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On a Ramanujan type entire function and its zeros

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ABSTRACT

In this paper, we derive some properties of a Ramanujan type entire function. A mild generalization of the Garret-Ismail-Stanton m-version of the Rogers-Ramanujan identities is obtained. Moreover, we investigate the zeros of the Ramanujan type entire function, and our results generalize those for the zeros of the Ramanujan function. Finally, an integral equation related to the Ramanujan type entire function is also derived.

1. Introduction

In his lost notebook [18, p. 57], Ramanujan wrote

$$A_q(z) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} (-z)^n = \prod_{n=1}^{\infty} \left[1 - \frac{zq^{2n-1}}{1 - \sum_{j=1}^{\infty} y_j q^{jn}} \right], \tag{1.1}$$

and gave explicit values for y_j for $1 \le j \le 4$. Andrews [1] interpreted (1.1) as a Weierstrass factor product representation and that the n-th zero has the asymptotic series $q^{1-2n}[1 - \sum_{j=1}^{\infty} y_j q^{jn}]$. This agrees with Hayman's results about asymptotic expansion for the zeros of entire functions of the form $\sum_{n=0}^{\infty} a_n q^{n^2} z^n$ with a_n bounded for all n; see [7]. Andrews proved the result only in the special case $0 < q < 1/2$. Al-Salam and Ismail proved that the zeros of $A_q(z)$ are real and simple and interlace with the zeros of $A_q(qz)$; see [12] for references. Ismail and C. Zhang [15] proved that if $z_n, n = 1, 2, \cdots$, denote the zeros of $A_q(z)$ in ascending order, then $q^{2n-1} z_n$ is analytic in q^n; namely, the asymptotic series $1 - \sum_{j=1}^{\infty} y_j q^{jn}$ is actually convergent, where q is allowed to be in the interval $(0, 1)$. They also investigated the structure of the coefficients and showed that the coefficients are in a polynomial ring with three generators involving two

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transcendental functions in q, and the coefficients in the polynomial are rational functions of q. The function $A_q(z)$ also appeared in the Rogers–Ramanujan identities, which actually give infinite product representations for $A_q(-1)$ and $A_q(-q)$. The Garrett–Ismail–Stanton generalization of the Rogers–Ramanujan identities expresses $A_q(-q^m)$ as a linear combination of $A_q(-1)$ and $A_q(-q)$ with coefficients being rational functions of q; see [5].

Ismail [11] pointed out that the Plancherel-Rotach asymptotics of the q^{-1}-Hermite polynomials, the Stieltjes–Wigert polynomials and the q-Laguerre polynomials involve the function $A_q(z)$ and the asymptotics of the k-th largest zero of any of these polynomials involve the k-th zero of $A_q(z)$. In other words, $A_q(z)$ plays the role like the Airy function in the asymptotics of the Hermite and Laguerre polynomials; see [12] and [19]. Recently, the Plancherel–Rotach asymptotics of the Al-Salam–Chihara polynomials were studied in [13], which led to a two-parameter function

$$F(w, A, B; q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-w)^n q^{\binom{n}{2}} A^k B^{n-k}}{(q;q)_k (q;q)_{n-k}}. \tag{1.2}$$

The same function appeared again in our work [3] where we established Plancherel–Rotach asymptotics for a class of orthogonal polynomials satisfying recurrence relations whose coefficients are polynomials in q^{-n}. This two-parameter function turned out to be similar in nature to a scaled form of the q-exponential function introduced by Ismail and R. Zhang [16].

The rest of the paper is arranged as follows. In Section 2, we study some elementary properties of the new function $F(w, A, B; q)$. In Section 3, we prove a mild generalization of the Garrett-Ismail-Stanton m-version of the Rogers-Ramanujan identities obtained in [5]. In our result, the Schur polynomials are also polynomials in q but their definition involves a double sum. Garrett [4] studied the very interesting combinatorics of the Garrett-Ismail-Stanton formula in terms of partitions. We expect our results will lead to a more elaborate combinatorial theory. Section 4 is devoted to a study of the zeros of $F(w, A, B; q)$ when $B/A = q^{1/2+k}$ with $k = 0, \pm 1, \pm 2, \cdots$. It turns out that the n-th zero is an analytic function of q^n, which generalizes the results for the zeros of $A_q(z)$ given in [15]. On the other hand, our generalization involves a positive integer parameter m and the structure of the j-th Taylor coefficient of the expansion of the n-th zero depends on the residue of j modulo m. In other words, there is certain sieving process involved. Finally, an integral equation for $F(-w, e^{-i\theta}, e^{i\theta}; q)$ is given in Section 5.

2. Elementary properties of $F(w, A, B; q)$

We first introduce the polynomials of two variables A and B:

$$u_n(A, B; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^k B^{n-k} = (AB)^{n/2} H_n(\cos\theta \,|\, q), \tag{2.1}$$

where $e^{2i\theta} = B/A$ and

$$H_n(\cos\theta \,|\, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta} \tag{2.2}$$

are the Rogers-Szegő polynomials or the q-Hermite polynomials; see [12]. Here,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad k = 0, 1, 2, \cdots, n$$

is the q-binomial coefficient; see the notations and terminology about q-series and relate functions in Andrews et al. [2], Gasper and Rahman [6]. It is readily seen from (1.2) and (2.1) that $F(w, A, B; q)$ is a generating function of $u_n(A, B; q)$:

$$F(w, A, B; q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(-w)^n}{(q;q)_n} u_n(A, B; q). \tag{2.3}$$

Actually, we have another generating function (see [17, (1.14.1)])

$$\sum_{n=0}^{\infty} u_n(A, B; q) \frac{t^n}{(q;q)_n} = \frac{1}{(At, Bt; q)_\infty}, \quad |At| < 1, \ |Bt| < 1, \tag{2.4}$$

from which we observe that $u_n(A, B; q)$ has a single-term closed form if and only if $B = -A$, or $B = Aq^{\pm 1/2}$. In the former case the generating function becomes $1/(A^2t^2; q^2)_\infty$, and in the case $B = Aq^{1/2}$ the generating function turns out to be $1/(At; q^{1/2})_\infty$. Therefore, we have

$$\begin{aligned} u_{2n+1}(A, -A; q) = 0, \qquad & u_{2n}(A, -A; q) = \frac{(q;q)_{2n}}{(q^2;q^2)_n} A^{2n}, \\ u_n(A, Aq^{1/2}; q) = \frac{(q;q)_n}{(q^{1/2};q^{1/2})_n} A^n &= u_n(Aq^{1/2}, A; q). \end{aligned} \tag{2.5}$$

The above formulas are known because they are essentially the evaluations of the q-Hermite polynomial $H_n(x \,|\, q)$ at $x = 0$ and $x = (q^{1/2} + q^{-1/2})/2$, respectively. From (2.3) and the above formulas, one can see that $F(w, A, -A; q)$ and $F(w, A, Aq^{1/2}; q)$ are indeed the q-Airy functions:

$$F(w, A, -A; q) = A_{q^2}\left(-\frac{A^2w^2}{q}\right), \qquad F(w, A, Aq^{1/2}; q) = A_{q^{1/2}}\left(\frac{Aw}{q^{1/2}}\right). \tag{2.6}$$

When $B = Aq^{-k+1/2}$ for $k \in \mathbb{N}_0$, we have the following representation.

Theorem 2.1. *With $u_n(A, B; q)$ defined in (2.1), we have, for $k \in \mathbb{N}_0$,*

$$u_n(A, Aq^{-k+1/2}; q) = (q;q)_n A^n q^{-kn} \sum_{j=0}^{\min\{k,n\}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}}}{(q^{1/2};q^{1/2})_{n-j}}. \tag{2.7}$$

Proof. From (2.4), it is clear that, for any $|t| < |q^k/A|$,

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(A, Aq^{-k+1/2}; q) \frac{t^n}{(q;q)_n} &= \frac{1}{(At, Atq^{-k+1/2}; q)_\infty} = \frac{(Atq^{-k}; q)_k}{(Atq^{-k}, Atq^{-k+1/2}; q)_\infty} \\ &= \frac{(Atq^{-k}; q)_k}{(Atq^{-k}; q^{1/2})_\infty}. \end{aligned}$$

Recalling formulas (1.9.8) and (1.14.1) in [17], we get

$$(Atq^{-k}; q)_k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2} - jk} (-At)^j \quad \text{and} \quad \frac{1}{(Atq^{-k}; q^{1/2})_\infty} = \sum_{m=0}^{\infty} \frac{(At)^m q^{-mk}}{(q^{1/2};q^{1/2})_m}.$$

Then, the above two formulas give us the desired result. □

In the following proposition, we give a recurrence relation satisfied by $u_n(A, B; q)$.

Proposition 2.2. *With $u_n(A, B; q)$ defined in (2.1), we have*

$$(A + B)u_n(A, B; q) = u_{n+1}(A, B; q) + AB(1 - q^n)u_{n-1}(A, B; q). \tag{2.8}$$

Proof. From the definition of $u_n(A, B; q)$ in (2.1), we get

$$\begin{aligned} Bu_n(A, B; q) - u_{n+1}(A, B; q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^k B^{n+1-k} - \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q A^k B^{n+1-k} \\ &= \sum_{k=0}^{n+1} \left(\frac{1 - q^{n+1-k}}{1 - q^{n+1}} - 1 \right) \begin{bmatrix} n+1 \\ k \end{bmatrix}_q A^k B^{n+1-k}. \end{aligned}$$

As the coefficient vanishes when $k = 0$, we change the index from k to $k + 1$ and obtain

$$Bu_n(A, B; q) - u_{n+1}(A, B; q) = -\sum_{k=0}^{n} q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q A^{k+1} B^{n-k}.$$

This gives us

$$(A + B)u_n(A, B; q) - u_{n+1}(A, B; q) = \sum_{k=0}^{n} (1 - q^{n-k}) \begin{bmatrix} n \\ k \end{bmatrix}_q A^{k+1} B^{n-k}.$$

Since the term vanishes when $k = n$, we extract the factor $AB(1 - q^n)$ out of the above summation and obtain (2.8). □

From the above proposition, we get a functional relation among $F(w, A, B; q)$, $F(qw, A, B; q)$ and $F(q^2w, A, B; q)$.

Proposition 2.3. *We have*

$$\Big[1 - (A + B)w\Big] F(qw, A, B; q) = F(w, A, B; q) + ABqw^2 F(q^2w, A, B; q). \tag{2.9}$$

Proof. The equation (2.3) can be considered as a Taylor expansion of $F(w, A, B; q)$ near $w = 0$. Let us consider the coefficients of w^n for the function $\Big[1 - (A + B)w\Big] F(qw, A, B; q) - F(w, A, B; q)$, which can be simplified as

$$\frac{q^{\binom{n+1}{2}}(-1)^{n+1}}{(q; q)_n} \Big[(A + B)u_n(A, B; q) - u_{n+1}(A, B; q)\Big].$$

Moreover, the coefficients of w^n for the last term $ABqw^2 F(q^2w, A, B; q)$ in (2.9) is given by

$$ABq \frac{q^{\binom{n-1}{2}}(-1)^{n-1}}{(q; q)_{n-1}} q^{2(n-1)} u_{n-1}(A, B; q) = \frac{q^{\binom{n+1}{2}}(-1)^{n+1}}{(q; q)_n} AB(1 - q^n)u_{n-1}(A, B; q).$$

Using the recurrence relation of $u_n(A, B; q)$ in (2.8), the above two formulas are indeed the same. This proves (2.9). □

It is more convenient to write (2.9) in the following form

$$\left[1 - \frac{(A+B)w}{q}\right] F(w,A,B;q) = F(w/q,A,B;q) + \frac{AB}{q} w^2 F(qw,A,B;q). \tag{2.10}$$

Now, we interchange the summations in the definition (1.2) and make use of the Euler's theorem [12, Theorem 12.2.6] to obtain

$$F(w,A,B;q) = (Aw;q)_\infty \sum_{n=0}^{\infty} \frac{(-Bw)^n q^{\binom{n}{2}}}{(Aw,q;q)_n}. \tag{2.11}$$

This implies that $F(w,A,B;q)$ is essentially a q-Bessel function of the type $J_\nu^{(3)}$, where ν depends on the variable w: $q^{\nu+1} = Aw$. This situation is similar to the functions arising from the spectral analysis of orthogonal polynomials generalizing the Lommel and the q-Lommel polynomials [10].

Ismail and R. Zhang [16] introduced the q-exponential function

$$\mathcal{E}_q(\cos\theta;t) := \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n=0}^{\infty} \left(-ie^{i\theta}q^{(1-n)/2}, -ie^{-i\theta}q^{(1-n)/2};q\right)_n \frac{(-it)^n}{(q;q)_n} q^{n^2/4}. \tag{2.12}$$

It was later proved that

$$(qt^2;q^2)_\infty \mathcal{E}_q(x;t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} t^n}{(q;q)_n} H_n(x\,|\,q); \tag{2.13}$$

see [12]. Note that the left-hand side of (2.13) is an entire function of t for fixed x and is an entire function of x for fixed t. Moreover, the t-pole singularities of $\mathcal{E}_q(x;t)$ are all canceled by the infinite product $(qt^2;q^2)_\infty$. The major difference between the series in (2.3) and (2.13) lies in the powers of q: one is $q^{\binom{n}{2}}$ and the other one is $q^{n^2/4}$.

3. Rogers-Ramanujan type identities

Garrett, Ismail and Stanton [5] proved the following generalization—which is usually referred to as the m-version—of the Rogers-Ramanujan identities

$$A_q(-q^m) = \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q,q^4;q^5)_\infty} + \frac{(-1)^{m+1} q^{-\binom{m}{2}} b_m(q)}{(q^2,q^3;q^5)_\infty}, \tag{3.1}$$

where

$$a_m(q) = \sum_{0\le 2j\le m-2} q^{j^2+j} \begin{bmatrix} m-j-2 \\ j \end{bmatrix}_q, \quad b_m(q) = \sum_{0\le 2j\le m-1} q^{j^2} \begin{bmatrix} m-j-1 \\ j \end{bmatrix}_q, \tag{3.2}$$

for $m > 1$, and

$$a_0(q) = b_1(q) = 1, \qquad a_1(q) = b_0(q) = 0. \tag{3.3}$$

The polynomials $a_m(q)$ and $b_m(q)$ were considered by Schur in conjunction with his proof of the Rogers–Ramanujan identities. They are solutions to the discrete system

$$y_{n+2} = y_{n+1} + q^n y_n, \tag{3.4}$$

with the initial conditions in (3.3).

We may also derive a similar Rogers-Ramanujan type identity for $F(-q^{m+1}, q^{2k}, q; q^2)$. To see this, by replacing A by Aq^k in (2.7), we find that

$$u_n(Aq^k, Aq^{1/2}; q) = (q;q)_n A^n \sum_{j=0}^{\min\{k,n\}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}}}{(q^{1/2};q^{1/2})_{n-j}}. \tag{3.5}$$

Substituting this into (2.3) leads to

$$F(w/A, Aq^k, Aq^{1/2}; q) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-w)^n \sum_{j=0}^{\min\{k,n\}} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}}}{(q^{1/2};q^{1/2})_{n-j}}.$$

We interchange the summations on the right-hand side of the above formula, and then shift the index $n - j = m$ to obtain

$$F(w/A, Aq^k, Aq^{1/2}; q) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q w^j q^{j(j-1)} A_{\sqrt{q}}(wq^{j-1/2}), \tag{3.6}$$

where A is a constant, but $A_{\sqrt{q}}$ is the Ramanujan function defined in (1.1). Note that the right-hand side in the above equation is independent of A. Without loss of generality, we may take $A = 1$. With $w = -q^{(m+1)/2}$, we find that

$$F(-q^{(m+1)/2}, q^k, q^{1/2}; q) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j q^{j(j-1)+j(m+1)/2} A_{\sqrt{q}}(-q^{j+m/2}). \tag{3.7}$$

Coupling this with the formula obtained by Garrett–Ismail–Stanton (3.1) gives us

$$F(-q^{m+1}, q^{2k}, q; q^2) = \frac{\bar{a}_m(q)}{(q, q^4; q^5)_\infty} + \frac{\bar{b}_m(q)}{(q^2, q^3; q^5)_\infty}, \tag{3.8}$$

where $\bar{a}_m(q)$ and $\bar{b}_m(q)$ are rational functions of q.

We next consider the function $F(-q^{(m+1)/2}, A, B; q)$. When $m = -2s$, $s \in \mathbb{N}$, is even, we set

$$F(-q^{-s+1/2}, A, B; q) = X_s\, q^{-\binom{s}{2}}. \tag{3.9}$$

Then, (2.10) becomes

$$\left[q^s + \frac{A+B}{\sqrt{q}}\right] X_s = X_{s+1} + \frac{AB}{q} X_{s-1}. \tag{3.10}$$

We solve this recursion using the generating function

$$G(z) := \sum_{s=0}^{\infty} X_s z^s. \tag{3.11}$$

It is easy to see that (3.10) implies

$$G(z)=\frac{zG(qz)}{(1-zA/\sqrt{q})(1-zB/\sqrt{q})}+\frac{X_0+X_1z-[1+(A+B)/\sqrt{q}]zX_0}{(1-zA/\sqrt{q})(1-zB/\sqrt{q})}.$$

Note that (3.10) also indicates that

$$X_1-[1+(A+B)/\sqrt{q}]X_0=-ABq^{-1}X_{-1}. \tag{3.12}$$

By iterating the q-difference equation for G, we conclude that

$$\begin{aligned} G(z)=X_0\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}z^n}{(Azq^{-1/2},Bzq^{-1/2};q)_{n+1}} \\ -\frac{AB}{q}X_{-1}\sum_{n=0}^{\infty}\frac{q^{\binom{n+1}{2}}z^{n+1}}{(Azq^{-1/2},Bzq^{-1/2};q)_{n+1}}. \end{aligned} \tag{3.13}$$

Using the q-binomial theorem

$$\frac{1}{(z;q)_m}=\sum_{n=0}^{\infty}\frac{(q^m;q)_n}{(q;q)_n}z^n, \tag{3.14}$$

we expand the generating function (3.13) and establish the explicit form

$$\begin{aligned} q^{-\binom{s}{2}}X_s=X_0\sum_{u,v\ge 0,u+v\le s}\begin{bmatrix} s-v\\ u\end{bmatrix}_q\begin{bmatrix} s-u\\ v\end{bmatrix}_q q^{-s(u+v)+(u+v)^2/2}A^uB^v \\ -\frac{AB}{q}X_{-1}\sum_{u,v\ge 0,u+v\le s-1}\begin{bmatrix} s-v-1\\ u\end{bmatrix}_q\begin{bmatrix} s-u-1\\ v\end{bmatrix}_q q^{-s(u+v)+(u+v)^2/2}A^uB^v \end{aligned} \tag{3.15}$$

for $s=0,1,2\cdots$, where the empty sum equals 0. It is easy to see that, when $s=1$, the above formula is the same as (3.12). From (3.9), we find the initial values

$$X_0=F(-q^{1/2},A,B;q)=\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q;q)_n}u_n(A,B;q), \tag{3.16}$$

$$X_{-1}=q\,F(-q^{3/2},A,B;q)=\sum_{n=0}^{\infty}\frac{q^{n^2/2+n+1}}{(q;q)_n}u_n(A,B;q). \tag{3.17}$$

When $B=Aq^{1/2}$, it follows from (1.1) and (2.5) that

$$X_0=\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q^{1/2};q^{1/2})_n}A^n=A_{\sqrt{q}}(-A),\ \ X_{-1}=\sum_{n=0}^{\infty}\frac{q^{n^2/2+n+1}}{(q^{1/2};q^{1/2})_n}A^n=q\,A_{\sqrt{q}}(-qA).$$

There is another representation for X_s which may be of interest. We go back to the generating function (3.13) and expand it using the q-binomial theorem to see that

$$\begin{aligned} X_s=X_0\sum_{n=0}^{s}q^{\binom{n}{2}+(n-s)/2}\sum_{u=0}^{s-n}\begin{bmatrix} n+u\\ u\end{bmatrix}_q\begin{bmatrix} s-u\\ n\end{bmatrix}_q A^uB^{s-n-u} \\ -\frac{AB}{q}X_{-1}\sum_{n=0}^{s-1}q^{\binom{n+1}{2}+(n+1-s)/2}\sum_{u=0}^{s-n-1}\begin{bmatrix} n+u\\ u\end{bmatrix}_q\begin{bmatrix} s-u-1\\ n\end{bmatrix}_q A^uB^{s-n-u-1}. \end{aligned} \tag{3.18}$$

We now come to the case $m = -2s+1, s \in \mathbb{N}$, is odd. Let

$$F(-q^{1-s}, A, B) = Y_s \, q^{-\binom{s}{2}}. \tag{3.19}$$

The functional equation (2.10) yields

$$(q^s + A + B)\, Y_s = Y_{s+1} + ABY_{s-1}. \tag{3.20}$$

This is exactly the recurrence relation (3.10) with $(A, B) \to (\sqrt{q}\, A, \sqrt{q}\, B)$. Therefore (3.18) implies

$$\begin{aligned} Y_s = Y_0 \sum_{n=0}^{s} q^{\binom{n}{2}} \sum_{u=0}^{s-n} \begin{bmatrix} n+u \\ u \end{bmatrix}_q \begin{bmatrix} s-u \\ n \end{bmatrix}_q A^u B^{s-n-u} \\ - AB\, Y_{-1} \sum_{n=0}^{s-1} q^{\binom{n+1}{2}} \sum_{u=0}^{s-n-1} \begin{bmatrix} n+u \\ u \end{bmatrix}_q \begin{bmatrix} s-u-1 \\ n \end{bmatrix}_q A^u B^{s-n-u-1}. \end{aligned} \tag{3.21}$$

In the case $B = Aq^{1/2}$, we conclude from (3.6) that

$$Y_0 = A_{\sqrt{q}}(-A\sqrt{q}), \qquad Y_{-1} = q\, A_{\sqrt{q}}(-Aq^{3/2}). \tag{3.22}$$

It must be noted that the recursions (3.10) and (3.20) have appeared earlier in the work [14] by Ismail and Mulla in the form of orthogonal polynomials generated by

$$p_0(x) = 1, p_1(x) = 2x - a, \quad p_{n+1}(x) = [2x - aq^n]p_n(x) - p_{n-1}(x). \tag{3.23}$$

The authors of [14] referred to these polynomials as the generalized Chebyshev polynomials.

Next, applying Darboux's method, we obtain the asymptotics of X_s and Y_s as $s \to \infty$.

Theorem 3.1. *Let X_s and Y_s be given in (3.18) and (3.21), respectively. When $|A| > |B|$, we have the following asymptotics for X_s and Y_s as $s \to \infty$:*

$$\begin{aligned} X_s \sim A^s q^{-s/2} \Bigg(X_0 \sum_{n=0}^{\infty} \frac{q^{n^2/2} A^{-n}}{(B/A; q)_{n+1}(q; q)_n} \\ - \frac{AB}{q} X_{-1} \sum_{n=0}^{\infty} \frac{q^{(n+1)^2/2} A^{-n-1}}{(B/A; q)_{n+1}(q; q)_n} \Bigg), \end{aligned} \tag{3.24}$$

$$\begin{aligned} Y_s \sim A^s \Bigg(Y_0 \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} A^{-n}}{(B/A; q)_{n+1}(q; q)_n} \\ - AB\, Y_{-1} \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}} A^{-n-1}}{(B/A; q)_{n+1}(q; q)_n} \Bigg). \end{aligned} \tag{3.25}$$

Proof. Recall the explicit expression of the generating function for X_s in (3.13). It is clear that $G(z)$ has simple poles at

$$z = \frac{q^{-j+1/2}}{A} \quad \text{and} \quad z = \frac{q^{-j+1/2}}{B} \qquad \text{for } j = 0, 1, \cdots. \tag{3.26}$$

When $|A| > |B|$, the pole closest to the origin is $q^{1/2}/A$. Then, the comparison function is

$$(1-Azq^{-1/2})^{-1}\left(X_0\sum_{n=0}^{\infty}\frac{q^{n^2/2}A^{-n}}{(B/A;q)_{n+1}(q;q)_n}-\frac{AB}{q}X_{-1}\sum_{n=0}^{\infty}\frac{q^{(n+1)^2/2}A^{-n-1}}{(B/A;q)_{n+1}(q;q)_n}\right). \tag{3.27}$$

By expanding the above formula near $z=0$ and comparing with (3.11), we obtain (3.24). Changing $(A,B)\to(\sqrt{q}\,A,\sqrt{q}\,B)$ gives the formula (3.25). □

We may also consider the recurrence relations (3.10) and (3.20) for X_s and Y_s when $s<0$. For this purpose, let

$$X_s=\left(\frac{AB}{q}\right)^s\widetilde{X}_s \quad\text{and}\quad Y_s=(AB)^s\,\widetilde{Y}_s. \tag{3.28}$$

Substituting the above formula into (3.10) and (3.20) gives us

$$\left[\left(\frac{1}{q}\right)^{-s}+\frac{A+B}{\sqrt{q}}\right]\widetilde{X}_s=\widetilde{X}_{s-1}+\frac{AB}{q}\widetilde{X}_{s+1},$$
$$\left[\left(\frac{1}{q}\right)^{-s}+A+B\right]\widetilde{Y}_s=\widetilde{Y}_{s-1}+AB\widetilde{Y}_{s+1}.$$

Replacing s by $-s$, we have

$$\left[\left(\frac{1}{q}\right)^{s}+\frac{A+B}{\sqrt{q}}\right]\widetilde{X}_{-s}=\widetilde{X}_{-(s+1)}+\frac{AB}{q}\widetilde{X}_{-(s-1)}, \tag{3.29}$$

$$\left[\left(\frac{1}{q}\right)^{s}+A+B\right]\widetilde{Y}_{-s}=\widetilde{Y}_{-(s+1)}+AB\widetilde{Y}_{-(s-1)}. \tag{3.30}$$

Comparing (3.29) with (3.10), they agree with each other through the relation $(A,B)\to(A/q,B/q)$. This, together with (3.18), gives us

$$\begin{aligned}\widetilde{X}_{-s}=\widetilde{X}_0\sum_{n=0}^{s}q^{-\binom{n}{2}+(n-s)/2}\sum_{u=0}^{s-n}\begin{bmatrix}n+u\\u\end{bmatrix}_{1/q}\begin{bmatrix}s-u\\n\end{bmatrix}_{1/q}A^uB^{s-n-u}\\ -\frac{AB}{q}\widetilde{X}_1\sum_{n=0}^{s-1}q^{-\binom{n+1}{2}+(n+1-s)/2}\sum_{u=0}^{s-n-1}\begin{bmatrix}n+u\\u\end{bmatrix}_{1/q}\begin{bmatrix}s-u-1\\n\end{bmatrix}_{1/q}A^uB^{s-n-u-1}.\end{aligned} \tag{3.31}$$

Recalling (3.28) and the relation

$$\begin{bmatrix}n\\k\end{bmatrix}_{1/q}=q^{-k(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q,$$

we obtain

$$\begin{aligned}X_{-s}=X_0\sum_{n=0}^{s}q^{-\binom{n}{2}+(n-s)(n+\frac12)-s}\sum_{u=0}^{s-n}\begin{bmatrix}n+u\\u\end{bmatrix}_q\begin{bmatrix}s-u\\n\end{bmatrix}_qA^{s+u}B^{2s-n-u}\\ -X_1\sum_{n=0}^{s-1}q^{-\binom{n+1}{2}+(n+1-s)(n+\frac12)-s}\sum_{u=0}^{s-n-1}\begin{bmatrix}n+u\\u\end{bmatrix}_q\begin{bmatrix}s-u-1\\n\end{bmatrix}_qA^{s+u}B^{2s-n-u-1}.\end{aligned} \tag{3.32}$$

Similarly, we also have

$$Y_{-s} = Y_0 \sum_{n=0}^{s} q^{-\binom{n}{2}+n(n-s)} \sum_{u=0}^{s-n} \begin{bmatrix} n+u \\ u \end{bmatrix}_q \begin{bmatrix} s-u \\ n \end{bmatrix}_q A^{s+u} B^{2s-n-u}$$
$$- Y_1 \sum_{n=0}^{s-1} q^{-\binom{n+1}{2}+n(n+1-s)} \sum_{u=0}^{s-n-1} \begin{bmatrix} n+u \\ u \end{bmatrix}_q \begin{bmatrix} s-u-1 \\ n \end{bmatrix}_q A^{s+u} B^{2s-n-u-1}. \tag{3.33}$$

4. Zeros

Using the Jacobi triple product identity we write

$$(-1)^n q^{\binom{n}{2}} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(q, z, q/z; q)_\infty}{z^{n+1}}\, dz,$$

where $\mathcal{C}$ is a positively oriented circular contour centered at $z = 0$ and containing the points A and B in its interior. Substituting this in the form (1.2) yields

$$F(w, A, B; q) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{(q, z, q/z; q)_\infty}{(Aw/z, Bw/z; q)_\infty} \frac{dz}{z}, \tag{4.1}$$

where we have made use of the Euler's theorem [12, Theorem 12.2.6]. The above integral gives us another expression of $F(w, A, B; q)$ in the following theorem.

Theorem 4.1. *The function $F(w, A, B; q)$ has the representation*

$$F(w, A, B; q) = \frac{(Aw, \frac{q}{Aw}; q)_\infty}{(B/A; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q, qA/B; q)_n} \left(\frac{q}{Bw}\right)^n$$
$$+ \frac{(Bw, \frac{q}{Bw}; q)_\infty}{(A/B; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q, qB/A; q)_n} \left(\frac{q}{Aw}\right)^n. \tag{4.2}$$

Note that (4.2) can be written in the hypergeometric notation

$$F(w, A, B; q) = \frac{(Aw, \frac{q}{Aw}; q)_\infty}{(B/A; q)_\infty} {}_1\phi_1(0; qA/B; q; \frac{q^2}{Bw})$$
$$+ \frac{(Bw, \frac{q}{Bw}; q)_\infty}{(A/B; q)_\infty} {}_1\phi_1(0; qB/A; q; \frac{q^2}{Aw}). \tag{4.3}$$

Proof of Theorem 4.1. The following standard identity will be used repeatedly in the proof:

$$(cq^{-k}; q)_k = (-1)^k c^k q^{-\binom{k+1}{2}} (q/c; q)_k. \tag{4.4}$$

Let $\mathcal{C}_n$ be a contour centered at $z = 0$ and lies in the interior of $\mathcal{C}$ with radius cq^n. Moreover we assume that neither c/A nor c/B is of the form q^m for any integer m. Let $f(z)$ denote the integrand in (4.1). On the integration contour $\mathcal{C}_n$, we have

$$|f(z)| \le \left| \frac{(q; q)_\infty (z, q^{n+1}/z; q)_\infty}{2\pi (q^n Aw/z, q^n Bw/z; q)_\infty} \right| \cdot \left| \frac{(q/z; q)_n}{(Aw/z, Bw/z; q)_n} \right|.$$

The first factor is clearly bounded and we now show that the second factor tends to zero as $n \to \infty$. Indeed, with $|z| = cq^n$, the second factor is at most

$$\left|\frac{(-q^{1-n}/c;q)_n}{(q^{-n}Aw/c,q^{-n}Bw/c;q)_n}\right| = \left|\frac{(-c;q)_n}{(\frac{qc}{Aw},\frac{qc}{Bw};q)_n}\right| \cdot q^{\binom{n+1}{2}} \cdot \left|\frac{qc}{ABw^2}\right|^n.$$

This shows that $\oint_{\mathcal{C}_n} f(z)dz \to 0$ as $n \to \infty$. Therefore, the integral in (4.1) is the sum of the residues at $z = Awq^n$ and Bwq^n, $n = 0, 1, \cdots$. A residue calculation then establishes (4.2). □

Recall the definitions of the four theta functions [20, §21.3],

$$\vartheta_1(z,q) = 2q^{1/4}\sin z\left(q^2, q^2e^{2iz}, q^2e^{-2iz}; q^2\right)_\infty, \tag{4.5}$$

$$\vartheta_2(z,q) = 2q^{1/4}\cos z\left(q^2, -q^2e^{2iz}, -q^2e^{-2iz}; q^2\right)_\infty, \tag{4.6}$$

$$\vartheta_3(z,q) = \left(q^2, -qe^{2iz}, -qe^{-2iz}; q^2\right)_\infty, \tag{4.7}$$

$$\vartheta_4(z,q) = \left(q^2, qe^{2iz}, qe^{-2iz}; q^2\right)_\infty. \tag{4.8}$$

Moreover with the notations [20, §21.61]

$$k = \vartheta_2^2(0,q)/\vartheta_3^2(0,q), \quad k' = \vartheta_4^2(0,q)/\vartheta_3^2(0,q), \tag{4.9}$$

the Jacobian elliptic functions sn, dn are [20, §22.11-12]:

$$\operatorname{sn}(u\,\vartheta_3^2(0,q),k) := \frac{\vartheta_3(0,q)}{\vartheta_2(0,q)}\frac{\vartheta_1(u,q)}{\vartheta_4(u,q)}, \tag{4.10}$$

$$\operatorname{dn}(u\,\vartheta_3^2(0,q),k) := \frac{\vartheta_4(0,q)}{\vartheta_3(0,q)}\frac{\vartheta_3(u,q)}{\vartheta_4(u,q)}. \tag{4.11}$$

Furthermore, we have [20, §21.11-12]

$$u = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \text{if} \ \ y = \operatorname{sn}(u,k); \tag{4.12}$$

$$u = \int_y^1 \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}, \quad \text{if} \ \ y = \operatorname{dn}(u,k). \tag{4.13}$$

Let us consider the zeros of $F(w,A,B;q^2)$. From (4.3), we have

$$\frac{(A/B, Aw, \frac{q^2}{Aw}; q^2)_\infty}{(B/A, Bw, \frac{q^2}{Bw}; q^2)_\infty} = -\frac{{}_1\phi_1(0; q^2B/A; q^2; \frac{q^4}{Aw})}{{}_1\phi_1(0; q^2A/B; q^2; \frac{q^4}{Bw})}. \tag{4.14}$$

Let

$$B/A = q^m, \qquad \text{with } m \text{ an odd integer}, \tag{4.15}$$

then we have

$$\frac{(q^{-m}, Aw, \frac{q^2}{Aw}; q^2)_\infty}{(q^m, q^mAw, \frac{q^{-m+2}}{Aw}; q^2)_\infty} = -\frac{{}_1\phi_1(0; q^{m+2}; q^2; \frac{q^4}{Aw})}{{}_1\phi_1(0; q^{-m+2}; q^2; \frac{q^{-m+4}}{Aw})}. \tag{4.16}$$

Note that

$$c_m(q) := \frac{(q^{-m};q^2)_\infty}{(q^m;q^2)_\infty} = \begin{cases} \frac{1}{(q^m;q^2)_{-m}} & \text{if } m<0, \\ (q^{-m};q^2)_m & \text{if } m>0. \end{cases} \tag{4.17}$$

Furthermore, we put

$$Aw = q^{-2n+2}e^{2iz} \quad \text{and} \quad \xi_n = e^{2iz}, \tag{4.18}$$

then the left-hand side of (4.16) becomes

$$\begin{aligned} &c_m(q)\cdot\frac{(q^{-2n+2}e^{2iz},q^{2n}e^{-2iz};q^2)_\infty}{(q^{-2n+m+2}e^{2iz},q^{2n-m}e^{-2iz};q^2)_\infty} \\ &= c_m(q)(-1)^{\frac{-m-3}{2}}q^{(n-\frac{m+1}{2})^2-n(n-1)}\,\frac{1-e^{2iz}}{e^{i(-m+1)z}}\cdot\frac{(q^2e^{2iz},q^2e^{-2iz};q^2)_\infty}{(qe^{2iz},qe^{-2iz};q^2)_\infty}. \end{aligned} \tag{4.19}$$

From (4.10), the last term in the above formula yields

$$\begin{aligned} \frac{(q^2e^{2iz},q^2e^{-2iz};q^2)_\infty}{(qe^{2iz},qe^{-2iz};q^2)_\infty} &= \frac{1}{2q^{1/4}\sin z}\frac{\vartheta_1(z,q)}{\vartheta_4(z,q)} \\ &= \frac{1}{2q^{1/4}\sin z}\frac{\vartheta_2(0,q)}{\vartheta_3(0,q)}\operatorname{sn}(z\,\vartheta_3^2(0,q),k). \end{aligned} \tag{4.20}$$

Combining (4.16), (4.19) and the above formula, we have

$$\begin{aligned} \operatorname{sn}(z\,\vartheta_3^2(0,q),k) &= (-1)^{\frac{-m-1}{2}}\frac{iq^{1/4}e^{-imz}}{c_m(q)q^{(n-\frac{m+1}{2})^2-n(n-1)}}\frac{\vartheta_3(0,q)}{\vartheta_2(0,q)}\frac{{}_1\phi_1(0;q^{m+2};q^2;\frac{q^4}{Aw})}{{}_1\phi_1(0;q^{-m+2};q^2;\frac{q^{-m+4}}{Aw})} \\ &= (-1)^{\frac{-m-1}{2}}\frac{iq^{mn-m(2+m)/4}e^{-imz}}{c_m(q)}\frac{\vartheta_3(0,q)}{\vartheta_2(0,q)}\frac{{}_1\phi_1(0;q^{m+2};q^2;\frac{q^4}{Aw})}{{}_1\phi_1(0;q^{-m+2};q^2;\frac{q^{-m+4}}{Aw})}. \end{aligned} \tag{4.21}$$

Typically, when $m=1$, we have $c_1(q) = 1-q^{-1}$. Then, the above formula reduces to

$$\operatorname{sn}(z\,\vartheta_3^2(0,q),k) = \frac{iq^{n+1/4}e^{-iz}}{1-q}\frac{\vartheta_3(0,q)}{\vartheta_2(0,q)}\frac{{}_1\phi_1(0;q^3;q^2;\frac{q^4}{Aw})}{{}_1\phi_1(0;q;q^2;\frac{q^3}{Aw})}, \tag{4.22}$$

which is similar to [15, eq. (3.8)]. The next theorem follows immediately from (4.12) and (4.21).

Theorem 4.2. *Let ξ_n be given in (4.18), which are the (scaled) zeros of $F(w,A,Aq^m;q^2)$ for odd m. Then, ξ_n satisfies the following integral equation*

$$\ln\xi_n = -2\int_0^{q^{mn-m(2+m)/4}\phi(\xi_n)}\frac{dt}{\sqrt{(1+a^2t^2)(1+b^2t^2)}} \tag{4.23}$$

with

$$a = \vartheta_3^2(0,q), \qquad b = \vartheta_2^2(0,q), \tag{4.24}$$

$$\phi(\xi_n) = (-1)^{\frac{-m-1}{2}}\frac{\xi_n^{-m/2}}{c_m(q)\vartheta_2(0,q)\vartheta_3(0,q)}\frac{{}_1\phi_1(0;q^{m+2};q^2;q^{2n+2}/\xi_n)}{{}_1\phi_1(0;q^{-m+2};q^2;q^{2n-m+2}/\xi_n)}. \tag{4.25}$$

Denote

$$\xi_n = \eta^{-2}, \quad Z = 1/\sqrt{ab}, \quad \text{and } L = a/b + b/a. \tag{4.26}$$

We then have

$$\ln \eta = \int_0^{\alpha} \frac{Z dt}{\sqrt{1 + Lt^2 + t^4}},$$

where

$$\alpha = \frac{(-1)^{(m+1)/2} q^{nm - m(m+2)/4} {}_1\phi_1(0; q^{m+2}; q^2; q^{2+2n}/\xi_n)}{\sqrt{\xi_n^m} c_m(q) {}_1\phi_1(0; q^{2-m}; q^2; q^{2-m+2n}/\xi_n)} = \sum_{j=0}^{\infty} \alpha_j (q^n \eta)^{2j+m}. \tag{4.27}$$

The coefficients α_j are rational functions in q. Now, we set

$$\exp\left[\int_0^{\alpha} \frac{Z dt}{\sqrt{1 + Lt^2 + t^4}}\right] = \sum_{k=0}^{\infty} h_k \alpha^k, \tag{4.28}$$

where h_k are polynomials in Z and L. It follows that

$$\eta = \sum_{k=0}^{\infty} h_k \left[\sum_{j=0}^{\infty} \alpha_j (q^n \eta)^{2j+m}\right]^k.$$

We further let

$$\eta = \sum_{l=0}^{\infty} \eta_l q^{nl}, \quad \left(\sum_{l=0}^{\infty} \eta_l q^{nl}\right)^{2j+mk} = \sum_{l=0}^{\infty} \eta_l^{(2j+mk)} q^{nl},$$

and

$$\left(\sum_{j=0}^{\infty} \alpha_j z^j\right)^k = \sum_{j=0}^{\infty} \alpha_j^{(k)} z^j,$$

where the coefficients $\alpha_j^{(k)}$ are rational functions in q. We then have

$$\eta = 1 + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} h_k \alpha_j^{(k)} (q^n \eta)^{2j+mk} = 1 + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} h_k \alpha_j^{(k)} \eta_l^{(2j+mk)} q^{n(2j+mk+l)}.$$

This implies $\eta_0 = 1$ and

$$\eta_s = \sum_{\substack{k \geq 1, j \geq 0, l \geq 0 \\ 2j+mk+l=s}} h_k \alpha_j^{(k)} \eta_l^{(2j+mk)}. \tag{4.29}$$

By induction, we learn that η_s are polynomials in Z and L with coefficients being rational functions in q. Consequently, we have

$$\xi_n = \left(\sum_{l=0}^{\infty} \eta_l q^{nl}\right)^{-2} = \sum_{l=0}^{\infty} \eta_l^{(-2)} q^{nl},$$

where $\eta_l^{(-2)}$ are also polynomials in Z and L with coefficients being rational functions in q. It can be easily calculated from (4.27), (4.28) and (4.29) that

$$\alpha_0 = \frac{(-1)^{(m+1)/2} q^{-m(m+2)/4}}{c_m(q)}, \; \alpha_1 = \alpha_0 \left[\frac{q^2}{(1-q^{m+2})(1-q^2)} - \frac{q^{2-m}}{(1-q^{2-m})(1-q^2)}\right], \cdots$$
$$h_0 = 1, \; h_1 = Z, \; h_2 = Z^2/2, \; h_3 = Z(Z^2 - L)/6, \; h_4 = Z^2(Z^2 - 4L)/24, \cdots;$$
$$\eta_0 = 1, \; \eta_1 = \cdots = \eta_{m-1} = 0, \; \eta_m = h_1\alpha_0, \; \eta_{m+2k-1} = 0, \; \eta_{m+2k} = h_1\alpha_l, \; 0 < k < m/2.$$

Similar to Theorem 3.4 and Theorem 4.1 in Ismail and C. Zhang [15], we have the following theorem.

Theorem 4.3. *Let $m > 0$ be an odd integer, then ξ_n is an analytic function of q^n and has the Taylor series expansion*

$$\xi_n = 1 + \sum_{j=1}^{\infty} d_j q^{jn}, \tag{4.30}$$

where $d_j = \eta_j^{(-2)}$ are polynomials in Z and L with coefficients being rational functions in q.

It is easily seen from the above two theorems that $d_1 = \cdots = d_{m-1} = 0$ and

$$d_m = 2(-1)^{\frac{-m+1}{2}} \frac{q^{-m(2+m)/4}}{(q^{-m}; q^2)_m} \frac{1}{\vartheta_2(0,q)\vartheta_3(0,q)}. \tag{4.31}$$

When $m = 1$, this agrees with [15, eq. (3.14)]. We may also compute the following a few coefficients:

$$\begin{aligned}
d_{m+1} &= 0, \\
d_{m+2} &= 2(-1)^{\frac{-m-1}{2}} \frac{q^{-m(2+m)/4}}{(1-q^2)\,(q^{-m}; q^2)_m} \frac{1}{\vartheta_2(0,q)\vartheta_3(0,q)} \left(\frac{1}{1-q^{m-2}} + \frac{q^2}{1-q^{m+2}}\right), \\
d_{m+3} &= 0, \\
d_{m+4} &= 2(-1)^{\frac{-m+1}{2}} \frac{q^{-m(2+m)/4}}{(1-q^2)^2\,(q^{-m}; q^2)_m} \frac{1}{\vartheta_2(0,q)\vartheta_3(0,q)} \\
&\quad \left(\frac{1}{(1-q^{m-2})^2} - \frac{1}{(1+q^2)(1-q^{m-2})(1-q^{m-4})}\right. \\
&\quad \left. + \frac{q^2}{(1-q^{m-2})(1-q^{m+2})} + \frac{q^6}{(1+q^2)(1-q^{m+2})(1-q^{m+4})}\right), \\
d_{2m} &= \frac{1-m}{2} d_m^2.
\end{aligned}$$

Remark 4.4. Note that the coefficients d_j satisfy similar structure as that in [15, Thm 4.1], namely they are polynomials in terms of Z and L given in (4.26) with coefficients rational in q. It is worthwhile to point out that combinatorial interpretations of the coefficients in [15, Thm 4.1] have been found by Huber [8] and Huber and Yee [9]. We expect some elaborate combinatorial interpretation for the coefficients d_j may also be possible. Moreover, from the above calculation, the formula of d_j depends on the residue of j modulo m, which indicates that certain sieving process may be involved.

5. An integral equation

Let

$$K(w,x) = F(-w, e^{-i\theta}, e^{i\theta}; q) = \sum_{n=0}^{\infty} \frac{w^n q^{\binom{n}{2}}}{(q;q)_n} H_n(x|q) \tag{5.1}$$

with $x = \cos\theta$. We have the following integral equation for $K(w,x)$.

Theorem 5.1. *The function $K(w,x)$ defined above satisfies the integral equation*

$$K(st, \cos\theta) = \frac{(t^2;q)_\infty (q;q)_\infty}{2\pi} \int_0^\pi \frac{K(s,\cos\phi)(e^{2i\phi}, e^{-2i\phi};q)_\infty \, d\phi}{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)}, te^{-i(\theta-\phi)};q)_\infty}. \tag{5.2}$$

Proof. We recall the Poisson kernel of the q-Hermite polynomials [12, (13.1.24)]

$$\sum_{n=0}^{\infty} \frac{H_n(\cos\theta \,|\, q) H_n(\cos\phi \,|\, q)}{(q;q)_n} t^n = \frac{(t^2;q)_\infty}{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}, te^{-i(\theta+\phi)}, te^{-i(\theta-\phi)};q)_\infty} \tag{5.3}$$

and their orthogonality relation

$$\int_{-1}^{1} H_m(x \,|\, q) H_n(x \,|\, q) w(x \,|\, q)\, dx = \frac{2\pi (q;q)_n}{(q;q)_\infty} \delta_{m,n}, \tag{5.4}$$

where

$$w(x \,|\, q) = \frac{(e^{2i\phi}, e^{-2i\phi};q)_\infty}{\sqrt{1-x^2}}, \quad x = \cos\phi,\ 0 \le \phi \le \pi. \tag{5.5}$$

Note that, for fixed $\theta \in (0,\pi)$, both $K(s,\cos\theta)$ and the Poisson kernel are in the space $L^2[w(x \,|\, q); [-1,1]]$ with $x = \cos\phi$. As the q-Hermite polynomials are complete in $L^2[w(x \,|\, q); [-1,1]]$, we have (5.2) from the Parseval's theorem. □

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