J. Math. Anal. Appl. 474 (2019) 1178–1197

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On quasi-orthogonal polynomials: Their differential equations, discriminants and electrostatics

Mourad E.H. Ismail [a], Xiang-Sheng Wang [b,*]

[a] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA*
[b] *Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70503, USA*

A R T I C L E   I N F O

Article history:
Received 25 August 2018
Available online 7 February 2019
Submitted by M.J. Schlosser

This paper is dedicated to our good friend Frank Stenger on the occasion of his 80th birthday

Keywords:
Quasi-orthogonal polynomials
Discriminant
Electrostatics
Hamburger moment problem
Nonlinear orthogonality preserving transformation

A B S T R A C T

In this paper, we develop a general theory of quasi-orthogonal polynomials. We first derive three-term recurrence relation and second-order differential equations for quasi-orthogonal polynomials. We also give an expression for their discriminants in terms of the recursion coefficients of the corresponding orthogonal polynomials. In addition, we investigate an electrostatic equilibrium problem where the equilibrium position of movable charges is attained at the zeros of the quasi-orthogonal polynomials. The examples of the Freud weight $w(x) = e^{-x^4+2tx^2}$ and the Jacobi weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ are discussed in some detail. Finally, we consider the nonlinear orthogonality preserving transformation and related matrix problem.

1. Introduction

Let $\{p_n(x) = \gamma_n x^n + \cdots\}$ be the orthonormal polynomials with respect to the weight $w(x) = e^{-v(x)}$ on an interval $[a, b]$. Throughout this paper, we assume that v is real and differentiable in (a, b), and all moments of the weight exist. By orthogonality, we have the three-term recurrence relation:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + \alpha_n p_n(x) + a_n p_{n-1}(x),\ n \geq 1, \tag{1.1}$$

with initial conditions: $p_0(x) = \gamma_0$ and $p_1(x) = \gamma_1(x - \alpha_0)$, where

$$a_n = \int_a^b xp_n(x)p_{n-1}(x)w(x)dx = \frac{\gamma_{n-1}}{\gamma_n},\ n \geq 1,$$

* Corresponding author.
E-mail addresses: mourad.eh.ismail@gmail.com (M.E.H. Ismail), xswang@louisiana.edu (X.-S. Wang).

https://doi.org/10.1016/j.jmaa.2019.02.011

$$\alpha_n = \int_a^b x p_n^2(x) w(x) dx, \ n \geq 0.$$

It is noted that the zeros of orthogonal polynomials are real and simple [38, Theorem 3.3.1], and the zeros of p_n and p_{n+1} interlace [38, Theorem 3.3.2]. Recall that the associated polynomials $p_k^{(n)}(x)$ corresponding to the recursion (1.1) are generated by $p_0^{(n)}(x) = 1, p_1^{(n)}(x) = (x - \alpha_n)/a_n$, and

$$x p_k^{(n)}(x) = a_{n+k+1} p_{k+1}^{(n)}(x) + \alpha_{n+k} p_k^{(n)}(x) + a_{n+k} p_{k-1}^{(n)}(x), \ k \geq 1. \tag{1.2}$$

Chen and Ismail [10] derived the lowering relation

$$p_n'(x) = -B_n(x) p_n(x) + A_n(x) p_{n-1}(x), \tag{1.3}$$

where

$$\frac{A_n(x)}{a_n} = \frac{w(y) p_n^2(y)}{y - x}\Big|_a^b + \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n^2(y) w(y) dy,$$

$$\frac{B_n(x)}{a_n} = \frac{w(y) p_n(y) p_{n-1}(y)}{y - x}\Big|_a^b + \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n(y) p_{n-1}(y) w(y) dy.$$

For convenience, we define a linear operator

$$L(f)(x) := \frac{w(y) f(y)}{y - x}\Big|_a^b + \int_a^b \frac{v'(x) - v'(y)}{x - y} f(y) w(y) dy. \tag{1.4}$$

It is readily seen that $A_n(x) = L(a_n p_n^2)(x)$ and $B_n(x) = L(a_n p_n p_{n-1})(x)$.

Stieltjes [36,37] found an electrostatic interpretation of the zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ with $\alpha > -1$ and $\beta > -1$. Place n unit movable charges randomly in the interval $(-1, 1)$, and two fixed charges $(1+\alpha)/2$ and $(1+\beta)/2$ at the end points 1 and -1, respectively. Assume the interaction force between charges e_1 and e_2 with a distance r apart is $2e_1e_2/r$. Then the equilibrium for the n movable charges is achieved at the zeros of $P_n^{(\alpha,\beta)}(x)$. To evaluate the energy at equilibrium one needs to compute the discriminant of $P_n^{(\alpha,\beta)}(x)$, which Stieltjes did in [37] and Hilbert later gave another proof in [24]. For an exposition of these results we refer the interested reader to [38, §6.7] or [26, Chapter 3]. A more general model for all polynomials orthogonal with respect to a weight of the form $w(x) = e^{-v(x)}$ was formulated by Ismail [25], which used the author's earlier evaluation of discriminants of these types of orthogonal polynomials. The key relations used in this approach are the lowering relation (1.3) and its adjoint (the raising relation). The details are available in [26, Chapter 3].

More recently, Dillcher and Stolarsky [13] used algebraic methods to evaluate the resultant of two linear combinations of Chebyshev polynomials of the second kind. Gishe and Ismail [20] gave an alternative method of computing the same resultant and resultants of more general combinations of Chebyshev polynomials of the first and second kinds. This work was generalized recently in [32] to quasi-Jacobi polynomials. The concept of quasi-orthogonal polynomials was introduced by Chihara in [11]. Starting with a sequence of orthonormal polynomials we generate quasi-orthogonal polynomials recursively by

$$q_n(x) = p_n(x) + c_n p_{n-1}(x), \tag{1.5}$$

where c_n is any given constant which may depend on n, and $q_0(x) = p_0(x)$. Dickinson [12] proved that there exists a sequence T_k such that

$$T_n p_n(x) = T_0 q_0(x) + \cdots + T_n q_n(x) \quad \text{for all } n \geq 0. \tag{1.6}$$

By [38, Theorem 3.3.4], $q_n(x)$ has n distinct real roots. Some properties of the zeros for quasi-orthogonal polynomials with respect to classical weights have been studied in [8,14–18,29]. A sample of an applied problem using quasi-orthogonal polynomials is in [42]. We refer to [21,39–41] and references therein for applications of quasi-orthogonal polynomials in quadrature and interpolation. Some studies on orthogonality of quasi-orthogonal polynomials when the sum in (1.6) has fixed length are given in [2,7].

In the present paper we provide a detailed study on quasi-orthogonal polynomials regarding their differential equations, discriminants and electrostatics. In Section 2 we derive a three-term recurrence relation for the quasi-orthogonal polynomials $\{q_n(x)\}$. We also find an inverse to (1.5) where we express $p_n(x)$ in terms of $q_n(x)$ and $q_{n-1}(x)$ and identify a lowering operator for the q_n's and use it to find a differential equation satisfied by the quasi-orthogonal polynomials. In Section 3 we evaluate the discriminant of quasi-orthogonal polynomials in terms of the recursion coefficients of the p_n's. Section 4 contains an electrostatic equilibrium model of n unit charges allowed to move on a straight line that are subjected to an external field. The equilibrium position is at the zeros of $q_n(x)$. Section 5 contains two important examples of the quasi-orthogonal polynomials associated with Freud and Jacobi weights.

The problem of characterizing orthogonal polynomials of a certain type has attracted the attention of many mathematicians. Al-Salam's interesting article [4] surveys the literature on the subject at the time of its writing. In Section 6, we give a necessary and sufficient condition, on c_n, for q_n to become orthogonal polynomials. Determining the sequence c_n which makes q_n orthogonal on $\mathbb{R}$ turned out to be related to the moment problems of the associated orthogonal polynomials of positive integer order of the polynomials $\{p_n\}$. The standard references to the moment problems are [1], [34]. The recent book [33] incorporates some of the recent results. Simon's article [35] explored the moment problem as a problem in spectral theory of operators. We also reformulate our results as a nonlinear problem in functional analysis. In other words we rephrase the problem of finding all the p_n's which make the q_n's orthogonal in terms of an equivalent spectral problem.

It must be noted that Section 6 provides an orthogonality preserving nonlinear transformation. It is expected to have applications within integrable systems.

2. Difference and differential equations

It follows from (1.1) and (1.5) that

$$q_{n-1}(x) = p_{n-1}(x) + c_{n-1}p_{n-2}(x) = p_{n-1}(x) + \frac{c_{n-1}}{a_{n-1}}[-a_n p_n(x) + (x - \alpha_{n-1})p_{n-1}(x)],$$

which, together with (1.5), implies the inverse relation

$$\begin{pmatrix} p_n(x) \\ p_{n-1}(x) \end{pmatrix} = \begin{pmatrix} \frac{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}}{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}+c_{n-1}c_n a_n} & \frac{-c_n a_{n-1}}{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}+c_{n-1}c_n a_n} \\ \frac{c_{n-1}a_n}{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}+c_{n-1}c_n a_n} & \frac{a_{n-1}}{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}+c_{n-1}c_n a_n} \end{pmatrix} \begin{pmatrix} q_n(x) \\ q_{n-1}(x) \end{pmatrix}.$$

The above equation can be used to express the recurrence relation

$$a_{n+1}q_{n+1}(x) = a_{n+1}p_{n+1}(x) + c_{n+1}a_{n+1}p_n(x) = (x - \alpha_n + c_{n+1}a_{n+1})p_n(x) - a_n p_{n-1}(x)$$

in terms of the q_n's. The result is

$$\begin{aligned}a_{n+1}q_{n+1}(x) =&(x-\alpha_n+c_{n+1}a_{n+1})\frac{(a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1})q_n(x)-c_na_{n-1}q_{n-1}(x)}{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}+c_{n-1}c_na_n}\\&-a_n\frac{c_{n-1}a_nq_n(x)+a_{n-1}q_{n-1}(x)}{a_{n-1}+c_{n-1}x-c_{n-1}\alpha_{n-1}+c_{n-1}c_na_n}.\end{aligned}$$

A further simplification yields the more compact form

$$a_{n+1}l_{n-1}(x)q_{n+1}(x)=[r_n(x)l_{n-1}(x)-c_{n-1}a_nl_n(x)]q_n(x)-a_{n-1}l_n(x)q_{n-1}(x), \tag{2.1}$$

where $r_n(x)=x-\alpha_n+c_{n+1}a_{n+1}$ and $l_n(x)=c_nr_n(x)+a_n$.

On account of (1.3), we express $q_n'(x)$ in terms of $p_n(x)$ and $p_{n-1}(x)$,

$$\begin{aligned}q_n'(x)&=p_n'(x)+c_np_{n-1}'(x)\\&=-B_n(x)p_n(x)+A_n(x)p_{n-1}(x)-c_nB_{n-1}(x)p_{n-1}(x)+c_nA_{n-1}(x)p_{n-2}(x)\\&=-\frac{a_{n-1}B_n(x)+c_na_nA_{n-1}(x)}{a_{n-1}}p_n(x)\\&\quad+\frac{a_{n-1}A_n(x)-c_na_{n-1}B_{n-1}(x)+c_nxA_{n-1}(x)-c_n\alpha_{n-1}A_{n-1}(x)}{a_{n-1}}p_{n-1}(x).\end{aligned}$$

Next, we shall express $q_n'(x)$ in terms of $q_n(x)$ and $p_{n-1}(x)$:

$$\begin{aligned}q_n'(x)=&-\frac{a_{n-1}B_n(x)+c_na_nA_{n-1}(x)}{a_{n-1}}q_n(x)\\&+\frac{a_{n-1}[A_n(x)+c_nB_n(x)-c_nB_{n-1}(x)]+(c_n^2a_n+c_nx-c_n\alpha_{n-1})A_{n-1}(x)}{a_{n-1}}p_{n-1}(x).\end{aligned} \tag{2.2}$$

Note that

$$\frac{a_{n-1}B_n(x)+c_na_nA_{n-1}(x)}{a_{n-1}}=L(a_np_np_{n-1}+c_na_np_{n-1}^2)(x)=L(a_nq_np_{n-1})(x),$$

and

$$\begin{aligned}&\frac{a_{n-1}[A_n(x)+c_nB_n(x)-c_nB_{n-1}(x)]+(c_n^2a_n+c_nx-c_n\alpha_{n-1})A_{n-1}(x)}{a_{n-1}}\\=&L(a_np_n^2+c_na_np_np_{n-1}-c_na_{n-1}p_{n-1}p_{n-2}+c_n^2a_np_{n-1}^2+c_n(x-\alpha_{n-1})p_{n-1}^2)(x)\\=&L(a_np_n^2+c_na_np_np_{n-1}+c_n^2a_np_{n-1}^2+c_na_np_np_{n-1}+c_n(x-\cdot)p_{n-1}^2)(x)\\=&L(a_nq_n^2)(x)+c_nv'(x).\end{aligned}$$

We can rewrite (2.2) as

$$q_n'=-L(a_nq_np_{n-1})q_n+[L(a_nq_n^2)+c_nv']p_{n-1}=-\bar{B}_nq_n+(\bar{A}_n+c_nv')p_{n-1}, \tag{2.3}$$

where $\bar{A}_n=L(a_nq_n^2)$ and $\bar{B}_n=L(a_nq_np_{n-1})$. For the special case $c_n=0$, we have $\bar{A}_n=A_n$ and $\bar{B}_n=B_n$. Actually, we have following relations:

$$\frac{\bar{B}_n}{a_n}=L(q_np_{n-1})=L(p_np_{n-1})+c_nL(p_{n-1}^2)=\frac{B_n}{a_n}+\frac{c_nA_{n-1}}{a_{n-1}}, \tag{2.4}$$

$$\frac{\bar{A}_n}{a_n}=L(q_n^2)=L(p_n^2)+c_nL(p_np_{n-1})+c_nL(q_np_{n-1})=\frac{A_n}{a_n}+\frac{c_nB_n}{a_n}+\frac{c_n\bar{B}_n}{a_n}. \tag{2.5}$$

Now, we take a further differentiation on (2.3) to obtain

$$\begin{aligned}q_n'' =& -\bar{B}_n' q_n + (\bar{A}_n' + c_n v'')p_{n-1} - \bar{B}_n[-\bar{B}_n q_n + (\bar{A}_n + c_n v')p_{n-1}] \\ & + (\bar{A}_n + c_n v')[-B_{n-1}p_{n-1} + \frac{A_{n-1}}{a_{n-1}}(-a_n(q_n - c_n p_{n-1}) + (x-\alpha_{n-1})p_{n-1})] \\ =& [-\bar{B}_n' + \bar{B}_n^2 - (\bar{A}_n + c_n v')A_{n-1}\frac{a_n}{a_{n-1}}]q_n \\ & + [\bar{A}_n' + c_n v'' + (\bar{A}_n + c_n v')(-\bar{B}_n - B_{n-1} + \frac{A_{n-1}}{a_{n-1}}(c_n a_n + x - \alpha_{n-1}))]p_{n-1}.\end{aligned}$$

Note that

$$\begin{aligned}& -\bar{B}_n - B_{n-1} + \frac{A_{n-1}}{a_{n-1}}(c_n a_n + x - \alpha_{n-1}) \\ =& L(-a_n q_n p_{n-1} - a_{n-1}p_{n-1}p_{n-2} + p_{n-1}^2(c_n a_n + x - \alpha_{n-1})) \\ =& L(-a_n q_n p_{n-1} + c_n a_n p_{n-1}^2 + (x - \cdot)p_{n-1}^2 + a_n p_n p_{n-1}) = v'.\end{aligned}$$

We obtain

$$q_n'' = [-\bar{B}_n' + \bar{B}_n^2 - (\bar{A}_n + c_n v')A_{n-1}\frac{a_n}{a_{n-1}}]q_n + [\bar{A}_n' + c_n v'' + (\bar{A}_n + c_n v')v']p_{n-1}. \tag{2.6}$$

We rewrite (2.3) and (2.6) in matrix form,

$$\begin{pmatrix} q_n' \\ q_n'' \end{pmatrix} = \begin{pmatrix} -\bar{B}_n & \bar{A}_n + c_n v' \\ -\bar{B}_n' + \bar{B}_n^2 - (\bar{A}_n + c_n v')A_{n-1}a_n/a_{n-1} & \bar{A}_n' + c_n v'' + (\bar{A}_n + c_n v')v' \end{pmatrix} \begin{pmatrix} q_n \\ p_{n-1} \end{pmatrix}.$$

We may solve q_n and p_{n-1} in terms of q_n' and q_n'' by taking inverse of the coefficient matrix in the above equation. Especially, we obtain the second-order differential equation

$$(\bar{A}_n + c_n v')q_n'' - [\bar{A}_n' + c_n v'' + (\bar{A}_n + c_n v')v']q_n' + \Delta q_n = 0,$$

where

$$\Delta = -\bar{B}_n[\bar{A}_n' + c_n v'' + (\bar{A}_n + c_n v')v'] - (\bar{A}_n + c_n v')[-\bar{B}_n' + \bar{B}_n^2 - (\bar{A}_n + c_n v')A_{n-1}\frac{a_n}{a_{n-1}}].$$

A further simplification gives

$$q_n'' - (\frac{\bar{A}_n' + c_n v''}{\bar{A}_n + c_n v'} + v')q_n' + [\bar{B}_n' - \bar{B}_n^2 - \bar{B}_n(\frac{\bar{A}_n' + c_n v''}{\bar{A}_n + c_n v'} + v') + (\bar{A}_n + c_n v')A_{n-1}\frac{a_n}{a_{n-1}}]q_n = 0. \tag{2.7}$$

Especially, when $c_n = 0$, we have

$$p_n'' - (\frac{A_n'}{A_n} + v')p_n' + [B_n' - B_n^2 - B_n(\frac{A_n'}{A_n} + v') + A_n A_{n-1}\frac{a_n}{a_{n-1}}]p_n = 0,$$

which agrees with [26, (3.2.12)–(3.2.14)]. Now, we denote $V_n(x) = v(x) + \ln \frac{\bar{A}_n(x) + c_n v(x)}{a_n}$. By variation of parameters, we note that a general solution of the differential equation (2.7) should be a linear combination of $q_n(x)$ and $f(x)q_n(x)$, where $f'(x) = e^{V_n(x)}[q_n(x)]^{-2}$. Actually, if $w(x)$ vanishes at the end points a and b and it can be analytically continued to the complex plane, we follow the idea in [26, §3.6] to introduce the function of the second kind:

$$Q_n(x) = \frac{1}{w(x)} \int_a^b \frac{q_n(y)w(y)}{x-y} dy, \quad x \notin [a,b]. \tag{2.8}$$

Theorem 2.1. *Assume that $w(x)$ vanishes at the end points a and b and it can be analytically continued to the complex plane. The function of the second kind $Q_n(x)$ defined in (2.8) satisfies the differential equation (2.7) for x not in $[a,b]$. Moreover, the sequence of functions $Q_n(x)$ satisfies the recurrence relation (2.1) for $n \geq 2$.*

Proof. Let h be any polynomial of degree no more than m. We have by orthogonality

$$\int_a^b \frac{[h(x)-h(y)]p_m(y)w(y)}{x-y} dy = 0. \tag{2.9}$$

We will make frequent uses of the above identity in this proof. An integration by parts gives

$$[w(x)Q_n(x)]' = -\int_a^b \frac{q_n(y)w(y)}{(x-y)^2} dy = \int_a^b \frac{q_n'(y)w(y)+q_n(y)w'(y)}{x-y} dy.$$

Since $w'(x) = -v'(x)w(x)$, we have

$$\begin{aligned} w(x)[Q_n'(x)q_n(x) - Q_n(x)q_n'(x)] &= [w(x)Q_n(x)]'q_n(x) - w(x)Q_n(x)[q_n'(x) - q_n(x)v'(x)] \\ = \int_a^b \frac{q_n(x)[q_n'(y)-q_n(y)v'(y)]w(y)}{x-y} dy &- \int_a^b \frac{q_n(y)[q_n'(x)-q_n(x)v'(x)]w(y)}{x-y} dy \\ = \int_a^b \frac{q_n'(y)[q_n(x)-q_n(y)]w(y)}{x-y} dy &+ \int_a^b \frac{q_n(x)q_n(y)[v'(x)-v'(y)]w(y)}{x-y} dy, \end{aligned}$$

where, in the last step, we have made use of (2.9) with $h(x) = q_n'(x)$ and $m = n, n-1$ respectively. We make another integration by parts, and use the fact that $q_n(y)$ is orthogonal with the derivative of the polynomial $[q_n(x) - q_n(y)]/(x-y)$ when x is treated as a parameter, to obtain

$$\begin{aligned} & w(x)[Q_n'(x)q_n(x) - Q_n(x)q_n'(x)] \\ &= \int_a^b \frac{q_n(y)[q_n(x)-q_n(y)]v'(y)w(y)}{x-y} dy + \int_a^b \frac{q_n(x)q_n(y)[v'(x)-v'(y)]w(y)}{x-y} dy \\ &= \int_a^b \frac{q_n^2(y)[v'(x)-v'(y)]w(y)}{x-y} dy + \int_a^b \frac{[q_n(x)-q_n(y)]q_n(y)v'(x)w(y)}{x-y} dy \end{aligned}$$

The first integral in the last expression is $\bar{A}_n/a_n$. Since $[q_n(x)-q_n(y)]/(x-y)$ is a polynomial of degree $n-1$ with leading coefficient γ_n, we obtain from orthogonality that the second integral in the last expression is $c_n v'(x)\gamma_n/\gamma_{n-1} = c_n v'(x)/a_n$. Thus, we have

$$w(x)[Q_n'(x)q_n(x) - Q_n(x)q_n'(x)] = \frac{\bar{A}_n + c_n v'(x)}{a_n},$$

which, by a simple calculation, implies that

$$\frac{d}{dx}\left[\frac{Q_n(x)}{q_n(x)}\right] = \frac{e^{V_n(x)}}{q_n^2(x)}.$$

Thus, $Q_n(x)$ is a solution to the differential equation (2.7).

To show that $Q_n(x)$ also satisfies the recurrence relation (2.1), we note from (2.9) that

$$x^k Q_n(x) = \frac{x^k}{w(x)} \int_a^b \frac{q_n(y)w(y)}{x-y} dy = \frac{1}{w(x)} \int_a^b \frac{y^k q_n(y)w(y)}{x-y} dy$$

for any $n > k$. It is readily seen that

$$\begin{aligned} a_{n+1}l_{n-1}(x)Q_{n+1}(x) &= \frac{1}{w(x)} \int_a^b \frac{a_{n+1}l_{n-1}(y)q_{n+1}(y)}{x-y} w(y)dy \\ =& \frac{1}{w(x)} \int_a^b \frac{[r_n(y)l_{n-1}(y) - c_{n-1}a_n l_n(y)]q_n(y) - a_{n-1}l_n(y)q_{n-1}(y)}{x-y} w(y)dy \\ =& [r_n(x)l_{n-1}(x) - c_{n-1}a_n l_n(x)]Q_n(x) - a_{n-1}l_n(x)Q_{n-1}(x) \end{aligned}$$

for all $n \geq 3$. When $n = 2$, we subtract the left-hand side from the right-hand side of (2.1) and make use of (2.9) to express the difference as:

$$\frac{1}{w(x)} \int_a^b \frac{[r_2(x)l_1(x) - r_2(y)l_1(y)]c_2p_1(y) - a_1[l_2(x) - l_2(y)]c_1p_0(y)}{x-y} w(y)dy.$$

Recall that $r_n(x) = x - \alpha_n + c_{n+1}a_{n+1}$ and $l_n(x) = c_n r_n(x) + a_n$. This quantity equals

$$\frac{1}{w(x)} \int_a^b [c_1(x+y)c_2p_1(y) - a_1c_2c_1p_0(y)]w(y)dy = \frac{c_1c_2/\gamma_1 - a_1c_2c_1/\gamma_0}{w(x)} = 0$$

because $a_1 = \gamma_0/\gamma_1$. Hence, $Q_n(x)$ also satisfies (2.1) for $n = 2$. This completes the proof. □

3. Discriminants

By [38, Theorem 3.3.4], $q_n(x)$ has n distinct real roots, denoted by $x_1, \cdots, x_n$. The discriminant of $q_n(x)$ is defined as [26, (3.1.8)]

$$D(q_n) = \gamma_n^{2n-2} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 = (-1)^{n(n-1)/2} \gamma_n^{n-2} \prod_{j=1}^n q_n'(x_j).$$

In view of $q_n(x_j) = 0$, we substitute (2.2) into the above equation to obtain

$$D(q_n) = (-1)^{n(n-1)/2} \gamma_n^{n-2} \prod_{j=1}^n p_{n-1}(x_j)$$

$$\times \prod_{j=1}^{n} \frac{a_{n-1}[A_n(x_j) + c_n B_n(x_j) - c_n B_{n-1}(x_j)] + (c_n^2 a_n + c_n x_j - c_n \alpha_{n-1}) A_{n-1}(x_j)}{a_{n-1}}.$$

Now, we let $\pi_n(x) = p_n(x)/\gamma_n$ be the monic orthogonal polynomials and write $q_n(x)$ in terms of $\pi_{n-1}(x)$ and $\pi_{n-2}(x)$:

$$\begin{aligned} q_n(x) = p_n(x) + c_n p_{n-1}(x) &= \frac{(x - \alpha_{n-1}) p_{n-1}(x) - a_{n-1} p_{n-2}(x)}{a_n} + c_n p_{n-1}(x) \\ &= \frac{x - \alpha_{n-1} + c_n a_n}{a_n} \gamma_{n-1} \pi_{n-1}(x) - \frac{a_{n-1}}{a_n} \gamma_{n-2} \pi_{n-2}(x). \end{aligned}$$

Recall that $a_n = \gamma_{n-1}/\gamma_n$. We rewrite the recurrence relation (1.1) as

$$\pi_m(x) = (x - \alpha_{m-1}) \pi_{m-1} - a_{m-1}^2 \pi_{m-2}(x)$$

for all $m \geq 2$, with initial conditions $\pi_1(x) = x - \alpha_0$ and $\pi_0(x) = 1$. Schur's formula [38, (6.71.2)] gives

$$\begin{aligned} (-1)^{n(n-1)/2} \prod_{j=1}^{n} p_{n-1}(x_j) &= \gamma_{n-1}^n (\frac{\gamma_{n-1}}{a_n})^{1-n} (\frac{a_{n-1}\gamma_{n-2}}{a_n})^{n-1} \prod_{j=2}^{n-1} a_{j-1}^{2j-2} \\ &= \gamma_{n-1} (a_{n-1} \gamma_{n-2})^{n-1} \prod_{j=1}^{n-2} a_j^{2j}. \end{aligned}$$

It then follows that

$$\begin{aligned} D(q_n) = {} & \gamma_n^{n-1} a_n / a_{n-1} \gamma_{n-2}^{n-1} \prod_{j=1}^{n-2} a_j^{2j} \\ & \times \prod_{j=1}^{n} \{ a_{n-1} [A_n(x_j) + c_n B_n(x_j) - c_n B_{n-1}(x_j)] + (c_n^2 a_n + c_n x_j - c_n \alpha_{n-1}) A_{n-1}(x_j) \}. \end{aligned}$$

A further calculation shows that

$$D(q_n) = \frac{1}{\gamma_n^2} \prod_{j=0}^{n-1} \gamma_j^2 \times \prod_{j=1}^{n} [\frac{A_n(x_j)}{a_n} + \frac{c_n B_n(x_j)}{a_n} - \frac{c_n B_{n-1}(x_j)}{a_n} + (c_n^2 + \frac{c_n x_j}{a_n} - \frac{c_n \alpha_{n-1}}{a_n}) \frac{A_{n-1}(x_j)}{a_{n-1}}].$$

We use the operator in (1.4) to rewrite

$$\begin{aligned} & \frac{A_n(x)}{a_n} + \frac{c_n B_n(x)}{a_n} - \frac{c_n B_{n-1}(x)}{a_n} + (c_n^2 + \frac{c_n x}{a_n} - \frac{c_n \alpha_{n-1}}{a_n}) \frac{A_{n-1}(x)}{a_{n-1}} \\ = & L \left(p_n^2 + c_n p_n p_{n-1} - \frac{c_n a_{n-1} p_{n-1} p_{n-2}}{a_n} + c_n^2 p_{n-1}^2 + \frac{c_n x p_{n-1}^2}{a_n} - \frac{c_n \alpha_{n-1} p_{n-1}^2}{a_n} \right)(x). \end{aligned}$$

Since $-a_{n-1} p_{n-2}(y) + y p_{n-1}(y) - \alpha_{n-1} p_{n-1}(y) = a_n p_n(y)$. We rewrite the right-hand side of the above formula as

$$\begin{aligned} L(p_n^2 + c_n p_n p_{n-1} + c_n p_{n-1} p_n + c_n^2 p_{n-1}^2 + c_n (x - \cdot) p_{n-1}^2 / a_n)(x) &= L(q_n^2)(x) + \frac{c_n v'(x)}{a_n} \\ &= \frac{\bar{A}_n(x) + c_n v'(x)}{a_n}, \end{aligned}$$

where we have made use of the identity

$$L((x-\cdot)p_{n-1}^2)(x) = -w(y)p_{n-1}^2(y)\Big|_a^b + \int_a^b [v'(x)-v'(y)]p_{n-1}^2(y)w(y)dy = v'(x).$$

Thus, we reach at the following formula for the discriminant

$$\begin{aligned} D(q_n) &= \gamma_n^{-2}\left(\prod_{j=0}^{n-1}\gamma_j^2\right)\left[\prod_{j=1}^{n}\frac{\bar{A}_n(x_j)+c_nv'(x_j)}{a_n}\right] \\ &= \gamma_0^{2n-2}\left(\prod_{j=1}^{n}a_j^{2+2j-2n}\right)\left[\prod_{j=1}^{n}\frac{\bar{A}_n(x_j)+c_nv'(x_j)}{a_n}\right]. \end{aligned} \tag{3.1}$$

Especially, let $\bar{q}_n(x) = q_n(x)/\gamma_n$ be the monic quasi-orthogonal polynomial. We have from $\gamma_n = \gamma_0/(a_1\cdots a_n)$ that

$$D(\bar{q}_n) = \gamma_n^{2-2n}D(q_n) = \left(\prod_{j=1}^{n}a_j^{2j}\right)\left[\prod_{j=1}^{n}\frac{\bar{A}_n(x_j)+c_nv'(x_j)}{a_n}\right].$$

4. Electrostatics

We consider the model of n movable unit charged particles $\{x_1,\cdots,x_n\}$ in an external field $V_n(x)$. The particles are restricted to a linear segment (a,b), where $[a,b]$ is the smallest interval containing the support of $e^{-V_n(x)}$. Here, we use V_n instead of V, to indicate the possible dependence of external field on the number of particles. The energy is given as $E_n = -\ln T_n(x_1,\cdots,x_n)$ where

$$T_n(x_1,\cdots,x_n) = \left[\prod_{j=1}^{n}e^{-V_n(x_j)}\right]\prod_{1\le j<k\le n}(x_j-x_k)^2.$$

The equations describing the equilibrium positions are

$$\frac{\partial \ln T_n(x_1,\cdots,x_n)}{\partial x_j} = 0,\ \ j=1,\cdots,n.$$

Equivalently, we have

$$V_n'(x_j) = \sum_{k=1,k\neq j}^{n}\frac{2}{x_j-x_k} = \frac{q_n''(x_j)}{q_n'(x_j)},$$

where $q_n(x) = \gamma_n(x-x_1)\cdots(x-x_n)$. By (2.7), we obtain

$$V_n'(x_j) = \frac{\bar{A}_n'(x_j)+c_nv''(x_j)}{\bar{A}_n(x_j)+c_nv'(x_j)} + v'(x_j). \tag{4.1}$$

We obtain an analogue result as in [25].

Theorem 4.1. *If the external field*

$$V_n(x) = v(x) + \ln \frac{\bar{A}_n(x) + c_n v'(x)}{a_n}$$

is a convex function (i.e., $V_n''(x) \geq 0$) in (a, b), then the equilibrium position of n movable unit charges in (a, b) in the presence of this external field is uniquely attained at the zeros of $q_n(x)$, provided that the particle interaction obeys a logarithmic potential and that $T_n(y_1, \cdots, y_n) \to 0$ as $(y_1, \cdots, y_n)^T$ tends to any boundary point of $[a, b]^n$, where

$$T_n(y_1, \cdots, y_n) = \exp\left[-\sum_{j=1}^{n} V_n(y_j)\right] \prod_{1 \leq j < k \leq n} (y_j - y_k)^2.$$

The maximum value of T_n and the equilibrium energy are given by

$$T_n(x_1, \cdots, x_n) = \exp\left[-\sum_{j=1}^{n} v_n(x_j)\right] \left(\prod_{j=1}^{n} a_j^{2j}\right),$$

and

$$E_n = \sum_{j=1}^{n} [v_n(x_j) - 2j \ln a_j],$$

respectively, where $x_1, \cdots, x_n$ are the zeros of $q_n(x)$.

The motion of unit particles in the external field $V_n(x)$ is governed by the following dynamical system:

$$x_j''(t) = -\frac{V_n'(x_j(t))}{2} + \sum_{k=1, k\neq j}^{n} \frac{1}{x_j(t) - x_k(t)}, \ j = 1, \cdots, n,$$

where V_n' is given in (4.1). Clearly, the zeros of $q_n(x)$ correspond to an electrostatic equilibrium. However, the dynamics of above system is usually oscillating. To calculate the numerical values of the zeros of $q_n(x)$ for an arbitrary external field, we impose an artificial damping coefficient $\delta > 0$ and solve the following friction system numerically:

$$x_j''(t) = -\frac{V_n'(x_j(t))}{2} + \sum_{k=1, k\neq j}^{n} \frac{1}{x_j(t) - x_k(t)} - \delta x_j'(t), \ j = 1, \cdots, n.$$

The value of δ can be chosen by trial and error method. In general, a larger δ could better help to damp the oscillation, but may require more time to converge to the equilibrium.

5. Examples

5.1. Freud weight

We consider the Freud weight $w(x) = e^{-v(x)}$ with $v(x) = x^4 - 2tx^2$. It can be calculated that [26, page 57]

$$\frac{A_n}{a_n} = 4(x^2 - t + a_n^2 + a_{n+1}^2), \ \frac{B_n}{a_n} = 4xa_n.$$

Consequently, we obtain

$$\begin{aligned}\frac{\bar{B}_n}{a_n} &= \frac{B_n}{a_n} + c_n\frac{A_{n-1}}{a_{n-1}} = 4(xa_n + c_nx^2 - c_nt + c_na_n^2 + c_na_{n-1}^2),\\ \frac{\bar{A}_n}{a_n} &= \frac{A_n}{a_n} + c_n\frac{B_n}{a_n} + c_n\frac{\bar{B}_n}{a_n} = 4(x^2 - t + a_n^2 + a_{n+1}^2 + 2c_nxa_n + c_n^2x^2 - c_n^2t + c_n^2a_n^2 + c_n^2a_{n-1}^2)\\ &= 4[(1+c_n^2)x^2 + 2c_na_nx + (1+c_n^2)(a_n^2 - t) + a_{n+1}^2 + c_n^2a_{n-1}^2].\end{aligned}$$

The above formulas are valid for all $n \geq 1$, and when $n = 1$, we need the compatibility condition $a_0 = 0$. Note that $L(q_n^2) = \bar{A}_n/a_n$. The discriminant (3.1) is given as

$$\begin{aligned}D(q_n) = \gamma_0^{2n-2}&\left(\prod_{j=1}^{n} a_j^{2+2j-2n}\right)\\ &\times\prod_{j=1}^{n} 4[(1+c_n^2)x_j^2 + 2c_na_nx_j + (1+c_n^2)(a_n^2 - t) + a_{n+1}^2 + c_n^2a_{n-1}^2 + \frac{c_nx_j^3 - c_ntx_j}{a_n}].\end{aligned}$$

The differential equation is $q_n''(x) - V_n'(x)q_n'(x) + S_n(x)q_n(x) = 0$, where

$$\begin{aligned}V_n'(x) =&\frac{2(1+c_n^2)x + 2c_na_n + c_n(3x^2 - t)/a_n}{(1+c_n^2)x^2 + 2c_na_nx + (1+c_n^2)(a_n^2 - t) + a_{n+1}^2 + c_n^2a_{n-1}^2 + c_n(x^3 - tx)/a_n}\\ &+ 4x^3 - 4tx\end{aligned}$$

is the derivative of external field for electrostatic equilibrium problem, and

$$\begin{aligned}S_n(x) = a_n^2\Bigg[&\frac{4(a_n + 2c_nx)}{a_n} - 16(xa_n + c_nx^2 - c_nt + c_na_n^2 + c_na_{n-1}^2)^2\\ &- \frac{4(xa_n + c_nx^2 - c_nt + c_na_n^2 + c_na_{n-1}^2)}{a_n}V_n'(x) + 16(x^2 - t + a_n^2 + a_{n-1}^2)((1+c_n^2)x^2\\ &+ 2c_na_nx + (1+c_n^2)(a_n^2 - t) + a_{n+1}^2 + c_n^2a_{n-1}^2 + \frac{c_n(x^3 - tx)}{a_n})\Bigg].\end{aligned}$$

It is noted that $V_n''(x)$ may not be non-negative if $c_n \neq 0$. Moreover, when $c_n \neq 0$, one can observe that $V_n'(x)$ has at least one simple pole, which corresponds to a real zero of the cubic polynomial $\bar{A}_n + c_nv'$. Numerical computation suggests that electrostatic equilibrium may not be unique even in the simple case of $n = 1$, $t = 0$ and $c_n \neq 0$.

Let $p_n(x) = \gamma_nx^n + \beta_nx^{n-1} + \lambda_nx^{n-2} + \mu_nx^{n-3} + \cdots$. Recall the lower relation $q_n' = -\bar{B}_nq_n + (\bar{A}_n + c_nv')p_{n-1}$ in (2.3). By matching the coefficients of $x^{n+2}, x^{n+1}, x^n, x^{n-1}$ on both sides, we obtain

$$\begin{aligned}0 =& -4c_na_n\gamma_n + 4c_n\gamma_{n-1},\\ 0 =& -4a_n^2\gamma_n - 4c_na_n(\beta_n + c_n\gamma_{n-1}) + 4(1+c_n^2)a_n\gamma_{n-1} + 4c_n\beta_{n-1},\\ 0 =& -4c_na_n(\lambda_n + c_n\beta_{n-1}) - 4a_n^2(\beta_n + c_n\gamma_{n-1}) - 4c_n(a_n^2 + a_{n-1}^2 - t)a_n\gamma_n\\ &+ 4c_n\lambda_{n-1} + 4(1+c_n^2)a_n\beta_{n-1} + 4(2c_na_n^2 - c_nt)\gamma_{n-1},\\ n\gamma_n =& -4c_na_n(\mu_n + c_n\lambda_{n-1}) - 4a_n^2(\lambda_n + c_n\beta_{n-1}) - 4c_n(a_n^2 + a_{n-1}^2 - t)a_n(\beta_n + c_n\gamma_{n-1})\\ &+ 4c_n\mu_{n-1} + 4(1+c_n^2)a_n\lambda_{n-1} + 4(2c_na_n^2 - c_nt)\beta_{n-1}\\ &+ 4[(1+c_n^2)(a_n^2 - t) + a_{n+1}^2 + c_n^2a_{n-1}^2]a_n\gamma_{n-1}.\end{aligned}$$

By symmetry, we have $\beta_n = \mu_n = \alpha_n = 0$, where α_n is the coefficient in three-term recurrence relation (1.1). The first three equations above are trivial if combined with the following relations which are obtained by comparing the coefficients of $x^n, x^{n-1}, x^{n-2}, x^{n-3}$ in the recurrence relation $xp_{n-1}(x) = a_n p_n(x) + a_{n-1}p_{n-2}(x)$:

$$\gamma_{n-1} = a_n\gamma_n,\ \beta_{n-1} = a_n\beta_n,\ \lambda_{n-1} = a_n\lambda_n + a_{n-1}\gamma_{n-2},\ \mu_{n-1} = a_n\mu_n + a_{n-1}\beta_{n-2}.$$

The fourth equation gives a nonlinear relation between the recurrence coefficients:

$$n = 4a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2 - t),$$

which agrees with [26, (3.2.20)].

5.2. Jacobi weight

For the Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta > -1$, we have $v(x) = -\alpha\ln(1-x) - \beta\ln(1+x)$ and $v'(x) = \frac{\alpha}{1-x} - \frac{\beta}{1+x} = \frac{\alpha(1+x)-\beta(1-x)}{1-x^2}$. It follows from [26, (3.3.13)–(3.3.15)] that

$$\frac{A_n}{a_n} = \frac{\alpha+\beta+1+2n}{1-x^2},$$
$$\frac{B_n}{a_n} = \frac{n(\alpha+\beta+1+2n)}{2(1-x^2)}\frac{\beta-\alpha+x(2n+\alpha+\beta)}{(n+\alpha)(n+\beta)}\sqrt{\frac{h_n}{h_{n-1}}},$$

where

$$a_n = \frac{2}{\alpha+\beta+2n}\sqrt{\frac{n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta-1+2n)(\alpha+\beta+1+2n)}},$$
$$h_n = \frac{(\alpha+\beta+1)(\alpha+1)_n(\beta+1)_n}{(2n+\alpha+\beta+1)n!(\alpha+\beta+1)_n}.$$

By (2.4) and (2.5), we have

$$\frac{\bar{B}_n}{a_n} = -\frac{n(\alpha+\beta+1+2n)}{2(1-x^2)}\frac{\beta-\alpha+x(2n+\alpha+\beta)}{(n+\alpha)(n+\beta)}\sqrt{\frac{h_n}{h_{n-1}}} + \frac{c_n(\alpha+\beta-1+2n)}{1-x^2},$$
$$\frac{\bar{A}_n}{a_n} = \frac{(\alpha+\beta+1+2n)+c_n^2(\alpha+\beta-1+2n)}{1-x^2} - \frac{c_n n(\alpha+\beta+1+2n)}{1-x^2}\frac{\beta-\alpha+x(2n+\alpha+\beta)}{(n+\alpha)(n+\beta)}\sqrt{\frac{h_n}{h_{n-1}}}.$$

The discriminant (3.1) is given by

$$D(q_n) = \gamma_0^{2n-2}\left(\prod_{j=1}^n a_j^{2+2j-2n}\right)\left[\prod_{j=1}^n \frac{\bar{A}_n + c_n v'(x_j)}{a_n}\right],$$

where

$$\gamma_n = \frac{(n+\alpha+\beta+1)_n}{2^n n!}\sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}.$$

It can be calculated that

$$(1-x^2)\frac{\bar{A}_n + c_n v'(x)}{a_n} = \frac{c_n(2n+\alpha+\beta)}{a_n}(x+\sigma_n),$$

where

$$\sigma_n = \frac{(\alpha+\beta+1+2n) + c_n^2(\alpha+\beta-1+2n)}{c_n(2n+\alpha+\beta)}a_n - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)^2}.$$

Consequently,

$$\prod_{j=1}^{n} \frac{\bar{A}_n + c_n v'(x_j)}{a_n} = \frac{c_n^n(2n+\alpha+\beta)^n \gamma_n q_n(-\sigma_n)}{a_n^n q_n(1) q_n(-1)}.$$

We then have the following formula for the discriminant

$$D(q_n) = \gamma_0^{2n-2}\left(\prod_{j=1}^{n} a_j^{2+2j-2n}\right)\frac{c_n^n(2n+\alpha+\beta)^n \gamma_n q_n(-\sigma_n)}{a_n^n q_n(1) q_n(-1)}.$$

Moreover, the derivative of external field is given by

$$V_n'(x) = \frac{\alpha+1}{1-x} - \frac{\beta+1}{1+x} + \frac{1}{x+\sigma_n}.$$

Theorem 5.1. *If $V_n''(x) > 0$, then there exists a unique electrostatic equilibrium. If $-V_n'(x)$ is a decreasing function of α, so are the zeros of $q_n(x)$. If $-V_n'(x)$ is an increasing function of β, so are the zeros of $q_n(x)$.*

Remark 5.2. Consider the simple case when $n = 1$. It is easily seen that the zero of $q_1(x)$ is

$$x_1 = \frac{\beta-\alpha-2c_n\sqrt{(\alpha+1)(\beta+1)/(\alpha+\beta+3)}}{\beta+\alpha+2}.$$

Obviously, if $c_n = 0$ or $|c_n|$ is sufficiently small, then x_1 is decreasing in α and increasing in β. However, if $|c_n|$ is large, then one may not expect monotonicity property of the zero x_1 in α or β.

Consider another case when n is large. Assume $c_n \to c$ as $n \to \infty$. We note that $a_n \sim 1/2$ and $\sigma_n \sim (1+c_n^2)/(2c_n)$. It follows that

$$V_n'(x) \sim \frac{\alpha+1}{1-x} - \frac{\beta+1}{1+x} + \frac{1}{x+\frac{1+c^2}{2c}},$$

and

$$V_n''(x) \sim \frac{\alpha+1}{(x-1)^2} + \frac{\beta+1}{(x+1)^2} - \frac{1}{(x+\frac{1+c^2}{2c})^2}.$$

We conclude that the zeros of $q_n(x)$ are decreasing in α and increasing in β. If further, $\alpha \geq 0$ and $\beta \geq 0$, we have $V_n''(x) > 0$ and the electrostatic equilibrium is unique.

6. An orthogonality preserving transformation

Before we describe our results we just mention that the operator theoretic transformation will be mentioned at the end of this section.

6.1. Orthogonality of quasi-orthogonal polynomials

Our first task is to determine when the quasi-orthogonal polynomials are also orthogonal. To do this we note that (2.1) gives a recurrence relation for three successive terms $q_{n+1}(x)$, $q_n(x)$ and $q_{n-1}(x)$ with linear, quadratic, and linear coefficients in x, respectively; see [11]. By choosing appropriate c_n, we may reduce the degree of the coefficients by one and obtain the following three-term recurrence relation

$$a_{n+1}q_{n+1}(x) = (x - \beta_n)q_n(x) - b_n q_{n-1}(x), \tag{6.1}$$

where b_n and β_n are determined by the equations

$$\begin{aligned} &b_n(a_{n-1} + c_{n-1}x - c_{n-1}\alpha_{n-1} + c_{n-1}c_n a_n) = c_n a_{n-1}(x - \alpha_n + c_{n+1}a_{n+1}) + a_{n-1}a_n, \\ &(x - \beta_n)(a_{n-1} + c_{n-1}x - c_{n-1}\alpha_{n-1} + c_{n-1}c_n a_n) \\ &\quad = (x - \alpha_n + c_{n+1}a_{n+1})(a_{n-1} + c_{n-1}x - c_{n-1}\alpha_{n-1}) - c_{n-1}a_n^2. \end{aligned}$$

By treating x as a variable and matching the coefficients of linear and constant terms in the above two equations, we obtain four equations

$$\begin{aligned} b_n c_{n-1} &= c_n a_{n-1}, \\ b_n(a_{n-1} - c_{n-1}\alpha_{n-1} + c_{n-1}c_n a_n) &= c_n a_{n-1}(-\alpha_n + c_{n+1}a_{n+1}) + a_{n-1}a_n, \\ -\beta_n c_{n-1} + a_{n-1} - c_{n-1}\alpha_{n-1} + c_{n-1}c_n a_n &= c_{n-1}(-\alpha_n + c_{n+1}a_{n+1}) + a_{n-1} - c_{n-1}\alpha_{n-1}, \\ -\beta_n(a_{n-1} - c_{n-1}\alpha_{n-1} + c_{n-1}c_n a_n) &= (-\alpha_n + c_{n+1}a_{n+1})(a_{n-1} - c_{n-1}\alpha_{n-1}) - c_{n-1}a_n^2. \end{aligned}$$

If the q_n's are orthogonal, it follows from the first and third equations that

$$b_n = c_n a_{n-1}/c_{n-1}, \beta_n = \alpha_n + c_n a_n - c_{n+1}a_{n+1}. \tag{6.2}$$

Now, we substitute the values from (6.2) into the second and the fourth equations respectively to find the same nonlinear three-term recurrence relation for c_n:

$$a_{n+1}c_{n+1} = a_n c_n + \frac{a_{n-1}}{c_{n-1}} - \frac{a_n}{c_n} + \alpha_n - \alpha_{n-1} \tag{6.3}$$

for $n > 1$. The case $n = 1$ requires a special treatment. We first define $q_0(x) = p_0(x)$ and observe that

$$\begin{aligned} a_2q_2(x) = a_2p_2(x) + a_2c_2p_1(x) &= (x - \alpha_1 + a_2c_2)p_1(x) - a_1p_0(x) \\ &= (x - \alpha_1 + a_2c_2)(q_1(x) - c_1q_0(x)) - a_1q_0(x) \\ &= (x - \alpha_1 + a_2c_2)q_1(x) - c_1q_0(x)(x - \alpha_1 + a_2c_2) - a_1q_0(x). \end{aligned}$$

Since $(x - \alpha_0)q_0(x) = (x - \alpha_0)p_0(x) = a_1p_1(x) = a_1q_1(x) - a_1c_1q_0(x)$, we obtain from the above equation

$$\begin{aligned} a_2q_2(x) &= (x - \alpha_1 + a_2c_2)q_1(x) - a_1c_1q_1(x) + a_1c_1^2q_0(x) \\ &\quad - c_1q_0(x)(\alpha_0 - \alpha_1 + a_2c_2) - a_1q_0(x) \\ &= (x - \alpha_1 - a_1c_1 + a_2c_2)q_1(x) - [a_1 + c_1(\alpha_0 - \alpha_1 - a_1c_1 + a_2c_2)]q_0(x). \end{aligned}$$

When $n > 1$, it follows from (6.3) that

$$b_n = \frac{c_n a_{n-1}}{c_{n-1}} = a_n + c_n(\alpha_{n-1} - \alpha_n - a_n c_n + a_{n+1}c_{n+1}).$$

Thus, we arrive at the following result.

Theorem 6.1. *Let c_n be a sequence satisfying (6.3) for $n \geq 2$. The quasi-orthogonal polynomials $q_n(x) = p_n(x) + c_n p_{n-1}(x)$ satisfy the following three-term recurrence relation*

$$\begin{aligned} a_{n+1}q_{n+1}(x) =&(x - \alpha_n - a_n c_n + a_{n+1}c_{n+1})q_n(x) \\ &- [a_n + c_n(\alpha_{n-1} - \alpha_n - a_n c_n + a_{n+1}c_{n+1})]q_{n-1}(x) \end{aligned} \tag{6.4}$$

for $n \geq 1$. If, in addition, $a_n + c_n(\alpha_{n-1} - \alpha_n - a_n c_n + a_{n+1}c_{n+1}) > 0$ for all $n \geq 1$, then $q_n(x)$ is a sequence of orthogonal polynomials.

Proof. If $a_{n-1} + c_{n-1}x - c_{n-1}\alpha_{n-1} + c_{n-1}c_n a_n \neq 0$, then (6.4) follows from (2.1) and the arguments before the statement of the theorem. Since both sides of (6.4) are polynomials in x, this recurrence relation holds for all x. Let $\bar{q}_n(x) = q_n(x)/\gamma_n$ be the monic quasi-orthogonal polynomial. On account of $a_n = \gamma_{n-1}/\gamma_n$, we can rewrite (6.4) as

$$\begin{aligned} \bar{q}_{n+1}(x) =&(x - \alpha_n - a_n c_n + a_{n+1}c_{n+1})\bar{q}_n(x) \\ &- a_n[a_n + c_n(\alpha_{n-1} - \alpha_n - a_n c_n + a_{n+1}c_{n+1})]\bar{q}_{n-1}(x). \end{aligned}$$

The orthogonality of q_n follows from the spectral theorem for orthogonal polynomials [26]. □

Consider a special case when $p_n(x)$ are orthonormal Jacobi polynomials and the coefficient $c_n \equiv c$ is independent of n. If $q_n(x) = p_n(x) + cp_{n-1}(x)$ are also orthogonal polynomials, it then follows from (6.3) that $ca_{n+1} + a_n/c - \alpha_n$ is independent of n. A simple calculation shows that $\alpha_n = 0$ and $a_n = 1/2$; or equivalently, the Jacobi polynomials are reduced to the Chebyshev polynomials of the first, second, third, or fourth kind. This coincides with result in [7, Theorem 3.1] for the special case when $k = 1$ in [7, (3.1)]. Actually, it was proved in [7] that the four Chebyshev sequences are the only classical orthogonal polynomials whose linear combinations with finite length and constant coefficients are also orthogonal polynomials.

Our next task is to identify the choices of c_n. Let $u_n = a_n c_n$. It follows from (6.3) that

$$u_{n+1} - u_n + \frac{a_n^2}{u_n} - \frac{a_{n-1}^2}{u_{n-1}} = \alpha_n - \alpha_{n-1}.$$

Adding these equations for consecutive values of n we find that

$$u_{n+1} + \frac{a_n^2}{u_n} = \alpha_n + C,$$

where $C = a_2c_2 + a_1/c_1 - \alpha_1$ is a constant. In other words

$$u_n = \frac{a_n^2}{\alpha_n + C - u_{n+1}},$$

which leads to the continued fraction

$$u_n = \cfrac{a_n^2}{\alpha_n + C - \cfrac{a_{n+1}^2}{\alpha_{n+1} + C - \cfrac{a_{n+2}^2}{\alpha_{n+2} + C - \dots}}}. \tag{6.5}$$

For any fixed n, the above continued fraction is associated with the recurrence

$$\pi_{k+1}^{(n)}(x) = (x + \alpha_{n+k})\pi_k^{(n)}(x) - a_{n+k}^2\pi_{k-1}^{(n)}(x), \quad k \geq 1, \tag{6.6}$$

with initial conditions $\pi_0^{(n)}(x) = 1$ and $\pi_1^{(n)}(x) = x + \alpha_n$. It is clear from (6.6) that the monic polynomials of (1.2) are $(-1)^k\pi_k^{(n)}(-x)$.

To avoid technical difficulties we shall assume that the moment problem of the polynomials p_n is determinate, that is, it has a unique solution [1,34,33]. In this case, the moment problems of the associated polynomials $\pi_k^{(n)}(x)$ are determinate for every $n = 1, 2, \cdots$. By the spectral theorem of orthogonal polynomials, $\pi_k^{(n)}(x)$ are orthogonal with respect to a unique probability measure $\psi^{(n)}(x)$. The continued fraction in (6.5) can be expressed as the Stieltjes transform of this measure:

$$\frac{u_n}{a_n^2} = X^{(n)}(C) := \int_{\mathbf{R}} \frac{d\psi^{(n)}(y)}{C - y}.$$

It then follows that $c_n = u_n/a_n = a_n X^{(n)}(C)$. Thus, we choose C outside the supports of $\psi^{(n)}(x)$ for all $n, n = 1, 2, \cdots$.

It must be noted the measure $\psi^{(n)}$ is complicated for many classical polynomials but what is needed is the Stieltjes function $\int_{\mathbb{R}} d\psi^{(n)}(x)/(z - x)$. The known techniques of finding $\psi^{(n)}$ actually compute the Stieltjes transform first. This technique was pioneered by Pollaczek and further modified by Askey and Ismail [5]. The Stieltjes functions $\int_{\mathbb{R}} d\psi^{(n)}(x)/(z-x)$ have been computed for all classical orthogonal polynomials. The Askey–Wilson case is in [28]. For a sample of other cases we refer the reader to [22], [23], and the references cited in these works.

In [31], Maroni considered the polynomials which are orthogonal with respect to the perturbed measure μ defined as

$$\mu(f) := \int_a^b f(x)d\mu(x) = \int_a^b [f(x_0) + \lambda\frac{f(x) - f(x_0)}{x - x_0}]w(x)dx, \tag{6.7}$$

where x_0 and λ are two constants. The transform from $w(x)dx$ to $d\mu(x)$ is also called Geronimus transform in the literature; see [3,43]. By choosing $x_0 = -C = \alpha_1 - a_1/c_1 - a_2c_2$ and $\lambda = \alpha_0 - a_1c_1 - x_0$, we find that μ defined above is the orthogonal measure for q_n in Theorem 6.1. Actually, by the choice of x_0, we can prove by induction that $\mu(a_{n+1}q_{n+1}) = -\mu(a_nq_n)$. On the other hand, it follows from the choice of λ that $\mu(q_1) = 0$. Thus, $\mu(q_n) = 0$ for all $n \geq 1$. For any $k < n$, we then have $\mu(x^kq_n) = \mu((x^k - x_0^k)(p_n + c_np_{n-1})) + x_0^k\mu(q_n) = 0$, which implies that q_n are the orthogonal polynomials with respect to the measure μ.

6.2. Polynomials arising from birth and death process

A birth and death process with birth and death rates λ_n, and μ_n, respectively, gives rise to two families of orthogonal polynomials, [26]. The first is the family of birth and death process polynomials generated by $Q_0(x) = 1, Q_1(x) = (\lambda_0 + \mu_0 - x)/\lambda_0$, and

$$-xQ_n(x) = \lambda_nQ_{n+1}(x) + \mu_nQ_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x). \tag{6.8}$$

The second is the family of random walk polynomials

$$xR_n(x) = \frac{\lambda_n}{\lambda_n + \mu_n}R_{n+1}(x) + \frac{\mu_n}{\lambda_n + \mu_n}R_{n-1}(x),$$

with $R_0(x) = 1, R_1(x) = x(\lambda_0 + \mu_0)/l_0$. It is assumed that $\lambda_n > 0, \mu_{n+1} > 0$ for all $n \geq 0$ while $\mu_0 \geq 0$. Under this assumption, it is known that the zeros of Q_n are in $(0, \infty)$ for all n, while the zeros of R_n lie in $(-1, 1)$, for all n.

The orthonormal birth and death process polynomials are

$$p_n(x) = (-1)^n Q_n(x) \sqrt{\prod_{k=1}^{n} \frac{\mu_k}{\lambda_{k-1}}}.$$

The two equations (6.6) and (6.8) are related by $\alpha_n = \lambda_n + \mu_n, a_{n+1} = \sqrt{\lambda_n \mu_{n+1}}$. Therefore, for any fixed n, the zeros of $\pi_k^{(n)}(x)$ lie in $(-\infty, 0)$ for all k, which implies that the support of $\psi^{(n)}(x)$ is also in $(-\infty, 0]$. Thus, C can be chosen as any positive real number. In the special case of the Laguerre polynomials $\lambda_n = n + 1, \mu_n = \alpha + n$, the measure $\psi^{(n)}$ are the orthogonality measures of the associated Laguerre polynomials and were found by Askey and Wimp in [6]. A second model for associated Laguerre polynomials is in [27]. The latter work also has the case of Meixner polynomials where $\lambda_n = c(n + \beta), \mu_n = n$. It must be noted that in both cases of the associated Laguerre and associated Meixner polynomials the measure $\psi^{(n)}$ is complicated but as we already said earlier what is needed is the Stieltjes function $\int_{\mathbb{R}} d\psi^{(n)}(x)/(z - x)$.

For example, in the case of Laguerre polynomials, we have

$$\int_0^\infty \frac{d\psi^{(n)}(x)}{x + z} = \frac{\Psi(n + 1, 1 - \alpha; z)}{\Psi(n, 1 - \alpha; z)},$$

where Ψ is the Tricomi ψ function, [19, Chapter 6]. On the other hand, the Meixner polynomials give rise to

$$\int_0^\infty \frac{d\psi^{(n)}(x)}{x + z} = (n + z) \frac{{}_2F_1(n + 1, 1 + z - \beta; n + 1 + z; c)}{{}_2F_1(n, 1 + z - \beta; n + z; c)},$$

see (4.6) in [27].

The orthonormal random walk polynomials are

$$p_n(x) = R_n(x) \sqrt{\prod_{k=1}^{n} \frac{\mu_k}{\lambda_{k-1}}} \sqrt{\frac{\lambda_0 + \mu_0}{\lambda_n + \mu_n}},$$

and in this case

$$\alpha_n = 0, a_n = \sqrt{\frac{\lambda_n \mu_{n+1}}{(\lambda_n + \mu_n)(\lambda_{n+1} + \mu_{n+1})}}.$$

When $\lambda_n = n + 1, \mu_n = n + 2\nu$, the random walk polynomials become ultraspherical polynomials and the $\psi^{(n)}$ is the orthogonality measure of the associated ultraspherical polynomials of order n, These measures as well as the orthogonality measures of their q-analogues were identified by Bustoz and Ismail in [9]. We mention the example of the q-ultraspherical polynomials $C_n(x; \beta|q)$. In this case

$$\int_{\mathbb{R}} \frac{d\psi^{(n)}(t)}{z - t} = \frac{2(1 - \beta q^n)_2\phi_1(\beta, \beta B/A; qB/A; q, a^{n+1})}{(1 - q^n) A_2\phi_1(\beta, \beta B/A; qB/A; q, a^n)}, \tag{6.9}$$

for $z \notin [-1, 1]$. Here

$$A, B = x \pm \sqrt{x^2 - 1}, \quad \text{and} \quad |A| \geq |B|. \tag{6.10}$$

6.3. A matrix problem

It is known, see for example [30, §31], that a polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

is the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{pmatrix}$$

There is no such construction known to write f as a characteristic polynomial of a Hermitian matrix.

We now rephrase the results of §6.1 in terms of tridiagonal matrices. Start with an infinite tridiagonal matrix A_∞,

$$H_\infty = \begin{pmatrix} \alpha_0 & a_1 & 0 & \ldots \\ a_1 & \alpha_1 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and define H_n to be the $n \times n$ matrix formed by truncating H_∞ after n rows and n columns. We further assume that $a_{k+1} > 0, \alpha_k \in \mathbb{R}$, for $k \geq 0$. Let $\bar{p}_n(x)$ be the characteristic polynomial of H_n. It is known that the orthonormal polynomials p_n in (1.1) are related to $\bar{p}_n$ via

$$\gamma_0 \bar{p}_n(x) = a_1 a_2 \cdots a_n p_n(x),$$

where $\gamma_0 = p_0(x)$ is a constant. We also note that when H_∞ has a unique self-adjoint extension to ℓ^2 then the measure of orthogonality of the p_n's is the spectral measure of H_∞. The polynomial q_n in this case is $\gamma_n \bar{q}_n$, where $\gamma_n = \gamma_0/(a_1 \cdots a_n)$ is the leading coefficient of q_n, and

$$\bar{q}_n(x) = \bar{p}_n(x) + a_n c_n \bar{p}_{n-1}(x).$$

We now determine the sequence c_n such that there is an infinite tridiagonal matrix whose truncations have the characteristic polynomials $\{\bar{q}_n(x) : n = 1, 2, \cdots\}$. The solution given in §6.1 uses $\psi^{(n)}$, which is the spectral measure of the Jacobi operator

$$H_\infty^{(n)} = \begin{pmatrix} -\alpha_n & a_{n+1} & 0 & \ldots \\ a_{n+1} & -\alpha_{n+1} & a_{n+2} & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

If $c_n = a_n X^{(n)}(C)$ with $X^{(n)}$ being the Stieltjes transform of the spectral measure $\psi^{(n)}$ and C being a constant outside the supports of $\psi^{(n)}(x)$ for all $n = 1, 2, \cdots$, and if $a_n + c_n(\alpha_{n-1} - \alpha_n - a_n c_n + a_{n+1} c_{n+1}) > 0$ for all $n = 1, 2, \cdots$, then by Theorem 6.1, $\bar{q}_n$ are the characteristic polynomials of the following infinite tridiagonal matrix truncated after n rows and n columns:

$$J_\infty = \begin{pmatrix} \beta_0 & b_1 & 0 & \dots \\ b_1 & \beta_1 & b_2 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $\beta_n = \alpha_n + a_n c_n - a_{n+1}c_{n+1}$ and $b_n^2 = a_n[a_n + c_n(\alpha_{n-1} - \alpha_n - a_n c_n + a_{n+1}c_{n+1})]$. Note that $\bar{q}_n(x) = \pi_n(x) + u_n\pi_{n-1}(x)$, where $u_n = a_n c_n$ satisfies the recurrence relation $u_{n+1} = \alpha_n - a_n^2/u_n + C$ and $\pi_n(x)$ are the given monic orthogonal polynomials. We have the following relation between Jacobi matrices for $\bar{q}_n(x)$ and $\pi_n(x)$.

$$\begin{pmatrix} 1 & 0 & \dots \\ u_1 & 1 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \alpha_0 & 1 & 0 & \dots \\ a_1^2 & \alpha_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \beta_0 & 1 & 0 & \dots \\ b_1^2 & \beta_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots \\ u_1 & 1 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}. \tag{6.11}$$

In the special case of $C = 0$, we set $v_n = \alpha_n - u_{n+1}$ to simplify $a_n^2 = u_n v_n$, $\alpha_n = u_{n+1} + v_n$, $b_n^2 = u_n v_{n-1}$ and $\beta_n = u_n + v_n$, which imply the following factorizations [3]:

$$\begin{pmatrix} \alpha_0 & 1 & 0 & \dots \\ a_1^2 & \alpha_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} v_0 & 1 & \dots \\ 0 & v_1 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots \\ u_1 & 1 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix},$$

$$\begin{pmatrix} \beta_0 & 1 & 0 & \dots \\ b_1^2 & \beta_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots \\ u_1 & 1 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} v_0 & 1 & \dots \\ 0 & v_1 & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Acknowledgments

We are grateful to two anonymous referees for careful reading and helpful suggestions which have led to an improvement to our original manuscript.

References

[1] N.I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis, 1965, English translation by N. Kemmer, Oliver and Boyd, Edinburgh.
[2] M. Alfaro, R. Marcellán, A. Peña, M.L. Rezola, When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials?, J. Comput. Appl. Math. 233 (2010) 1446–1452.
[3] M. Alfaro, R. Marcellán, A. Peña, M.L. Rezola, On linearly related orthogonal polynomials and their functionals, J. Math. Anal. Appl. 287 (2003) 307–319.
[4] W.A. Al-Salam, Characterization theorems for orthogonal polynomials, in: P. Nevai (Ed.), Orthogonal Polynomials, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 294, Kluwer Acad. Publ., Dordrecht, 1990, pp. 1–24.
[5] R.A. Askey, M.E.H. Ismail, Recurrence Relations, Continued Fractions and Orthogonal Polynomials, Memoirs Amer. Math. Soc., vol. 300, 1984, 112 pp.
[6] R.A. Askey, J. Wimp, Associated Laguerre and Hermite polynomials, Proc. Roy. Soc. Edinburgh Sect. A 96 (1984) 15–37.
[7] E. Berriochoa, A. Cachafeiro, J.M. Garcia-Amor, A characterization of the four Chebyshev orthogonal families, Int. J. Math. Math. Sci. (2005) 2071–2079.
[8] C. Brezinski, K.A. Driver, M. Redivo-Zaglia, Quasi-orthogonality with applications to some families of classical orthogonal polynomials, Appl. Numer. Math. 48 (2) (2004) 157–168.
[9] J. Bustoz, M.E.H. Ismail, The associated ultraspherical polynomials and their q-analogues, Canad. J. Math. 34 (1982) 718–736.
[10] Y. Chen, M.E.H. Ismail, Ladder operators and differential equations for orthogonal polynomials, J. Phys. A 30 (1997) 7818–7829.
[11] T.S. Chihara, On quasi-orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 765–767.
[12] D. Dickinson, On quasi-orthogonal polynomials, Proc. Amer. Math. Soc. 12 (1961) 185–194.
[13] K. Dilcher, K.B. Stolarsky, Resultants and discriminants of Chebyshev and related polynomials, Trans. Amer. Math. Soc. 357 (2004) 965–981.
[14] K. Driver, K. Jordaan, Zeros of quasi-orthogonal Jacobi polynomials, SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016) 042, 13 pp.

[15] K. Driver, M.E. Muldoon, Bounds for extreme zeros of quasi-orthogonal ultraspherical polynomials, J. Class. Anal. 9 (2016) 69–78.
[16] K. Driver, M.E. Muldoon, Zeros of quasi-orthogonal ultraspherical polynomials, Indag. Math. (N.S.) 27 (2016) 930–944.
[17] K. Driver, M.E. Muldoon, Zeros of quasi-orthogonal Jacobi polynomials, SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016) 042, 13 pp.
[18] K. Driver, M.E. Muldoon, Interlacing properties and bounds for zeros of some quasi-orthogonal Laguerre polynomials, Comput. Methods Funct. Theory 15 (2015) 645–654.
[19] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Tables of Integral Transforms, Vol. I, McGraw-Hill Book Company, New York, 1954.
[20] J. Gishe, M.E.H. Ismail, Resultants of Chebyshev polynomials, Z. Anal. Anwend. 27 (2008) 491–500.
[21] B.-Y. Guo, T. Sun, C. Zhang, Jacobi and Laguerre quasi-orthogonal approximations and related interpolations, Math. Comp. 82 (2013) 413–441.
[22] D.P. Gupta, D.R. Masson, Contiguous relations, continued fractions and orthogonality, Trans. Amer. Math. Soc. 350 (1998) 769–808.
[23] D.P. Gupta, M.E.H. Ismail, D.R. Masson, Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual q-Hahn polynomials, J. Comput. Appl. Math. 68 (1996) 115–149.
[24] D. Hilbert, Ueber die Discriminante der im Endlichen abbrechenden hypergeometrischen Reihe, J. Reine Angew. Math. 103 (1888) 337–345.
[25] M.E.H. Ismail, An electrostatics model for zeros of general orthogonal polynomials, Pacific J. Math. 193 (2000) 355–369.
[26] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, With two chapters by Walter Van Assche. With a foreword by Richard A. Askey. Reprint of the 2005 original Encyclopedia of Mathematics and Its Applications, vol. 98, Cambridge University Press, Cambridge, 2009.
[27] M.E.H. Ismail, J. Letessier, G. Valent, Linear birth and death models and associated Laguerre and Meixner polynomials, J. Approx. Theory 55 (1988) 337–348.
[28] M.E.H. Ismail, M. Rahman, The associated Askey–Wilson polynomials, Trans. Amer. Math. Soc. 328 (1991) 201–237.
[29] H. Joulak, A contribution to quasi-orthogonal polynomials and associated polynomials, Appl. Numer. Math. 54 (2005) 65–78.
[30] M. Marden, Geometry of Polynomials, American Mathematical Society, Providence, RI, 1966.
[31] P. Maroni, Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x - c)^{-1}L$, Period. Math. Hungar. 21 (1990) 223–248.
[32] M. Sawa, Y. Uchida, Compact formulas for discriminants of classical quasi-orthogonal polynomials, with Hausdorff-type equations, J. Math. Soc. Japan (2019), in press.
[33] K. Schmüdgen, The Moment Problem, Springer, New York, 2017.
[34] J. Shohat, J.D. Tamarkin, The Problem of Moments, revised edition, American Mathematical Society, Providence, RI, 1950.
[35] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Adv. Math. 137 (1998) 82–203.
[36] T.J. Stieltjes, Sur quelques théorèmes d'algèbre, C. R. Acad. Sci. Paris 100 (1885) 439–440; Oeuvres Complètes. Vol. 1, pp. 440–441.
[37] T.J. Stieltjes, Sur les polynômes de Jacobi, C. R. Acad. Sci. Paris 100 (1885) 620–622; Oeuvres Complètes. Vol. 1, pp. 442–444.
[38] G. Szegő, Orthogonal Polynomials, fourth edition, American Mathematical Society, Colloquium Publications, vol. XXIII, American Mathematical Society, Providence, RI, 1975.
[39] Y. Xu, Quasi-orthogonal polynomials, quadrature, and interpolation, J. Math. Anal. Appl. 182 (1994) 779–799.
[40] Y. Xu, On zeros of multivariate quasi-orthogonal polynomials and Gaussian cubature formulae, SIAM J. Math. Anal. 25 (1994) 991–1001.
[41] Y. Xu, A characterization of positive quadrature formulae, Math. Comp. 62 (1994) 703–718.
[42] A. Zarzo, R.J. Yáñez, A. Ronveaux, J.S. Dehesa, Algebraic and spectral properties of some quasiorthogonal polynomials encountered in quantum radiation, J. Math. Phys. 36 (1995) 5179–5197.
[43] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67–86.