that the limit of $\sum_{i\in Q_1} S_i(t) + \sum_{j\in Q_2} I_j(t)$ as $t \to +\infty$ exists. According to the Fluctuations Lemma (Hirsch et al. 1985), we conclude that

$$\lim_{t\to+\infty} \frac{d(\sum_{i\in Q_1} S_i(t) + \sum_{j\in Q_2} I_j(t))}{dt} = 0,$$

which implies $\lim_{t\to+\infty} I_j(t) = 0$, for any $j \in Q_2$ via (3.1). That is to say for any $\epsilon > 0$, there exists a $T > 0$ such that $I_j(t) < \epsilon/\gamma$ holds for $t > T$. In addition, from the equations of R_{w_i} and R_{w_s} in System (2.2), we have

$$\frac{dR_{w_i}(t)}{dt} < \epsilon - \alpha R_{w_i}(t), \text{ and } \frac{dR_{w_s}(t)}{dt} < \epsilon - \alpha R_{w_s}(t)$$

for $t > T$. Consequently, the comparison principle gives $\limsup_{t\to+\infty} R_{w_i}(t) \le \epsilon/\alpha$ and $\limsup_{t\to+\infty} R_{w_s}(t) \le \epsilon/\alpha$. Since $\epsilon > 0$ can be arbitrarily small, we obtain $R_{w_i}(t), R_{w_s}(t) \to 0$, as $t \to +\infty$. □

In the following theorem, we investigate the dependence of the basic reproduction number $\mathcal{R}_0$ on the model parameters. The calculation of $\mathcal{R}_0$ and proof of this theorem are given in the appendix.

Theorem 3.2 *The partial derivatives of the basic reproduction number $\mathcal{R}_0$ given in (A.1) with respect to model parameters have the following properties.*

(1) $D_{\eta_w}\mathcal{R}_0 \geq 0$ and $D_{\eta_p}\mathcal{R}_0 \geq 0$;
(2) If $\eta_w \geq \eta_p$, then $D_{\rho_i}\mathcal{R}_0 \leq 0$ and $D_{\rho_s}\mathcal{R}_0 \geq 0$;
(3) Assume $\rho_i = \rho_s = 1$ and $\eta_p = 0$. Then $D_\theta \mathcal{R}_0 \leq 0$, $D_{\omega_1}\mathcal{R}_0 \leq 0$ and $D_{\omega_2}\mathcal{R}_0 \geq 0$. Further, if $\eta_w = 0$, then $D_\alpha \mathcal{R}_0 \geq 0$ and if $\eta_w = 1$, then $D_\alpha \mathcal{R}_0 \leq 0$.

Remark 3.3 Theorem 3.2 implies that improving isolation efficacy of individuals receiving positive test results or quarantine efficacy of individuals awaiting for their test results, by decreasing η_p and/or η_w, respectively, always decreases the basic reproduction number. If individuals with positive test results have better isolation than those in the waiting period ($\eta_p \leq \eta_w$), then increasing accuracy of tests for infectious individuals (increasing test sensitivity) or reducing accuracy of tests for susceptible ones (reducing test specificity) lowers $\mathcal{R}_0$. If isolation of individuals with positive test results is perfect and testing is 100% accurate, increasing the testing rate, θ, or lengthening the isolation period (decreasing ω_2) decreases $\mathcal{R}_0$. Meanwhile, increasing ω_1, the rate back to untested group for individuals with negative results, will also help reducing the value of $\mathcal{R}_0$. Further, the relation between $\mathcal{R}_0$ and test turnaround time, α, depends on the quarantine efficacy, η_w: if individuals are perfectly quarantined while waiting for their test results ($\eta_w = 0$), then a slower test turnaround (a longer wait for test results, or equivalently, a lower value for α) leads to a lower $\mathcal{R}_0$; while if individuals do not self-quarantine at all while waiting for their test results, then a faster return of test results leads to a lower $\mathcal{R}_0$.

Theorem 3.1 implies the disease dies out eventually. After the epidemic passes, it is of great importance to find the number of individuals who were infected during the epidemic period, that is, the final epidemic size, $\mathcal{F}$, defined as

$$\mathcal{F} = \sum_{i\in Q_1} S_i(0) - \lim_{t\to+\infty} \sum_{i\in Q_1} S_i(t).$$

Let $S(t) = \sum_{i\in Q_1} S_i(t)$, $I(t) = \sum_{j\in Q_2} I_j(t)$, and $R(t) = \sum_{k\in Q_3} R_k(t)$. From System (2.2), the dynamics of S, I and R are then governed by

$$\begin{aligned}
\frac{dS(t)}{dt} &= -\Lambda\left(S_u(t) + \eta_w S_{w_s}(t) + S_n(t) + \eta_p S_p(t)\right),\\
\frac{dI(t)}{dt} &= \Lambda\left(S_u(t) + \eta_w S_{w_s}(t) + S_n(t) + \eta_p S_p(t)\right) - \gamma I(t),\\
\frac{dR(t)}{dt} &= \gamma I(t),
\end{aligned} \tag{3.2}$$

together with the nonnegative initial conditions $S(0) = S_u(0)$, $I(0) = I_u(0)$, $R(0) = 0$. Theorem 3.1 states that the solution of (2.2) is nonnegative, which implies that

$dS(t)/dt \leq 0$. This, together with $0 \leq S(t) \leq 1$, shows that $\lim_{t\to+\infty} S(t) := S(\infty)$ exists.

Theorem 3.4 *Suppose $\eta_w > 0$ and $\eta_p > 0$ and set $\eta = \min\{\eta_w, \eta_p\}$. Then the final epidemic size $\mathcal{F}$ of System (2.2) satisfies*

$$S_u(0) - S_2(\infty) \leq \mathcal{F} \leq S_u(0) - S_1(\infty),$$

where $S_1(\infty)$ and $S_2(\infty)$ are, respectively, the roots of the two closely related final epidemic size relations given below

$$\ln \frac{S_u(0)}{S_1(\infty)} = \frac{\beta}{\gamma}\left[\eta^2(S_u(0) - S_1(\infty)) + I_u(0)\right],$$
$$\ln \frac{S_u(0)}{S_2(\infty)} = \frac{\beta}{\gamma}\left[S_u(0) - S_2(\infty) + \eta^2 I_u(0)\right].$$

Proof Assume $\eta_w \neq 0$ and $\eta_p \neq 0$ and set $\eta = \min\{\eta_w, \eta_p\}$. We next estimate the lower and upper bounds of $S(\infty)$. Let $(S_1(t), I_1(t), R_1(t))$ be the solution to the following system

$$\begin{aligned} S_1' &= -\beta S_1 I_1, \\ I_1' &= \beta\eta^2 S_1 I_1 - \gamma I_1, \\ R_1' &= \gamma I_1, \end{aligned} \tag{3.3}$$

with the nonnegative initial conditions

$$S_1(0) = S_u(0), I_1(0) = I_u(0), R_1(0) = 0. \tag{3.4}$$

By a similar approach as in the proof of Theorem 3.1, we obtain

$$\lim_{t\to+\infty} I_1(t) = 0.$$

Adding the first two equations of (3.3) leads to

$$\eta^2 S_1' + I_1' = -\gamma I_1. \tag{3.5}$$

It follows from (3.5) that

$$\int_0^\infty (\eta^2 S_1'(t) + I_1'(t))dt = -\gamma \int_0^\infty I_1(t)dt.$$

By (3.4), we get

$$\int_0^\infty I_1(t)dt = \frac{\eta^2 S_1(0) + I_1(0) - \eta^2 S_1(\infty) - I_1(\infty)}{\gamma} = \frac{\eta^2 S_u(0) + I_u(0) - \eta^2 S_1(\infty)}{\gamma}.$$

Integrating the first equation of System (3.3) then yields

$$\ln \frac{S_u(0)}{S_1(\infty)} = \beta \int_0^\infty I_1(t)dt = \frac{\beta}{\gamma}\left[\eta^2(S_u(0) - S_1(\infty)) + I_u(0)\right]. \tag{3.6}$$

A comparison implies $S(t)$, the solution of (3.2), satisfies $S(t) \geq S_1(t)$ for $t \geq 0$, which yields $\lim_{t\to+\infty} S(t) \geq \lim_{t\to+\infty} S_1(t) := S_1(\infty)$. In a similar manner, we can prove $S(t) \leq S_2(t)$, that is, $S(\infty) \leq \lim_{t\to+\infty} S_2(t) := S_2(\infty)$, where $S_2(t)$ is the solution to the following system

$$\begin{aligned} S_2' &= -\beta\eta^2 S_2 I_2, \\ I_2' &= \beta S_2 I_2 - \gamma I_2, \\ R_2' &= \gamma I_2, \end{aligned}$$

with the nonnegative initial conditions

$$S_2(0) = S_u(0), I_2(0) = I_u(0), R_2(0) = 0;$$

and $S_2(\infty)$ satisfies

$$\ln \frac{S_u(0)}{S_2(\infty)} = \frac{\beta}{\gamma}[S_u(0) - S_2(\infty) + \eta^2 I_u(0)]. \tag{3.7}$$

This completes the proof. □

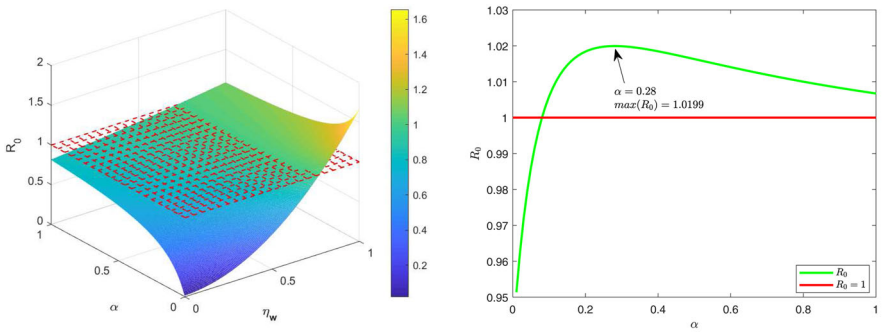
4 Numerical simulations

In this section, we numerically explore the dynamics of System (2.2). The parameter values used for simulations are listed in Table 1, of which, four parameter values are assumed for illustration purposes as no related references on these parameters are available. In practice, the individuals with positive results will be isolated and the individuals with negative results will not be isolated. Thus, we are more interested in the effects of parameters related to the testing states of waiting, (α, η_w, θ), on the dynamics of System (2.2).

We first take $\beta = 0.5, \omega_1 = 0.25, \omega_2 = 0.071, \gamma = 0.3, \eta_p = 0.4$ and examine how $\mathcal{R}_0$ depends on $\eta_w \in [0, 1]$ and α. Figure 2a plots $\mathcal{R}_0$ as a function of α and η_w in three-dimensional space, where $\theta = 0.5, \rho_i = 0.8, \rho_s = 0.8$ from Table 1 are used. If we fix $\eta_w = 0.75$, then, as seen in Fig. 2b, $\mathcal{R}_0$ increases in α for $\alpha \in [0, 0.28]$ and decreases in α for $\alpha \in [0.28, 1]$.

Set $\alpha = 0.5, \eta_w = 0.7$. $\mathcal{R}_0$ is a decreasing function of θ under the following three cases:

(i) For any $\rho_s \in [0, 1]$, fix $\rho_i = 0.8$ and $\eta_p = 0.4$, see Fig. 3a;
(ii) For any $\rho_i \in [0, 1]$, fix $\rho_s = 0.8$ and $\eta_p = 0.4$, see Fig. 3b;



(a) The value of $\mathcal{R}_0$ for $0 \le \eta_w, \alpha \le 1$. (b) The value of $\mathcal{R}_0$ for $\eta_w = 0.75, 0 \le \alpha \le 1$.

Fig. 2 The dependency of $\mathcal{R}_0$ on α. Parameter values are $\beta = 0.5, \omega_1 = 0.25, \omega_2 = 0.071, \gamma = 0.3$, $\eta_p = 0.4$, $\rho_i = 0.8$, $\rho_s = 0.8$ and $\theta = 0.5$

(iii) For any $\eta_p \in [0, \eta_w]$, fix $\rho_i = 0, 8$ and $\rho_s = 0.8$, see Fig. 3c.

Fig. 3d depicts the values of $\mathcal{R}_0$ for different values of ρ_i, ρ_s, η_p selected from the above three cases.

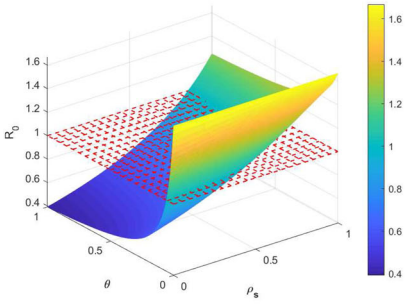
We denote the peak size by $\mathcal{P} = \max_{t \ge 0}(\sum_{j \in Q_2} I_j(t))$ and numerically demonstrate how α affects the final epidemic size, $\mathcal{F}$, and the peak size, $\mathcal{P}$. We choose the initial condition $S_u(0) = 1 - 10^{-4}$, $S_{w_s}(0) = S_n(0) = S_p(0) = 0$, $I_u(0) = 10^{-4}$, $I_{w_i}(0) = I_{w_s}(0) = I_n(0) = I_p(0) = 0$, $R_u(0) = R_{w_r}(0) = R_{w_i}(0) = R_{w_s}(0) = R_n(0) = R_p(0) = 0$, and take $\theta = 0.2, \beta = 0.5, \gamma = 0.2, \omega_1 = 0.25, \omega_2 = 0.071$, $\rho_i = 0.8, \rho_s = 0.8, \rho_r = 0.8, \eta_p = 0.4$. Figure 4a–d indicates that as α increases in [0.33, 1],

(i) $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{R}_0$ decrease for $\eta_w = 0.82$; see Fig. 4a–b.
(ii) $\mathcal{F}$, $\mathcal{P}$ and $\mathcal{R}_0$ increase for $\eta_w = 0.6$; see Fig. 4c–d.
(iii) $\mathcal{F}$ and $\mathcal{P}$ have a positive correlation with $\mathcal{R}_0$ for $\eta_w = 0.82$ or $\eta_w = 0.6$; see Fig. 4a–d.

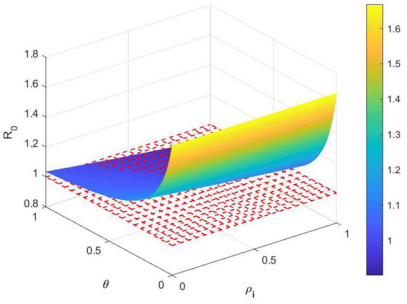
As shown in Fig. 4e–f, the final epidemic size $\mathcal{F}$ and the peak size $\mathcal{P}$ are non-monotonic with respect to α for $\eta_w = 0.75$. For $\alpha \in [0.33, 1]$, we observe that

(i) $\mathcal{F}$ is increasing for $\alpha \in [0, 33, 0.464]$ and decreasing for $\alpha \in (0.464, 1]$; $\mathcal{F}$ has a positive correlation with $\mathcal{R}_0$ for $\alpha \in [0.33, 0.464] \cup [0.6047, 1]$ and a negative correlation for $\alpha \in (0.464, 0.6047)$; see Fig. 4e.
(ii) $\mathcal{P}$ increases for $\alpha \in [0.33, 0.7923]$ and decreases for $\alpha \in (0.7923, 1]$; $\mathcal{P}$ and $\mathcal{R}_0$ are positively correlated for $\alpha \in [0.33, 0.6047] \cup [0.7923, 1]$ and negatively correlated for $\alpha \in (0.6047, 0.7923)$; see Fig. 4f.

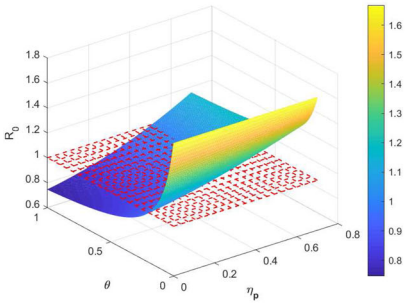
This observation indicates that $\mathcal{F}$ and $\mathcal{P}$ are not always positively related to $\mathcal{R}_0$ and the value of η_w plays an important role to determine the relation. Under some circumstances (better quarantine while awaiting test results), faster test reporting (shorter waiting period) may reduce the effectiveness of control (See Fig. 4c–d). The basic reproduction number, $\mathcal{R}_0$, alone does not completely characterize the disease dynamics and lowering the basic reproduction number may increase the final epidemic size and the peak size (see Fig. 4e–f).



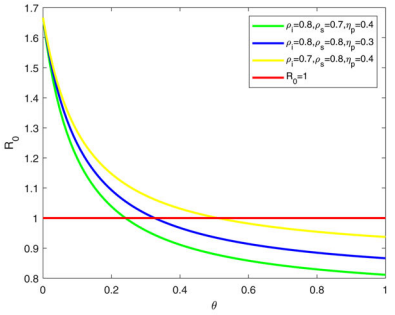
(a) The value of $\mathcal{R}_0$ for $0 \leq \rho_s, \theta \leq 1$.



(b) The value of $\mathcal{R}_0$ for $0 \leq \rho_i, \theta \leq 1$.



(c) The value of $\mathcal{R}_0$ for $0 \leq \eta_p, \theta \leq 1$.



(d) The value of $\mathcal{R}_0$ for fixed values of η_p, ρ_i and ρ_s.

Fig. 3 The effect of θ on the basic reproduction number. The parameter values are $\beta = 0.5$, $\alpha = 0.5$, $\omega_1 = 0.25$, $\omega_2 = 0.071$, $\gamma = 0.3$, $\eta_w = 0.7$; for **a**, $\rho_i = 0.8$, $\eta_p = 0.4$ and $\rho_s \in [0, 1]$; for **b**, $\rho_s = 0.8$, $\eta_p = 0.4$ and $\rho_i \in [0, 1]$ for **c**, $\rho_i = \rho_s = 0.8$ and $\eta_p \in [0, \eta_w]$

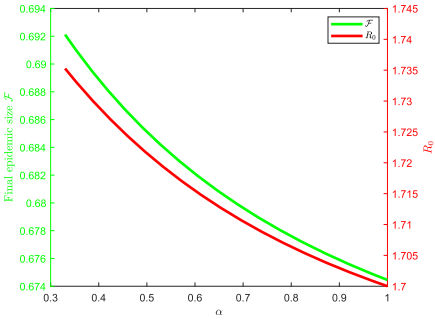
Finally, we explore how η_w, θ affect $\mathcal{F}, \mathcal{P}$ and peak time $\mathcal{T}$ at which $\sum_{j \in Q_2} I_j(t)$ achieves its peak value. We observe that

(i) $\mathcal{F}, \mathcal{P}$ and $\mathcal{R}_0$ increase, but $\mathcal{T}$ decreases as η_w increases; see Fig. 5a–c;
(ii) $\mathcal{F}, \mathcal{P}$ and $\mathcal{R}_0$ decrease, but $\mathcal{T}$ increases as θ increases; see Fig. 5d–f;
(iii) $\mathcal{F}$ and $\mathcal{P}$ are positively correlated with $\mathcal{R}_0$, and $\mathcal{T}$ has a negative correlation with $\mathcal{R}_0$ (see Fig. 5a–f).

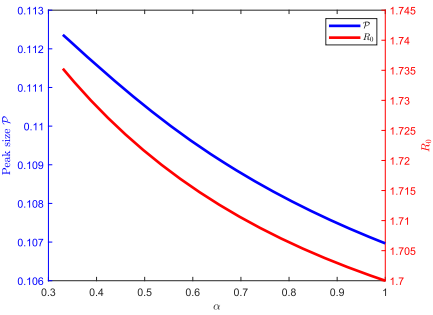
5 Summary and discussion

In this work, we incorporated testing processes into an SIR model to explore the dynamics of the COVID-19 epidemic. Focusing on some parameters related to the testing process, several results concerning the basic reproduction number, $\mathcal{R}_0$, the final epidemic size, $\mathcal{F}$, and the peak size, $\mathcal{P}$, were obtained.

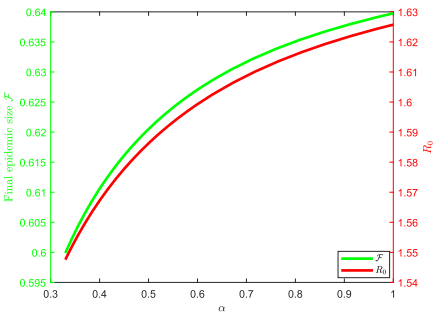
We find that if good quarantine while awaiting test results is implemented (small η_w), then slower but more accurate test reporting increases the effectiveness of control (lowers the basic reproduction number). One interesting finding is that under some



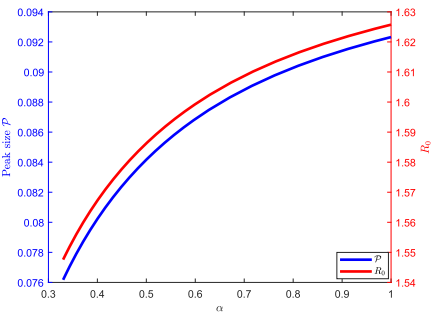
(a) The value of $\mathcal{F}$ and $\mathcal{R}_0$ for $\eta_w = 0.82$.



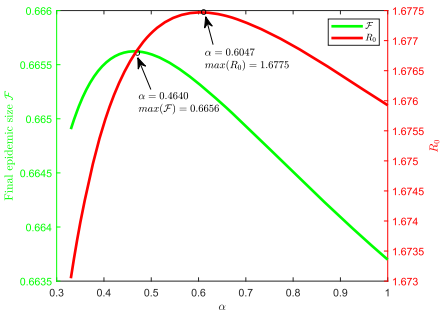
(b) The value of $\mathcal{P}$ and $\mathcal{R}_0$ for $\eta_w = 0.82$.



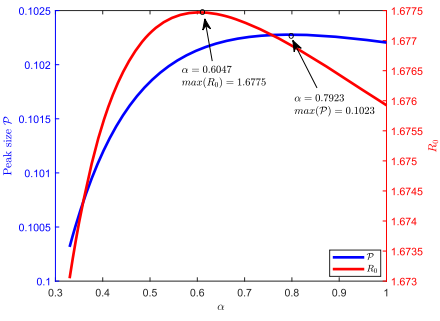
(c) The value of $\mathcal{F}$ and $\mathcal{R}_0$ for $\eta_w = 0.6$.



(d) The value of $\mathcal{P}$ and $\mathcal{R}_0$ for $\eta_w = 0.6$.



(e) The value of $\mathcal{F}$ and $\mathcal{R}_0$ for $\eta_w = 0.75$.

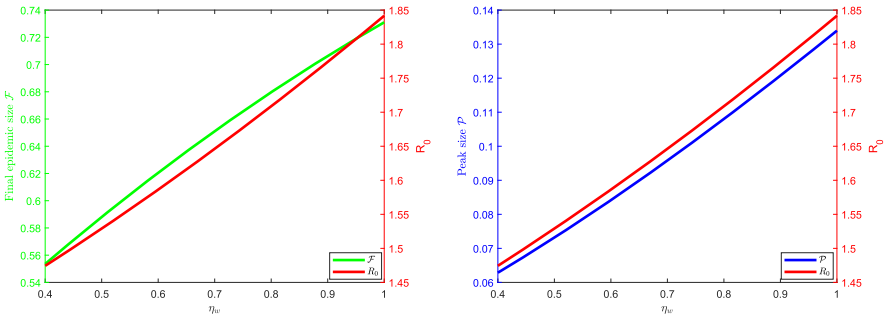


(f) The value of $\mathcal{P}$ and $\mathcal{R}_0$ for $\eta_w = 0.75$.

Fig. 4 The effect of α on final epidemic size and peak size for different η_w. The parameter values are $\alpha \in [0.33, 1], \theta = 0.2, \beta = 0.5, \gamma = 0.2, \omega_1 = 0.25, \omega_2 = 0.071, \rho_i = 0.8, \rho_s = 0.8, \rho_r = 0.8$ and $\eta_p = 0.4$

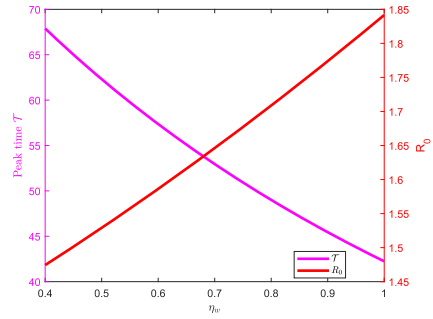
circumstances, the faster test reporting is (higher α), the smaller the basic reproduction number, $\mathcal{R}_0$, but the larger the final epidemic size, $\mathcal{F}$ (see Fig. 4e–f). This suggests that lowering the basic reproduction number may increase the final epidemic size if isolation and test reporting are not properly managed.

Our findings suggest that the effectiveness of control will be improved by increasing the testing rate and increasing test sensitivity. This is not surprising as both lead to

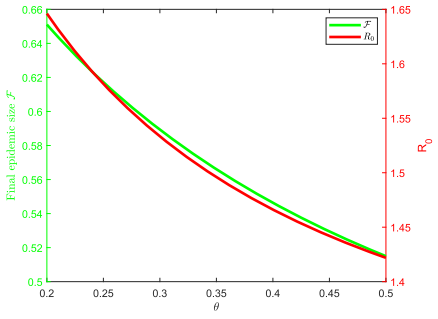


(a) The value of $\mathcal{F}$ and $\mathcal{R}_0$.

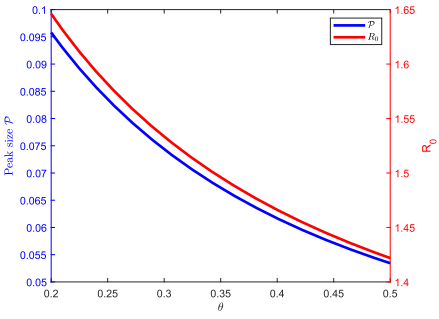
(b) The value of $\mathcal{P}$ and $\mathcal{R}_0$.



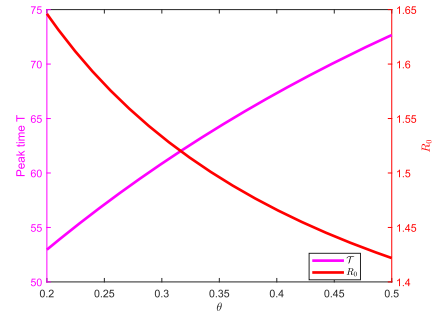
(c) The value of $\mathcal{T}$ and $\mathcal{R}_0$.



(d) The value of $\mathcal{F}$ and $\mathcal{R}_0$.



(e) The value of $\mathcal{P}$ and $\mathcal{R}_0$.



(f) The value of $\mathcal{T}$ and $\mathcal{R}_0$.

Fig. 5 Effect of η_w and θ on the final epidemic size, peak size, and peak time. Parameter values are $\alpha = 0.5, \beta = 0.5, \gamma = 0.2, \omega_1 = 0.25, \omega_2 = 0.071, \rho_i = 0.8, \rho_r = 0.8, \rho_s = 0.8, \eta_p = 0.4$; for **a**–**c**, $\eta_w \in [0.4, 1]$ and $\theta = 0.2$; for **d**–**f**, $\theta \in [0.2, 0.5]$ and $\eta_w = 0.7$

faster and more reliable identification of infected individuals. On the other hand, it is important that implement good self-quarantine by individuals who were tested but haven't received their testing results. Perhaps perversely, our model shows a benefit of delaying tests returns and reduced test specificity if strict self-quarantine is adhered

to while awaiting results. In practice, our expectation is that this would reduce public confidence in testing and lead to reduced adherence to self-quarantine.

We point out that our model can speculate on effect of including more disease states such as exposed, infectious but asymptomatic, and infectious with symptoms (Ge et al. 2020). Our analysis extends the results of Arino et al. (2007) to allow for transitions between susceptible states, but does not incorporate their generality of disease states. Our future work will close this gap.

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Appendix

A Calculation of basic reproduction number

We calculate the basic reproduction number by the next generation matrix method (van den Driessche and Watmough 2002). It is readily seen that System (2.2) has a unique disease-free equilibrium (DFE), which is the solution of

$$\begin{aligned}\theta S_u^* &= \omega_1 S_n^* + \omega_2 S_p^*,\\ \theta S_u^* &= \alpha S_{w_s}^*,\\ \rho_s \alpha S_{w_s}^* &= \omega_1 S_n^*,\\ (1-\rho_s)\alpha S_{w_s}^* &= \omega_2 S_p^*,\end{aligned}$$

with $S_u^* + S_{w_s}^* + S_n^* + S_p^* = 1$. Specifically, the DFE is given by

$$E_0 = (S_u^*, S_{w_s}^*, S_n^*, S_p^*, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$

where

$$S_{w_s}^* = \frac{\theta}{\alpha} S_u^*,\ S_n^* = \frac{\rho_s \theta}{\omega_1} S_u^*,\ S_p^* = \frac{(1-\rho_s)\theta}{\omega_2} S_u^*, \text{ and } S_u^* = \frac{1}{1+\theta\left(\frac{1}{\alpha} + \frac{\rho_s}{\omega_1} + \frac{1-\rho_s}{\omega_2}\right)}.$$

The Jacobian matrix associated with the infectious compartments at E_0 is given by

$$J = \begin{pmatrix} \beta S_u^* - \theta - \gamma & \beta\eta_w S_u^* & \beta\eta_w S_u^* & \beta S_u^* + \omega_1 & \beta\eta_p S_u^* + \omega_2 \\ \theta & -\alpha - \gamma & 0 & 0 & 0 \\ \beta\eta_w S_{w_s}^* & \beta\eta_w^2 S_{w_s}^* & \beta\eta_w^2 S_{w_s}^* - \alpha - \gamma & \beta\eta_w S_{w_s}^* & \beta\eta_w\eta_p S_{w_s}^* \\ \beta S_n^* & \beta\eta_w S_n^* + (1-\rho_i)\alpha & \beta\eta_w S_n^* + \rho_s\alpha & \beta S_n^* - \omega_1 - \gamma & \beta\eta_p S_n^* \\ \beta\eta_p S_p^* & \beta\eta_p\eta_w S_p^* + \rho_i\alpha & \beta\eta_p\eta_w S_p^* + (1-\rho_s)\alpha & \beta\eta_p S_p^* & \beta\eta_p^2 S_p^* - \omega_2 - \gamma \end{pmatrix}.$$

We decompose $J = F - V$, where

$$F = \beta \begin{pmatrix} S_u^* & \eta_w S_u^* & \eta_w S_u^* & S_u^* & \eta_p S_u^* \\ 0 & 0 & 0 & 0 & 0 \\ \eta_w S_{w_s}^* & \eta_w \eta_w S_{w_s}^* & \eta_w \eta_w S_{w_s}^* & \eta_w S_{w_s}^* & \eta_w \eta_p S_{w_s}^* \\ S_n^* & \eta_w S_n^* & \eta_w S_n^* & S_n^* & \eta_p S_n^* \\ \eta_p S_p^* & \eta_p \eta_w S_p^* & \eta_p \eta_w S_p^* & \eta_p S_p^* & \eta_p \eta_p S_p^* \end{pmatrix},$$

and

$$V = \begin{pmatrix} \theta + \gamma & 0 & 0 & -\omega_1 & -\omega_2 \\ -\theta & \alpha + \gamma & 0 & 0 & 0 \\ 0 & 0 & \alpha + \gamma & 0 & 0 \\ 0 & -(1-\rho_i)\alpha & -\rho_s \alpha & \omega_1 + \gamma & 0 \\ 0 & -\rho_i \alpha & -(1-\rho_s)\alpha & 0 & \omega_2 + \gamma \end{pmatrix}.$$

$\mathcal{R}_0$ is the spectral radius of FV^{-1} (van den Driessche and Watmough 2002), defined by $\mathcal{R}_0 = r(FV^{-1})$. Note that $F = \beta F_1^T F_2$, where $F_1 = (S_u^*, 0, \eta_w S_{w_s}^*, S_n^*, \eta_p S_p^*)$ and $F_2 = (1, \eta_w, \eta_w, 1, \eta_p)$ are two row vectors. It follows from (Osnaga 2005, Proposition 1) that $R_0 = r(\beta F_1^T F_2 V^{-1}) = \beta F_2 V^{-1} F_1^T$. Denote $\bar{\alpha} = \alpha/(\alpha + \gamma)$, $\bar{\theta} = \theta/(\theta + \gamma)$, $\bar{\omega}_1 = \omega_1/(\omega_1 + \gamma)$, and $\bar{\omega}_2 = \omega_2/(\omega_2 + \gamma)$. We also set $D = diag\left((1-\bar{\theta})^{-1}, (1-\bar{\alpha})^{-1}, (1-\bar{\alpha})^{-1}, (1-\bar{w}_1)^{-1}, (1-\bar{w}_2)^{-1}\right)$, $W_i = \bar{\omega}_1 \bar{\alpha}(1-\rho_i) + \bar{\omega}_2 \bar{\alpha} \rho_i$, $W_s = \bar{\omega}_2 \bar{\alpha}(1-\rho_s) + \bar{\omega}_1 \bar{\alpha} \rho_s$, $W_1 = \bar{\alpha}(1-\rho_s) + \bar{\omega}_1 \bar{\alpha}^2 \bar{\theta}(\rho_i + \rho_s - 1)$, $W_2 = \bar{\omega}_2 \bar{a}^2 \bar{\theta}(1 - \rho_i - \rho_s) + \bar{\alpha} \rho_s$, and

$$\bar{V} = \begin{pmatrix} 1 & 0 & 0 & -\bar{\omega}_1 & -\bar{\omega}_2 \\ -\bar{\theta} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -(1-\rho_i)\bar{a} & -\rho_s \bar{a} & 1 & 0 \\ 0 & -\rho_i \bar{a} & -(1-\rho_s)\bar{a} & 0 & 1 \end{pmatrix}.$$

We observe $V = \gamma \bar{V} D$ and $V^{-1} = (1/\gamma) D^{-1} \bar{V}^{-1}$, where

$$\bar{V}^{-1} = \frac{1}{1 - W_i \bar{\theta}} \times \begin{pmatrix} 1 & W_i & W_s & \bar{\omega}_1 & \bar{\omega}_2 \\ \bar{\theta} & 1 & W_s \bar{\theta} & \bar{\omega}_1 \bar{\theta} & \bar{\omega}_2 \bar{\theta} \\ 0 & 0 & 1 - W_i \bar{\theta} & 0 & 0 \\ \bar{\alpha}\bar{\theta}(1-\rho_i) & \bar{a}(1-\rho_i) & W_2 & 1 - \bar{\omega}_2 \bar{\alpha} \bar{\theta} \rho_i & \bar{\omega}_2 \bar{\alpha} \bar{\theta}(1-\rho_i) \\ \bar{\alpha}\bar{\theta}\rho_i & \rho_i \bar{a} & W_1 & \bar{\omega}_1 \bar{\alpha} \bar{\theta} \rho_i & 1 - \bar{\omega}_1 \bar{\alpha} \bar{\theta}(1-\rho_i) \end{pmatrix}.$$

Consequently,

$$\mathcal{R}_0 = \frac{\beta}{\gamma} F_2 D^{-1} \bar{V}^{-1} F_1^T. \tag{A.1}$$

B Proof of Theorem 3.2 (1)

We observe that in (A.1), $D^{-1}\bar{V}^{-1}$ is nonnegative and independent of the η_w and η_p. Then $\mathcal{R}_0$ is a quadratic polynomial in η_w or η_p. Hence, the derivative of $\mathcal{R}_0$ with respect to η_w or η_p is nonnegative.

C Proof of Theorem 3.2 (2)

By simple calculations, we obtain

$$D_{\rho_i}\mathcal{R}_0 = \frac{-\omega_1\omega_2\alpha^2\beta\theta c_3^{\rho_i} c_4^{\rho_i}}{\gamma(\alpha+\gamma)(c_1^{\rho_i})^2 c_2^{\rho_i}}, \tag{C.1}$$

where

$$\begin{aligned}
c_1^{\rho_i} &= (\omega_2+\gamma)\left[(\alpha+\gamma)(\theta+\gamma+\omega_1)+\theta\omega_1\right]+\alpha\rho_i\theta(\omega_1-\omega_2),\\
c_2^{\rho_i} &= \omega_1\left[\omega_2(\alpha+\theta)+\alpha\theta(1-\rho_s)+\omega_2\alpha\theta\rho_s\right],\\
c_3^{\rho_i} &= (1-\eta_p)(\alpha+\gamma)(\omega_1+\theta+\gamma)+\omega_2\theta(1-\eta_w)+\omega_1\theta(\eta_w-\eta_p),
\end{aligned}$$

and

$$c_4^{\rho_i} = \omega_1\left[\omega_2(\alpha+\gamma+\eta_w\theta)+c_{41}^{\rho_i}\right]+\omega_2 c_{42}^{\rho_i}+c_{40}^{\rho_i};$$

see expressions of $c_{4j}^{\rho_i}$, $j=0,1,2$ in Table 2. It is readily seen that for $j=0,1,2$, $c_{4j}^{\rho_i}\geq 0$ since $\rho_s\in[0,1]$, which implies $c_4^{\rho_i}\geq 0$. If further $\eta_w\geq\eta_p$ and $\rho_i,\eta_p,\eta_w\in[0,1]$, then $c_k^{\rho_i}\geq 0$, $k=1,2,3$ holds. Hence, by (C.1), we derive $D_{\rho_i}\mathcal{R}_0\leq 0$ if $\rho_i,\rho_s,\eta_w,\eta_p\in[0,1]$ and $\eta_w\geq\eta_p$.

We next take the derivative of $\mathcal{R}_0$ with respect to ρ_s, and get

$$\begin{aligned}
D_{\rho_s}\mathcal{R}_0 = \frac{\omega_1\omega_2\alpha\beta\theta}{\gamma(\alpha+\gamma)c_1^{\rho_i}(c_2^{\rho_i})^2}&\left\{\omega_1^2\left[\omega_2 c_1^{\rho_s}+c_2^{\rho_s}\right]+\omega_1\left[\omega_2^2 c_3^{\rho_s}+\omega_2 c_4^{\rho_s}+c_5^{\rho_s}\right]\right.\\
&\left.+\theta\left[\alpha\gamma(\alpha+\gamma)^2(1-\eta_p^2)(\theta+\gamma)+\omega_2^2 c_6^{\rho_s}+\omega_2 c_7^{\rho_s}\right]\right\},
\end{aligned} \tag{C.2}$$

where

$$\begin{aligned}
c_1^{\rho_s} &= \alpha^3(1-\eta_p)+\alpha^2 c_{12}^{\rho_s}+\alpha c_{11}^{\rho_s}+c_{10}^{\rho_s},\\
c_2^{\rho_s} &= \gamma^2(\eta_w^2-\eta_p^2)\theta(\theta+\gamma)+\alpha^3(1-\eta_p)(\gamma+\gamma\eta_p+\eta_p\theta\rho_i)+\alpha^2 c_{22}^{\rho_s}+\alpha c_{21}^{\rho_s},\\
c_3^{\rho_s} &= \theta(1-\eta_w)\left\{\alpha^2+\alpha\left[2\gamma(1+\eta_w)+\eta_w\theta\right]+\gamma(\gamma+\gamma\eta_w+2\eta_w\theta)\right\},\\
c_4^{\rho_s} &= \alpha^3(1-\eta_p)\left[\gamma+\theta(2-\rho_i)\right]+\alpha^2\left[c_{40}^{\rho_s}+\theta c_{41}^{\rho_s}+\theta^2 c_{42}^{\rho_s}\right]+\alpha c_{43}^{\rho_s}+c_{44}^{\rho_s},\\
c_5^{\rho_s} &= (\alpha+\gamma)\left\{\gamma^2(\eta_w^2-\eta_p^2)\theta(\theta+\gamma)+\alpha c_{51}^{\rho_s}+\alpha^2 c_{52}^{\rho_s}\right\},
\end{aligned}$$

Table 2 The expressions of b_{ij}^{θ} and $c_{4j}^{\rho_i}$

Notation	Expression
b_{10}^{θ}	$\gamma^2(1+\eta_w)+2\gamma(1+\eta_w)\theta+2\eta_w\theta^2$
b_{11}^{θ}	$\gamma(3+\eta_w)+2(1+\eta_w)\theta$
b_{20}^{θ}	$2\gamma^2(1-\eta_w)\left[\gamma^2(1+\eta_w)+2\gamma\theta(1+\eta_w)+2\eta_w\theta^2\right]$
b_{21}^{θ}	$2\gamma(1-\eta_w)\left[2\gamma^2(2+\eta_w)+5\gamma\theta(1+\eta_w)+3\eta_w\theta^2\right]$
b_{22}^{θ}	$\gamma^2(11-8\eta_w-2\eta_w^2)+2\gamma(4-3\eta_w^2)\theta+(2-\eta_w)\eta_w\theta^2$
b_{30}^{θ}	$\gamma(1-\eta_w)\left[\gamma^2(1+\eta_w)+2\gamma\theta(1+\eta_w)+2\eta_w\theta^2\right]$
b_{31}^{θ}	$(1-\eta_w)\left[\gamma^2(3+\eta_w)+4\gamma\theta(1+\eta_w)+2\eta_w\theta^2\right]$
b_{41}^{θ}	$\gamma(1-\eta_w)\left[2\gamma^2(2+\eta_w)+2\gamma\theta(3+\eta_w)+(2+\eta_w)\theta^2\right]$
b_{42}^{θ}	$(1-\eta_w)\left[\gamma^2(5+\eta_w)+6\gamma\theta+\theta^2(1+\eta_w)\right]$
b_{51}^{θ}	$\gamma(1-\eta_w)\left[2\gamma^2(3+\eta_w)+2\gamma\theta(5+\eta_w)+\theta^2(3+2\eta_w)\right]$
b_{52}^{θ}	$\gamma^2(5-4\eta_w)+\gamma\theta(8-6\eta_w)+\theta^2(1+\eta_w)$
b_{61}^{θ}	$\gamma(1-\eta_w)\left[\gamma^2(3+\eta_w)+2\gamma\theta(3+\eta_w)+2\theta^2(1+\eta_w)\right]$
b_{62}^{θ}	$\gamma^2(3-2\eta_w)+\gamma\theta(6-4\eta_w)+\theta^2$
$c_{40}^{\rho_i}$	$\gamma(\alpha+\gamma)\left[\gamma+\theta(\eta_p+\rho_s-\eta_p\rho_s)\right]$
$c_{41}^{\rho_i}$	$\alpha\left[\gamma+\eta_p\theta(1-\rho_s)\right]+\gamma\left[\gamma+\theta(\eta_p-\eta_p\rho_s+\eta_w\rho_s)\right]$
$c_{42}^{\rho_i}$	$\alpha(\gamma+\theta\rho_s)+\gamma\left[\gamma+\theta(\eta_w+\rho_s-\eta_w\rho_s)\right]$

$$c_6^{\rho_s}=(1-\eta_w)\left[\alpha^2(2\gamma+\theta-\theta\rho_i)+\alpha c_{61}^{\rho_s}+\gamma^2(1+\eta_w)(\theta+\gamma)\right],$$

and

$$c_7^{\rho_s}=(\alpha+\gamma)\left[\alpha^2 c_{72}^{\rho_s}+\alpha c_{71}^{\rho_s}+\gamma^2(1-\eta_w^2)(\theta+\gamma)\right];$$

see the expressions of $c_{ij}^{\rho_s}$ in Table 3. We claim in $c_{43}^{\rho_s}$, $d_1=\eta_p(2\rho_i-1-4\eta_w)+\eta_w(4+\eta_w-2\eta_w\rho_i)\geq 0$. If $(2\rho_i-1-4\eta_w)\geq 0$, it obviously holds; if $2\rho_i-1-4\eta_w<0$, then $\eta_w\geq\eta_p$ and $\eta_w\in[0,1]$ yield $\eta_p\leq\eta_w\leq\eta_w(4+\eta_w-2\eta_w\rho_i)/(1+4\eta_w-2\rho_i)$, which is equivalent to $d_1>0$. Note that $\eta_w,\eta_p,\rho_i\in[0,1]$ and $\eta_w\geq\eta_p$. Therefore, $c_{ij}^{\rho_s}\geq 0$, $c_i^{\rho_s}\geq 0$ for $1\leq i\leq 7$, $1\leq j\leq 3$. It follows from (C.2) that $D_{\rho_s}\mathcal{R}_0\geq 0$.

D Proof of Theorem 3.2 (3)

We first calculate $D_\theta\mathcal{R}_0$, $D_{\omega_1}\mathcal{R}_0$ and $D_{\omega_2}\mathcal{R}_0$ when $\rho_i=\rho_s=1$ and $\eta_p=0$. Then,

Table 3 The expressions of $c_{ij}^{\rho_s}$

Notation	Expression
$c_{10}^{\rho_s}$	$\gamma\theta(\eta_w-\eta_p)(\gamma+\gamma\eta_w+2\eta_w\theta)$
$c_{11}^{\rho_s}$	$\gamma(1-\eta_p)[\gamma+\eta_w(\theta+\gamma)]+(\eta_w-\eta_p)\theta\left[2\gamma(1+\eta_w)+\eta_w\theta\right]$
$c_{12}^{\rho_s}$	$\gamma(1-\eta_p)(2+\eta_w)+\theta\left(\eta_w-\eta_p+\eta_w(1-\eta_p)\right)$
$c_{21}^{\rho_s}$	$\gamma\left\{\gamma^2(1-\eta_p^2)+\gamma\theta(\eta_w^2-\eta_p^2+2\eta_w-2\eta_p^2)+\theta^2(1+\rho_i)(\eta_w^2-\eta_p^2)\right\}$
$c_{22}^{\rho_s}$	$2\gamma^2(1-\eta_p^2)+\eta_p(\eta_w-\eta_p)\theta^2\rho_i+\gamma\theta\left[2(\eta_w-\eta_p^2)+\eta_p\rho_i(1-\eta_p)\right]$
$c_{40}^{\rho_s}$	$\gamma^2(1-\eta_p)(2+\eta_w)$
$c_{41}^{\rho_s}$	$\gamma(1-\eta_p)(4+3\eta_w-\rho_i)$
$c_{42}^{\rho_s}$	$(\eta_w-\eta_p)(1-\rho_i)+\eta_w(1-\eta_p)+\eta_p\rho_i(1-\eta_w)$
$c_{43}^{\rho_s}$	$\gamma\left\{\gamma(1-\eta_p)\left[\gamma(1+\eta_w)+(3+4\eta_w)\theta\right]+\theta^2[\eta_p(2\rho_i-1-4\eta_w)+\eta_w(4+\eta_w-2\eta_w\rho_i)]\right\}$
$c_{44}^{\rho_s}$	$\gamma^2(1-\eta_p)\theta(\gamma+\gamma\eta_w+2\eta_w\theta)$
$c_{51}^{\rho_s}$	$\gamma(\theta+\gamma)\left[\gamma(1-\eta_p^2)+2\theta(\eta_w-\eta_p^2)\right]$
$c_{52}^{\rho_s}$	$(1-\eta_p)\left[\gamma^2(1+\eta_p)+\gamma\theta(2+2\eta_p-\rho_i)+\eta_p\theta^2\rho_i\right]$
$c_{61}^{\rho_s}$	$\gamma\left[\gamma(3+\eta_w)+(1+\eta_w)\theta(2-\rho_i)\right]$
$c_{72}^{\rho_s}$	$(1-\eta_p)(\theta+\gamma+\gamma\rho_i-\theta\rho_i)$
$c_{71}^{\rho_s}$	$\gamma\left[\gamma(3-\eta_p-\eta_w-\eta_p\eta_w)+2\theta(1-\eta_p\eta_w)\right]$

$$D_\theta\mathcal{R}_0=\frac{-\omega_1\alpha\beta\left\{b_0^\theta+\omega_1^2\left[\omega_2^2b_1^\theta+\omega_2b_2^\theta+b_3^\theta\right]+\omega_1\left[\omega_2^2b_4^\theta+\omega_2b_5^\theta+b_6^\theta\right]\right\}}{\gamma(\omega_1+\gamma)(\alpha+\gamma)\left[\alpha\theta+\omega_1(\alpha+\theta)\right]^2\left[(\alpha+\gamma)(\theta+\gamma)+\omega_2(\alpha+\gamma+\theta)\right]^2},\tag{D.1}$$

where

$$\begin{aligned}
b_0^\theta&=\omega_2\alpha^2(\alpha+\gamma)\theta^2\left[(\omega_2+\gamma)(1-\eta_w)+\alpha\right],\\
b_1^\theta&=(\alpha+\gamma)(1-\eta_w)(2\alpha^2+b_{10}^\theta+\alpha b_{11}^\theta),\\
b_2^\theta&=\alpha^4+\alpha^3\left[\gamma(6-4\eta_w)+2\theta\right]+\alpha^2b_{22}^\theta+\alpha b_{21}^\theta+b_{20}^\theta,\\
b_3^\theta&=\gamma(\alpha+\gamma)\left\{\alpha^3+\alpha^2\left[\gamma(3-2\eta_w)+2\theta\right]+\alpha b_{31}^\theta+b_{30}^\theta\right\},\\
b_4^\theta&=2\alpha^3(1-\eta_w)(\theta+\gamma)+\alpha^2b_{42}^\theta+\alpha b_{41}^\theta+\gamma^2(1-\eta_w^2)(\theta+\gamma)^2,\\
b_5^\theta&=(\alpha+\gamma)\left[\alpha^3(\gamma+2\theta)+\alpha^2b_{52}^\theta+\alpha b_{51}^\theta+2\gamma^2(1-\eta_w^2)(\theta+\gamma)^2\right],
\end{aligned}$$

and

$$b_6^\theta=\gamma(\alpha+\gamma)\left[\gamma^2(1-\eta_w^2)(\theta+\gamma)^2+\alpha^3(\gamma+2\theta)+\alpha^2b_{62}^\theta+\alpha b_{61}^\theta\right];$$

see the expressions of b_{ij}^{θ} in Table 2. It follows from $\eta_w \in [0,1]$ that for $1 \le i \le 6$ and $0 \le j \le 2$ all the above $b_{ij}^{\theta} \ge 0$, which implies $b_i^{\theta} \ge 0$. Then, $D_\theta \mathcal{R}_0 \le 0$ if $\rho_i = \rho_s = 1$, $\eta_p = 0$. A simple calculation yields

$$D_{\omega_2}\mathcal{R}_0 = \frac{\omega_1\alpha^2\beta\theta(\alpha+\gamma+\eta_w\theta)\left[(\alpha+\gamma)(\theta+\gamma)+\omega_1(\alpha+\gamma+\eta_w\theta)\right]}{\gamma(\omega_1+\gamma)(\alpha+\gamma)\left[\alpha\theta+\omega(\alpha+\theta)\right]\left[(\alpha+\gamma)(\theta+\gamma)+\omega_2(\alpha+\gamma+\theta)\right]^2}.$$

We obtain from the above equation that $D_{\omega_2}\mathcal{R}_0 \ge 0$. Moreover, we have

$$D_{\omega_1}\mathcal{R}_0 = \frac{-\alpha\beta\theta^2\left\{\gamma\left[\omega_2 b_0^{\omega_1}+b_1^{\omega_1}\right]+\omega_1^2\left[\omega_2 b_2^{\omega_1}+b_3^{\omega_1}\right]+2\omega_1\gamma\left[\omega_2 b_4^{\omega_1}+b_5^{\omega_1}\right]\right\}}{\gamma(\omega_1+\gamma)^2(\alpha+\gamma)\left[\alpha\theta+\omega_1(\alpha+\theta)\right]^2\left[(\alpha+\gamma)(\theta+\gamma)+\omega_2(\alpha+\gamma+\theta)\right]}, \tag{D.2}$$

where

$$\begin{aligned}
b_0^{\omega_1} &= (1-\eta_w)\left\{\gamma^2(1+\eta_w)(\theta+\gamma)+\alpha^2(2\gamma+\theta)+\alpha\gamma\left[\gamma(3+\eta_w)+2\theta\right]\right\},\\
b_1^{\omega_1} &= (\alpha+\gamma)(\theta+\gamma)\left[\alpha^2+2\alpha\gamma(1-\eta_w)+\gamma^2(1-\eta_w^2)\right],\\
b_2^{\omega_1} &= (1-\eta_w)\left\{\alpha^2+\alpha\left[2\gamma(1+\eta_w)+\eta_w\theta\right]+\gamma(\gamma+\gamma\eta_w+2\eta_w\theta)\right\},\\
b_3^{\omega_1} &= \gamma\left\{\alpha^2+2\alpha(1-\eta_w)[\gamma+\eta_w(\theta+\gamma)]+\gamma(1-\eta_w)(\gamma+\gamma\eta_w+2\eta_w\theta)\right\},\\
b_4^{\omega_1} &= (\alpha+\gamma)(1-\eta_w)\left[2\alpha+(1+\eta_w)(\theta+\gamma)\right],
\end{aligned}$$

and

$$b_5^{\omega_1} = \alpha^3+\alpha^2\left[\gamma(3-2\eta_w)+\theta\right]+(1-\eta_w)\gamma\left\{\alpha\gamma(3+\eta_w)+(1+\eta_w)[2\theta\alpha+\gamma(\theta+\gamma)]\right\}.$$

Since $\eta_w \in [0,1]$, all the above $b_i^{\omega_1} \ge 0$ for $0 \le i \le 5$. That is to say $D_{\omega_1}\mathcal{R}_0 \le 0$ for $\rho_i = \rho_s = 1$, and $\eta_p = 0$.

If further $\eta_w = 0$, taking the derivative of $\mathcal{R}_0$ in α yields

$$D_\alpha \mathcal{R}_0 = \frac{\beta\theta\omega_1\left[\omega_1^2(\omega_2+\gamma)a_{11}+\theta(a_{12}+\omega_2^2 a_{13})+\omega_1(\omega_2+\gamma)a_{14}\right]}{\gamma(\omega_1+\gamma)\left[\alpha\theta+\omega_1(\alpha+\theta)\right]^2\left[(\alpha+\gamma)(\theta+\gamma)+\omega_2(\alpha+\gamma+\theta)\right]^2}, \tag{D.3}$$

where

$$\begin{aligned}
a_{11} &= (\alpha+\gamma)^2(\theta+\gamma)+\omega_2[2\alpha^2+2\alpha(\theta+\gamma)+\gamma(\theta+\gamma)],\\
a_{12} &= \gamma(\alpha+\gamma)^2(\theta+\gamma)^2+\omega_2\gamma(\theta+\gamma)\left[\alpha^2+2(\alpha+\gamma)(\theta+\gamma+\alpha)\right],\\
a_{13} &= 2\alpha\gamma(\theta+\gamma)+\gamma(\theta+\gamma)^2+\alpha^2(2\gamma+\theta),
\end{aligned}$$

and

$$a_{14} = (\alpha+\gamma)^2(\theta+\gamma)^2 + \omega_2\left[2\alpha(\theta+\gamma)^2 + \gamma(\theta+\gamma)^2 + \alpha^2(2\gamma+3\theta)\right].$$

Clearly, $D_\alpha \mathcal{R}_0 \geq 0$ since $a_{11}, a_{12}, a_{13}, a_{14} \geq 0$ in (D.3). On the other hand, if $\eta_w = 1$, then

$$D_\alpha \mathcal{R}_0 = \frac{-\omega_1\alpha\beta\theta\left\{\alpha(\omega_2+\gamma)(\alpha+\gamma)^2\theta(\theta+\gamma)^2 + \omega_1^2\left[\omega_2 a_{21} + a_{22}\right] + \omega_1\left[a_{23} + \omega_2 a_{24}\right]\right\}}{\gamma(\omega_1+\gamma)(\alpha+\gamma)^2\left[\alpha\theta + \omega_1(\alpha+\theta)\right]^2\left[(\alpha+\gamma)(\theta+\gamma) + \omega_2(\alpha+\gamma+\theta)\right]^2}, \tag{D.4}$$

where

$$\begin{aligned}
a_{21} &= (\alpha+\gamma+\theta)^2[2\gamma\theta + \alpha(\theta+\gamma)],\\
a_{22} &= \gamma(\theta+\gamma)\left[\alpha^3 + 2\gamma\theta(\theta+\gamma) + 2\alpha^2(\gamma+2\theta) + \alpha(\gamma^2+6\gamma\theta+2\theta^2)\right],\\
a_{23} &= \gamma(\alpha+\gamma)(\theta+\gamma)\left[2\gamma\theta(\theta+\gamma) + \alpha^2(\gamma+3\theta) + \alpha(\gamma^2+5\gamma\theta+4\theta^2)\right],
\end{aligned}$$

and

$$\begin{aligned}
a_{24} = {} & 2\gamma^2\theta(\theta+\gamma)^2 + \alpha^3(\gamma^2+4\gamma\theta+2\theta^2) + 2\alpha^2(\gamma^3+5\gamma^2\theta+5\gamma\theta^2+\theta^3)\\
& + \alpha\gamma(\gamma^3+8\gamma^2\theta+12\gamma\theta^2+5\theta^3).
\end{aligned}$$

Therefore, $D_\alpha \mathcal{R}_0 \leq 0$ follows from $a_{21}, a_{22}, a_{23}, a_{24} \geq 0$ in (D.4).

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