J. Math. Anal. Appl. 436 (2016) 1149–1164

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotics of Racah polynomials with varying parameters

X.-S. Wang [a,*], R. Wong [b]

[a] *Department of Mathematics, Southeast Missouri State University, Cape Girardeau, MO 63701, USA*
[b] *Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong*

ARTICLE INFO

Article history:
Received 6 September 2015
Available online 29 December 2015
Submitted by K. Driver

Keywords:
Asymptotics
Racah polynomials
Recurrence relation
Askey scheme

ABSTRACT

Within the Askey scheme of hypergeometric orthogonal polynomials, Racah polynomials stay on the top of the hierarchy and they generalize all of the discrete hypergeometric orthogonal polynomials. In this paper, we investigate asymptotic behaviors of Racah polynomials with varying parameters when the polynomial degree tends to infinity. Using the difference equation technique developed in our earlier papers, we obtain an asymptotic formula in the outer region via ratio asymptotics and then derive asymptotic formulas in the oscillatory region via a matching method. Our asymptotic formulas are explicitly given in terms of the polynomial degree, variable and parameters, using elementary functions such as logarithmic, exponential and rational functions. By taking limits, our results also yield asymptotic formulas for orthogonal polynomials in the lower hierarchy of the Askey scheme such as Hahn polynomials and Krawtchouk polynomials.

1. Introduction

The Racah polynomials are named after Racah, because the orthogonal relation is equivalent to that of Racah coefficients or 6-j symbols; see [1]. In [11], Wilson defined the Racah polynomials in terms of a ${}_4F_3$ hypergeometric function. Let $\lambda(x) := x(x+\gamma+\delta+1)$ and N be a nonnegative integer. Define

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) := {}_4F_3\left(\begin{matrix}-n, & n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1\\ & \alpha+1, & \beta+\delta+1, & \gamma+1\end{matrix}\middle|1\right), \tag{1.1}$$

where $n=0,\cdots,N$ and one of the three equalities is satisfied: $\alpha+1=-N$ or $\beta+\delta+1=-N$ or $\gamma+1=-N$. Set

* Corresponding author.
E-mail address: xswang@semo.edu (X.-S. Wang).

http://dx.doi.org/10.1016/j.jmaa.2015.12.035
0022-247X/

$$A_n := -\frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$
$$C_n := -\frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \tag{1.2}$$

The Racah polynomials (1.1) satisfy the recurrence relation [7, (9.2.3)]

$$\lambda(x)R_n(\lambda(x)) = -A_n R_{n+1}(\lambda(x)) + (A_n + C_n)R_n(\lambda(x)) - C_n R_{n-1}(\lambda(x)).$$

This recurrence relation can be normalized as

$$\pi_{n+1}(z) = (z - A_n - C_n)\pi_n(z) - A_{n-1}C_n\pi_{n-1}(z), \quad \pi_0(z) = 1, \ \pi_1(z) = z - A_0, \tag{1.3}$$

where $z = \lambda(x)$ and

$$\pi_n(\lambda(x)) := \frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(n+\alpha+\beta+1)_n} R_n(\lambda(x)). \tag{1.4}$$

In addition to the variable x and the degree n, the polynomials in (1.1) involve four free parameters. This inevitably makes the problem of deriving their asymptotic formulas much more complicated. The only result that we can find on this topic is that given by Chen, Ismail and Simeonov [2]. Their method starts with the hypergeometric representation in (1.1). By approximating the ratio of two shifted factorials, they obtain several asymptotic formulas in terms of hypergeometric function ${}_3F_2$ or ${}_2F_1$, when the parameters are fixed.

In the present paper, we are interested in the large-n behavior of $\pi_n(z)$ with varying parameters $\alpha, \beta, \gamma, \delta$. More precisely, we set

$$\alpha + 1 = Na, \ \beta = Nb, \ \gamma + 1 = Nc, \ \delta + 1 = Nd, \tag{1.5}$$

where either $a = -1$ or $b + d = -1$ or $c = -1$. For simplicity, we assume $A_n > 0$ and $C_n > 0$. By Favard's theorem, these conditions guarantee that the zeros of $\pi_n(z)$ are all real and simple; see [3, Sections 1.4 and 1.5]. Thus, we require some additional conditions:

1. when $a = -1$, we assume $b, c, d > 0$ and $b > c + 1$;
2. when $b + d = -1$, we assume $a, b, c > 0$ and $a + b + 1 < c$;
3. when $c = -1$, we assume $a, b, d > 0$ and $a + 1 < d$.

Let $n/N = p$ be a fixed number in $(0, 1)$. We shall derive asymptotic formulas for $\pi_n(N^2 y)$ as $N \to \infty$.

2. Main results

Define the ratio $w_k(z) := \pi_k(z)/\pi_{k-1}(z)$ for $k = 1, \cdots, n$. From (1.3), it follows that

$$w_{k+1}(z) = z - (A_k + C_k) - \frac{A_{k-1}C_k}{w_k(z)}. \tag{2.1}$$

Recall $p = n/N$. We obtain from (1.2) and (1.5) that

$$\lim_{N\to\infty} \frac{A_n}{N^2} = A(p); \quad \lim_{N\to\infty} \frac{C_n}{N^2} = C(p),$$

where

$$A(t) := -\frac{(t+a)(t+a+b)(t+b+d)(t+c)}{(2t+a+b)^2}, \quad 0 < t < p,$$
$$C(t) := -\frac{t(t+a+b-c)(t+a-d)(t+b)}{(2t+a+b)^2}, \quad 0 < t < p. \tag{2.2}$$

Now, we shall introduce the transition points (or turning points) when the characteristic roots of (1.3) coincide; see [9]. For $t \in [0, p]$, we define

$$y_{\pm}(t) := A(t) + C(t) \pm 2\sqrt{A(t)C(t)}; \tag{2.3}$$

these are the transition points when $t = p$. From (2.2), it is readily seen that $C(0) = 0$. Thus, $y_{\pm}(0) = A(0)$. For simplicity, we only consider the case when $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0, p]$. The other cases will be studied in forthcoming papers. If $z = N^2 y$ with $y \in \mathbf{C} \setminus [y_-(p), y_+(p)]$, we apply a standard successive approximation technique to (2.1) and obtain

$$w_k(z) = w_k^0[1 + w_k^1 + O(1/n^2)] \text{ as } N \to \infty,$$

uniformly for $k = 1, \cdots, n$; see [10]. Here, the leading term is

$$w_k^0 = \frac{z - A_k - C_k + \sqrt{(z - A_k - C_k)^2 - 4A_{k-1}C_k}}{2}, \tag{2.4}$$

and the first-order term is

$$w_k^1 = \frac{1}{2} + \frac{A_{k+1} - A_k + C_{k+1} - C_k - \sqrt{(z - A_{k+1} - C_{k+1})^2 - 4A_k C_{k+1}}}{2\sqrt{(z - A_k - C_k)^2 - 4A_{k-1}C_k}}. \tag{2.5}$$

We can rewrite (2.5) as

$$w_k^1 = \frac{A_{k+1} - A_k + C_{k+1} - C_k + \sqrt{(z - A_k - C_k)^2 - 4A_{k-1}C_k} - \sqrt{(z - A_{k+1} - C_{k+1})^2 - 4A_k C_{k+1}}}{2\sqrt{(z - A_k - C_k)^2 - 4A_{k-1}C_k}}.$$

Note that $z = N^2 y$ with $y \in \mathbf{C} \setminus [y_-(p), y_+(p)]$. From (1.2), it is readily seen that for any $k = 1, \cdots, n$, the denominator in the expression of w_k^1 is of order $O(n^2)$, while the numerator is of order $O(n)$. Thus, $w_k^1 = O(1/n)$. Since $\pi_n = w_1 \cdots w_n$, we have

$$\ln \pi_n = \sum_{k=1}^{n} \ln w_k = \sum_{k=1}^{n} \ln w_k^0 + \sum_{k=1}^{n} \ln(1 + w_k^1) + O(1/n) = \sum_{k=1}^{n} \ln w_k^0 + \sum_{k=1}^{n} w_k^1 + O(1/n). \tag{2.6}$$

To find an asymptotic formula for π_n, we only need to approximate $\sum_{k=1}^{n} \ln w_k^0$ and $\sum_{k=1}^{n} w_k^1$. First, we note that

$$\sqrt{[z - (A_k + C_k)]^2 - 4A_{k-1}C_k} \sim N^2\sqrt{[y - (A(k/N) + C(k/N))]^2 - 4A(k/N)C(k/N)}, \tag{2.7}$$

where $A(k/N)$ and $C(k/N)$ are given in (2.2) with $t = k/N$. Secondly, we have

$$\frac{A_{k+1} - A_k}{N} \sim A^*(k/N) := \frac{4\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4 - (\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 + \mathcal{A}_1\mathcal{A}_2\mathcal{A}_4 + \mathcal{A}_1\mathcal{A}_3\mathcal{A}_4 + \mathcal{A}_2\mathcal{A}_3\mathcal{A}_4)\mathcal{D}}{\mathcal{D}^3},$$
$$\frac{C_{k+1} - C_k}{N} \sim C^*(k/N) := \frac{4\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\mathcal{C}_4 - (\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3 + \mathcal{C}_1\mathcal{C}_2\mathcal{C}_4 + \mathcal{C}_1\mathcal{C}_3\mathcal{C}_4 + \mathcal{C}_2\mathcal{C}_3\mathcal{C}_4)\mathcal{D}}{\mathcal{D}^3}, \tag{2.8}$$

where

$$\mathcal{A}_1(t) := t + a,\ \mathcal{A}_2(t) := t + a + b,\ \mathcal{A}_3(t) := t + b + d, \tag{2.9a}$$

$$\mathcal{A}_4(t) := t + c,\ \mathcal{C}_1(t) := t,\ \mathcal{C}_2(t) := t + a + b - c, \tag{2.9b}$$

and

$$\mathcal{C}_3(t) := t + a - d,\ \mathcal{C}_4(t) := t + b,\ \mathcal{D}(t) := 2t + a + b. \tag{2.9c}$$

Using the above notations, we can rewrite

$$A(t) = -\mathcal{A}_1(t)\mathcal{A}_2(t)\mathcal{A}_3(t)\mathcal{A}_4(t)/\mathcal{D}(t)^2,$$

and

$$C(t) = -\mathcal{C}_1(t)\mathcal{C}_2(t)\mathcal{C}_3(t)\mathcal{C}_4(t)/\mathcal{D}(t)^2.$$

Finally, we approximate

$$\sqrt{[z - (A_k + C_k)]^2 - 4A_{k-1}C_k} - \sqrt{[z - (A_{k+1} + C_{k+1})]^2 - 4A_k C_{k+1}}$$

by multiplying and dividing by

$$\sqrt{[z - (A_k + C_k)]^2 - 4A_{k-1}C_k} + \sqrt{[z - (A_{k+1} + C_{k+1})]^2 - 4A_k C_{k+1}}.$$

In the resulting expression, we replace the denominator by using the approximant in (2.7), and replace $A_{k+1} - A_k$ and $C_{k+1} - C_k$ by using $A^*(k/N)$ and $C^*(k/N)$ given in (2.8). The final result is

$$\frac{N\{[y - A(k/N) - C(k/N)][A^*(k/N) + C^*(k/N)] + 2[A(k/N)C^*(k/N) + A^*(k/N)C(k/N)]\}}{\sqrt{[y - (A(k/N) + C(k/N))]^2 - 4A(k/N)C(k/N)}}. \tag{2.10}$$

For convenience, we introduce the notations

$$S(y;t) := \sqrt{(y - A(t) - C(t))^2 - 4A(t)C(t)} = \sqrt{[y - y_-(t)][y - y_+(t)]}, \tag{2.11}$$

$$T(y;t) := y - A(t) - C(t) + S(y;t) = [\sqrt{y - y_-(t)} + \sqrt{y - y_+(t)}]^2/2, \tag{2.12}$$

and

$$G(y;t) := T(y;t)[A^*(t) + C^*(t)] + 2[A(t)C^*(t) + A^*(t)C(t)]. \tag{2.13}$$

Note that $S(y;t)$ is just the denominator in (2.10) with $t = k/N$.

Substituting the estimates (2.7), (2.8) and (2.10) into (2.5), and using the notations (2.11)–(2.13), we obtain

$$w_k^1 = \frac{G(y;k/N)}{2NS(y;k/N)^2} + O(1/N^2).$$

It then follows from the trapezoidal rule (cf. [10]) that

$$\sum_{k=1}^{n} w_k^1 = \int_0^p \frac{G(y;t)}{2S(y;t)^2} dt + O(1/n). \tag{2.14}$$

Next, we approximate $\sum_{k=1}^{n} \ln w_k^0$. Note that

$$\frac{A_k}{N^2} \sim A(k/N)\left(1 - \frac{1}{N\mathcal{D}}\right), \qquad \frac{C_k}{N^2} \sim C(k/N)\left(1 + \frac{1}{N\mathcal{D}}\right),$$

and

$$\frac{A_{k-1}}{N^2} \sim A(k/N)\left(1 + \frac{3}{N\mathcal{D}} - \frac{1}{N\mathcal{A}_1} - \frac{1}{N\mathcal{A}_2} - \frac{1}{N\mathcal{A}_3} - \frac{1}{N\mathcal{A}_4}\right),$$

where $\mathcal{A}_1, \cdots, \mathcal{A}_4$ and $\mathcal{D}$ are given in (2.9).

Furthermore, we obtain from (2.4) that

$$\begin{aligned}\frac{2w_k^0}{N^2} &= \frac{z - A_k - C_k + \sqrt{(z - A_k - C_k)^2 - 4A_kC_k}}{N^2} \\ &\quad + \frac{\sqrt{(z - A_k - C_k)^2 - 4A_{k-1}C_k} - \sqrt{(z - A_k - C_k)^2 - 4A_kC_k}}{N^2}.\end{aligned} \tag{2.15}$$

The first fraction on the right-hand side of (2.15) can be approximated as

$$\begin{aligned}&y - A(k/N) + \frac{A(k/N)}{N\mathcal{D}} - C(k/N) - \frac{C(k/N)}{N\mathcal{D}} \\ &\quad + \sqrt{\left[y - A(k/N) - C(k/N) + \frac{A(k/N) - C(k/N)}{N\mathcal{D}}\right]^2 - A(k/N)C(k/N)} \\ &\sim y - A(k/N) - C(k/N) + \sqrt{[y - A(k/N) - C(k/N)]^2 - A(k/N)C(k/N)} + \frac{A(k/N) - C(k/N)}{N\mathcal{D}} \\ &\quad + \frac{[A(k/N) - C(k/N)][y - A(k/N) - C(k/N)]}{N\mathcal{D}\sqrt{[y - A(k/N) - C(k/N)]^2 - A(k/N)C(k/N)}} \\ &\sim T(y; k/N) + \frac{[A(k/N) - C(k/N)]T(y; k/N)}{N\mathcal{D}(k/N)S(y; k/N)}.\end{aligned}$$

The second fraction on the right-hand side of (2.15) can be approximated as

$$\begin{aligned}&\frac{-4(A_{k-1} - A_k)C_k}{N^2[\sqrt{(z - A_k - C_k)^2 - 4A_{k-1}C_k} + \sqrt{(z - A_k - C_k)^2 - 4A_kC_k}]} \\ &\sim \frac{-2A(k/N)C(k/N)}{S(y; k/N)}\left(\frac{4}{N\mathcal{D}} - \frac{1}{N\mathcal{A}_1} - \frac{1}{N\mathcal{A}_2} - \frac{1}{N\mathcal{A}_3} - \frac{1}{N\mathcal{A}_4}\right).\end{aligned}$$

Applying the above two estimates to (2.15) yields

$$\frac{2w_k^0}{N^2} \sim T(y; k/N) + \frac{[A(k/N) - C(k/N)]T(y; k/N)}{N\mathcal{D}(k/N)S(y; k/N)} - \frac{2A(k/N)C(k/N)\mathcal{E}(k/N)}{NS(y; k/N)},$$

where

$$\mathcal{E}(t) := \frac{4}{\mathcal{D}(t)} - \frac{1}{\mathcal{A}_1(t)} - \frac{1}{\mathcal{A}_2(t)} - \frac{1}{\mathcal{A}_3(t)} - \frac{1}{\mathcal{A}_4(t)}. \tag{2.16}$$

Thus, we obtain from the trapezoidal rule that

$$\sum_{k=1}^{n} \ln w_k^0 = n \ln(N^2/2) + N \int_0^p \ln T(y;t) dt + \frac{1}{2} \ln \frac{T(y;p)}{T(y;0)}$$
$$+ \int_0^p \left[\frac{A(t) - C(t)}{\mathcal{D}(t) S(y;t)} - \frac{2A(t)C(t)\mathcal{E}(t)}{S(y;t)T(y;t)} \right] dt + O(1/n). \tag{2.17}$$

Now, we are ready to state our first main result.

Theorem 2.1. *Let $\pi_n(z)$ be the monic Racah polynomials satisfying the recurrence relation (1.3). Assume $n/N = p$ is fixed in $(0, 1)$. Let $A(t)$, $C(t)$ and $y_\pm(t)$ be defined as in (2.2) and (2.3). Also, let $\Omega_1(y)$ and $\Omega_0(y)$ denote, respectively, the integrals in (2.14) and (2.17). Assume that $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0, p]$. Then, for $y \in \mathbf{C} \setminus [y_-(p), y_+(p)]$, we have*

$$\pi_n(N^2 y) = \left(\frac{N^2}{2}\right)^n e^{Ng(y)+r(y)} \left[1 + O\left(\frac{1}{n}\right)\right], \tag{2.18}$$

where the main term $g(y)$ is given by

$$g(y) := \int_0^p \ln T(y;t) dt, \tag{2.19}$$

and the correction term $r(y)$ is given by

$$r(y) := \frac{1}{2} \ln \frac{T(y;p)}{T(y;0)} + \Omega_0(y) + \Omega_1(y). \tag{2.20}$$

Proof. Use a combination of (2.6), (2.14) and (2.17). □

Next, we investigate $\pi_n(N^2 y)$ for y in a small neighborhood of the oscillatory interval $(y_-(p), y_+(p))$. For $y \in (y_-(p), y_+(p))$, we use the notation

$$F^\pm(y;t) := \lim_{\varepsilon \to 0^+} F(y \pm i\varepsilon; t). \tag{2.21}$$

Thus,

$$S^\pm(y) := \lim_{\varepsilon \to 0^+} S(y \pm i\varepsilon), \qquad g^\pm(y) := \lim_{\varepsilon \to 0^+} g(y \pm i\varepsilon), \cdots, \text{etc.} \tag{2.22}$$

Recall that $y_\pm(0) = A(0)$. We consider two cases: $y \in (A(0), y_+(p))$ and $y \in (y_-(p), A(0))$, separately.

If $A(0) = y_+(0) < y < y_+(p)$, we choose $t_y \in (0, p)$ such that

$$y = y_+(t_y) = A(t_y) + C(t_y) + 2\sqrt{A(t_y)C(t_y)};$$

cf. (2.3). In this case, we have

$$S^\pm(y;t) = \begin{cases} \sqrt{[y - y_-(t)][y - y_+(t)]}, & 0 \le t \le t_y, \\ \pm i\sqrt{[y - y_-(t)][y_+(t) - y]}, & t_y \le t \le p. \end{cases} \tag{2.23a}$$

If $y_-(p) < y < y_-(0) = A(0)$, we choose $t_y \in (0, p)$ such that

$$y = y_-(t_y) = A(t_y) + C(t_y) - 2\sqrt{A(t_y)C(t_y)},$$

in which case we have

$$S^{\pm}(y;t) = \begin{cases} -\sqrt{[y_-(t) - y][y_+(t) - y]}, & 0 \le t \le t_y, \\ \pm i\sqrt{[y - y_-(t)][y_+(t) - y]}, & t_y \le t \le p. \end{cases} \tag{2.23b}$$

Using the definition of $F^{\pm}(y;t)$ given in (2.21), it follows from (2.19) and (2.20) that

$$g^{\pm}(y) = \int_0^p \ln T^{\pm}(y;t)dt, \tag{2.24}$$

where

$$T^{\pm}(y;t) = y - A(t) - C(t) + S^{\pm}(y;t); \tag{2.25}$$

and

$$r^{\pm}(y) = \frac{1}{2}\ln\frac{T^{\pm}(y;p)}{T^{\pm}(y;0)} + \Omega_0^{\pm}(y) + \Omega_1^{\pm}(y), \tag{2.26}$$

where

$$\Omega_0^{\pm}(y) = \int_0^p \left[\frac{A(t) - C(t)}{\mathcal{D}(t)S^{\pm}(y;t)} - \frac{2A(t)C(t)\mathcal{E}(t)}{S^{\pm}(y;t)T^{\pm}(y;t)}\right] dt, \tag{2.27}$$

and

$$\Omega_1^{\pm}(y) = P.V. \int_0^p \frac{G(y;t)}{2S(y;t)^2} dt \mp i\pi \frac{G(y;t_y)}{2y_-'(t_y)[y - y_+(t_y)]} \tag{2.28a}$$

if $y_-(p) < y < y_-(0) = A(0)$, and

$$\Omega_1^{\pm}(y) = P.V. \int_0^p \frac{G(y;t)}{2S(y;t)^2} dt \mp i\pi \frac{G(y;t_y)}{2[y - y_-(t_y)]y_+'(t_y)} \tag{2.28b}$$

if $A(0) = y_+(0) < y < y_+(p)$. In (2.28), "P.V." denotes the Cauchy principal value.

An explanation is needed for the results in (2.28). In view of the second equality in (2.11), the integral $\Omega_1(y)$ in (2.14) (see the statement of Theorem 2.1) has a singularity at $t = t_y$. For this reason, we digress briefly to discuss the Stieltjes transform

$$F(z) = \int_a^b \frac{f(s)}{s - z} ds, \quad z \in \mathbf{C} \setminus [a, b].$$

By using Cauchy's theorem, one can show that

$$F^{\pm}(x) = P.V. \int_a^b \frac{f(s)}{s - x} dx \pm i\pi f(x), \ a < x < b. \tag{2.29}$$

We first consider the case $A(0) = y_+(0) < y < y_+(p)$. The integral $\Omega_1(y)$ can be written as

$$\Omega_1(y) = \int_0^p \frac{(t-t_y)G(y;t)}{2S(y;t)^2(t-t_y)}dt.$$

From (2.29), it follows that

$$\begin{aligned}\Omega_1^{\pm}(y) &= P.V. \int_0^p \frac{G(y;t)}{2S(y;t)^2}dt \pm i\pi \lim_{t\to t_y} \frac{t-t_y}{2S(y;t)^2}G(y;t) \\ &= P.V. \int_0^p \frac{G(y;t)}{2S(y;t)^2}dt \mp i\pi \frac{G(y;t_y)}{2[y-y_-(t_y)]y_+'(t_y)}.\end{aligned}$$

In the case $y_-(p) < y < y_-(0) = A(0)$, the result is

$$\Omega_1^{\pm}(y) = P.V. \int_0^p \frac{G(y;t)}{2S(y;t)^2}dt \mp i\pi \frac{G(y;t_y)}{2y_-'(t_y)[y-y_+(t_y)]}.$$

Our second main theorem is stated below.

Theorem 2.2. *Let $\pi_n(z)$ be the monic Racah polynomials satisfying the recurrence relation (1.3). Assume $n/N = p$ is fixed in $(0,1)$. Let $A(t)$, $C(t)$ and $y_\pm(t)$ be defined as in (2.2) and (2.3). We further assume that $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0,p]$. Then, we have*

$$\pi_n(N^2y) = \left(\frac{N^2}{2}\right)^n \left\{ e^{Ng^+(y)+r^+(y)}\left[1+O\left(\frac{1}{n}\right)\right] + e^{Ng^-(y)+r^-(y)}\left[1+O\left(\frac{1}{n}\right)\right]\right\} \tag{2.30}$$

for $y \in (y_-(p), A(0)) \cup (A(0), y_+(p))$.

Proof. First, we note that the functions $g^\pm(y)$ and $r^\pm(y)$ can be analytically continued to a small neighborhood of any compact subset of $(y_-(p), A(0)) \cup (A(0), y_+(p))$ in the complex plane. In such a neighborhood, by (2.19) and (2.12),

$$g_+(y) - g_-(y) = \int_{t_y}^p \ln \frac{T^+(y;t)}{T^-(y;t)}dt$$

has positive real part if y is in the upper-half plane and negative real part if y is in the lower-half plane. Thus, the function $e^{N[g^+(y)-g^-(y)]}$ is exponentially large on the upper-half plane and exponentially small on the lower-half plane, which together with (2.18) implies that the asymptotic formula (2.30) is valid on both upper- and lower-half planes. By analytical continuation, this formula is also valid for $y \in (y_-(p), A(0)) \cup (A(0), y_+(p))$; see [8,10]. □

Remark 2.3. A standard application of the uniform technique developed by Wang and Wong [9] allows us to provide uniform asymptotic formulas of Racah polynomials near the turning points $y_-(p)$ and $y_+(p)$. For the sake of simplicity, we omit the details in this paper.

3. Limiting cases: Hahn polynomials

The Hahn polynomials are defined by the ${}_3F_2$ hypergeometric series [7, (9.5.1)]

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -n, & n+\alpha+\beta+1, & -x \\ & \alpha+1, & -N \end{matrix} \middle| 1\right), \quad n = 0, \cdots, N. \tag{3.1}$$

They are the limiting cases of Racah polynomials in the following sense [7, (9.2.15)]

$$\lim_{\delta\to\infty} R_n(\lambda(x), \alpha, \beta, -N-1, \delta) = Q_n(x;\alpha,\beta,N). \tag{3.2}$$

Here $\lambda(x) = x(x+\gamma+\delta+1)$ and $\gamma = -N-1$. Since asymptotic formulas of Racah polynomials hold uniformly with respect to the parameter δ, asymptotic formulas of the Hahn polynomials can be deduced from the formulas in (2.18) and (2.30). Let

$$\bar{\pi}_n(x) := \frac{(\alpha+1)_n(-N)_n}{(n+\alpha+\beta+1)_n} Q_n(x;\alpha,\beta,N) \tag{3.3}$$

be the monic Hahn polynomials. From (1.4), (3.1), (3.2) and (3.3), it follows that

$$\bar{\pi}_n(x) = \lim_{\delta\to\infty} \frac{\pi_n(\lambda(x))}{(\beta+\delta+1)_n}. \tag{3.4}$$

Recall that

$$z = N^2 y = \lambda(x),\ \alpha+1 = Na,\ \beta = Nb,\ \gamma+1 = Nc,\ \delta+1 = Nd.$$

With $\gamma = -N-1$ in (3.2), we must take $c = -1$. Since $\delta \to \infty$, we also have $d \to \infty$; cf. the equalities in the preceding line. Set $x = N\bar{y}$ and take logarithms on both sides of (3.4). Since $(\beta+\delta+1)_n = (Nb+Nd)_n \sim (Nd)^n$ as $d \to \infty$, we obtain

$$\ln \bar{\pi}_n(N\bar{y}) = \lim_{d\to\infty} [\ln \pi_n(N^2 y) - n \ln N - n \ln d]. \tag{3.5}$$

Furthermore, since $z = N^2 y = \lambda(N\bar{y}) = N\bar{y}(N\bar{y} + Nc + Nd - 1)$, we have $y/d \to \bar{y}$ as $d \to \infty$. From (2.2), it follows that for $t \in [0, p]$,

$$\begin{aligned} \lim_{d\to\infty} \frac{A(t)}{d} &= \frac{(t+a)(t+a+b)(1-t)}{(2t+a+b)^2} =: \bar{A}(t), \\ \lim_{d\to\infty} \frac{C(t)}{d} &= \frac{t(t+a+b+1)(t+b)}{(2t+a+b)^2} =: \bar{C}(t). \end{aligned} \tag{3.6}$$

Note from (2.9) that $\mathcal{A}_3(t)/d \to 1$ and $\mathcal{C}_3(t)/d \to -1$ as $d \to \infty$. Thus, we have from (2.8) and (2.9)

$$\begin{aligned} \bar{A}^*(t) &:= \lim_{d\to\infty} \frac{A^*(t)}{d} = \frac{4\mathcal{A}_1(t)\mathcal{A}_2(t)\mathcal{A}_4(t) - (\mathcal{A}_1(t)\mathcal{A}_2(t) + \mathcal{A}_1(t)\mathcal{A}_4(t) + \mathcal{A}_2(t)\mathcal{A}_4(t))\mathcal{D}(t)}{\mathcal{D}^3(t)}, \\ \bar{C}^*(t) &:= \lim_{d\to\infty} \frac{C^*(t)}{d} = \frac{(\mathcal{C}_1(t)\mathcal{C}_2(t) + \mathcal{C}_1(t)\mathcal{C}_4(t) + \mathcal{C}_2(t)\mathcal{C}_4(t))\mathcal{D}(t) - 4\mathcal{C}_1(t)\mathcal{C}_2(t)\mathcal{C}_4(t)}{\mathcal{D}^3(t)}, \end{aligned} \tag{3.7}$$

where $\mathcal{A}_i(t), \mathcal{C}_i(t)$ (with $i = 1, 2, 4$) and $\mathcal{D}(t)$ are defined as in (2.9) with c replaced by -1. Finally, by (2.11) and (2.12), we obtain

$$\bar{S}(\bar{y};t) := \lim_{d\to\infty} \frac{S(y;t)}{d} = \sqrt{[\bar{y} - \bar{A}(t) - \bar{C}(t)]^2 - 4\bar{A}(t)\bar{C}(t)} \tag{3.8}$$

and

$$\bar{T}(\bar{y};t) := \lim_{d\to\infty} \frac{T(y;t)}{d} = \bar{y} - \bar{A}(t) - \bar{C}(t) + \bar{S}(\bar{y};t). \tag{3.9}$$

On account of (2.13) and (2.16), we also have

$$\bar{G}(\bar{y};t) := \lim_{d\to\infty} \frac{G(y;t)}{d^2} = \bar{T}(\bar{y};t)[\bar{A}^*(t) + \bar{C}^*(t)] + 2[\bar{A}(t)\bar{C}^*(t) + \bar{A}^*(t)\bar{C}(t)] \tag{3.10}$$

and

$$\bar{\mathcal{E}}(t) := \lim_{d\to\infty} \mathcal{E}(t) = \frac{4}{\mathcal{D}(t)} - \frac{1}{\mathcal{A}_1(t)} - \frac{1}{\mathcal{A}_2(t)} - \frac{1}{\mathcal{A}_4(t)}. \tag{3.11}$$

Combining (2.3) and (3.6) gives

$$\bar{y}_\pm(t) := \lim_{d\to\infty} \frac{y_\pm(t)}{d} = \bar{A}(t) + \bar{C}(t) \pm 2\sqrt{\bar{A}(t)\bar{C}(t)}. \tag{3.12}$$

From (2.18) and (3.5), we have the following result.

Corollary 3.1. *Let $\alpha = Na$ and $\beta = Nb$ with $a, b > 0$. Assume $n/N = p$ is a fixed number in $(0, 1)$. Let $\bar{y}_\pm(t)$ be defined as in (3.12). Assume $\bar{y}_+(t)$ is increasing and $\bar{y}_-(t)$ is decreasing for $t \in [0, p]$. As $n \to \infty$, we have the following asymptotic formula for the monic Hahn polynomials $\bar{\pi}_n$ in (3.3):*

$$\bar{\pi}_n(N\bar{y}) = \left(\frac{N}{2}\right)^n e^{N\bar{g}(\bar{y})+\bar{r}(\bar{y})} \left[1 + O\left(\frac{1}{n}\right)\right] \tag{3.13}$$

for $\bar{y} \in \mathbf{C} \setminus [\bar{y}_-(p), \bar{y}_+(p)]$. Here, the main term $\bar{g}(\bar{y})$ is given by

$$\bar{g}(\bar{y}) := \lim_{d\to\infty} [g(y) - p \ln d] = \int_0^p \ln \bar{T}(\bar{y};t)dt, \tag{3.14}$$

and the correction term $\bar{r}(\bar{y})$ is given by

$$\bar{r}(\bar{y}) := \lim_{d\to\infty} r(y) = \frac{1}{2} \ln \frac{\bar{T}(\bar{y};p)}{\bar{T}(\bar{y};0)} + \bar{\Omega}_0(\bar{y}) + \bar{\Omega}_1(\bar{y}), \tag{3.15}$$

where

$$\bar{\Omega}_0(\bar{y}) := \lim_{d\to\infty} \Omega_0(y) = \int_0^p \left[\frac{\bar{A}(t) - \bar{C}(t)}{\mathcal{D}(t)\bar{S}(\bar{y};t)} - \frac{2\bar{A}(t)\bar{C}(t)\bar{\mathcal{E}}(t)}{\bar{S}(\bar{y};t)\bar{T}(\bar{y};t)}\right] dt, \tag{3.16}$$

and

$$\bar{\Omega}_1(\bar{y}) := \lim_{d\to\infty} \Omega_1(y) = \int_0^p \frac{\bar{G}(\bar{y};t)}{2\bar{S}(\bar{y};t)^2} dt. \tag{3.17}$$

Equation (3.14) follows from (2.19) and (3.9), and equation (3.15) is obtained from (2.20), (2.17) and (2.14); see also (3.8), (3.10) and (3.11).

The asymptotic formula of monic Hahn polynomials $\bar{\pi}_n(N\bar{y})$ in the oscillatory interval can be obtained by using either the matching method or by taking a limit in the corresponding formula of Racah polynomials.

Corollary 3.2. *Let $\alpha = Na$ and $\beta = Nb$ with $a, b > 0$. Assume $n/N = p$ is a fixed number in $(0,1)$. Let $\bar{y}_\pm(t)$ be defined as in (3.12). Assume $\bar{y}_+(t)$ is increasing and $\bar{y}_-(t)$ is decreasing for $t \in [0,p]$. Let $\bar{S}^\pm(\bar{y};t)$ and $\bar{T}^\pm(\bar{y};t)$ be defined similarly as in (2.23) and (2.25). As $n \to \infty$, we have*

$$\bar{\pi}_n(N\bar{y}) = \left(\frac{N}{2}\right)^n \left\{ e^{N\bar{g}^+(\bar{y})+\bar{r}^+(\bar{y})} \left[1 + O\left(\frac{1}{n}\right)\right] + e^{N\bar{g}^-(\bar{y})+\bar{r}^-(\bar{y})} \left[1 + O\left(\frac{1}{n}\right)\right]\right\} \tag{3.18}$$

for $\bar{y} \in (\bar{y}_-(p), \bar{A}(0)) \cup (\bar{A}(0), \bar{y}_+(p))$. Here,

$$\bar{g}^\pm(\bar{y}) = \int_0^p \ln \bar{T}^\pm(\bar{y};t)dt, \tag{3.19}$$

and

$$\bar{r}^\pm(\bar{y}) = \frac{1}{2}\ln\frac{\bar{T}^\pm(\bar{y};p)}{\bar{T}^\pm(\bar{y};0)} + \bar{\Omega}_0^\pm(\bar{y}) + \bar{\Omega}_1^\pm(\bar{y}), \tag{3.20}$$

where

$$\bar{\Omega}_0^\pm(\bar{y}) = \int_0^p \left[\frac{\bar{A}(t)-\bar{C}(t)}{\mathcal{D}(t)\bar{S}^\pm(\bar{y};t)} - \frac{2\bar{A}(t)\bar{C}(t)\bar{\mathcal{E}}(t)}{\bar{S}^\pm(\bar{y};t)\bar{T}^\pm(\bar{y};t)}\right]dt, \tag{3.21}$$

and

$$\bar{\Omega}_1^\pm(\bar{y}) = P.V.\int_0^p \frac{\bar{G}(\bar{y};t)}{2\bar{S}(\bar{y};t)^2}dt \mp i\pi\frac{\bar{G}(\bar{y};t_{\bar{y}})}{2\bar{y}_-'(t_{\bar{y}})[\bar{y}-\bar{y}_+(t_{\bar{y}})]} \tag{3.22a}$$

if $\bar{y}_-(p) < \bar{y} < \bar{y}_-(0) = \bar{A}(0)$ with $\bar{y} = \bar{y}_-(t_{\bar{y}})$ for some $t_{\bar{y}} \in (0,p)$, and

$$\bar{\Omega}_1^\pm(\bar{y}) = P.V.\int_0^p \frac{\bar{G}(\bar{y};t)}{2\bar{S}(\bar{y};t)^2}dt \mp i\pi\frac{\bar{G}(\bar{y};t_{\bar{y}})}{2[\bar{y}-\bar{y}_-(t_{\bar{y}})]\bar{y}_+'(t_{\bar{y}})} \tag{3.22b}$$

if $\bar{A}(0) = \bar{y}_+(0) < \bar{y} < \bar{y}_+(p)$ with $\bar{y} = \bar{y}_+(t_{\bar{y}})$ for some $t_{\bar{y}} \in (0,p)$.

4. Limiting cases: Krawtchouk polynomials

The Krawtchouk polynomials are defined by the ${}_2F_1$ hypergeometric series [7, (9.11.1)]

$$K_n(x;q,N) = {}_2F_1\left(\begin{matrix}-n, & -x \\ & -N\end{matrix}\middle|\frac{1}{q}\right), \quad n = 0,\cdots,N. \tag{4.1}$$

These polynomials are limiting cases of Hahn polynomials in the following sense. Fix $q \in (0,1)$, and let $\alpha/\beta = q/(1-q)$ and $\beta \to \infty$. We have [7, (9.5.16)]

$$\lim_{\beta\to\infty} Q_n(x;\beta q/(1-q),\beta,N) = K_n(x;q,N). \tag{4.2}$$

The monic Krawtchouk polynomials are given by

$$\tilde{\pi}_n(x) := (-N)_n q^n K_n(x;q,N). \tag{4.3}$$

Recall that $\alpha+1 = Na$ and $\beta = Nb$. We choose $a = mq$ and $b = m(1-q)$, and then let $m \to \infty$. In view of (3.3), (4.1), (4.2) and (4.3), we have the following limiting relation between monic Hahn polynomials and monic Krawtchouk polynomials.

$$\lim_{m\to\infty} \bar{\pi}_n(x) = \tilde{\pi}_n(x). \tag{4.4}$$

Furthermore, from (3.6), (3.7) and (2.9) with $a = mq$ and $b = m(1-q)$, we obtain for $t \in [0,p]$,

$$\begin{aligned}\lim_{m\to\infty} \bar{A}(t) &= q(1-t) =: \tilde{A}(t),\\ \lim_{m\to\infty} \bar{C}(t) &= t(1-q) =: \tilde{C}(t),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\lim_{m\to\infty} \bar{A}^*(t) &= -q,\\ \lim_{m\to\infty} \bar{C}^*(t) &= 1-q.\end{aligned} \tag{4.6}$$

Recall that in the previous section, we have chosen $c = -1$ to obtain Hahn polynomials from Racah polynomials. In view of (2.9), we have as $m \to \infty$

$$1/\mathcal{D}(t) \to 0,\ 1/\mathcal{A}_1(t) \to 0,\ 1/\mathcal{A}_2(t) \to 0,\ 1/\mathcal{A}_4(t) = 1/(t-1). \tag{4.7}$$

Let $\tilde{y} = \bar{y}$. From (3.8), (3.9) and (4.5), it follows that

$$\tilde{S}(\tilde{y};t) := \lim_{m\to\infty} \bar{S}(\bar{y};t) = \sqrt{[\tilde{y} - \tilde{A}(t) - \tilde{C}(t)]^2 - 4\tilde{A}(t)\tilde{C}(t)}, \tag{4.8}$$

and

$$\tilde{T}(\tilde{y};t) := \lim_{m\to\infty} \bar{T}(\bar{y};t) = \tilde{y} - \tilde{A}(t) - \tilde{C}(t) + \tilde{S}(\tilde{y};t). \tag{4.9}$$

Similarly, from (3.10), (4.6) and (4.5), we obtain

$$\tilde{G}(\tilde{y};t) := \lim_{m\to\infty} \bar{G}(\bar{y};t) = \tilde{T}(\tilde{y};t)(1-2q) + 2q(1-q)(1-2t). \tag{4.10}$$

Furthermore, in view of (3.11) and (4.7), we also have

$$\tilde{\mathcal{E}}(t) := \lim_{m\to\infty} \bar{\mathcal{E}}(t) = -\frac{1}{t-1}. \tag{4.11}$$

Coupling (3.12) and (4.5) gives

$$\tilde{y}_\pm(t) := \lim_{m\to\infty} \bar{y}_\pm(t) = \tilde{A}(t) + \tilde{C}(t) \pm 2\sqrt{\tilde{A}(t)\tilde{C}(t)}, \tag{4.12}$$

and from (3.14) and (4.9) it follows

$$\tilde{g}(\tilde{y}) := \lim_{m\to\infty} \bar{g}(\bar{y}) = \int_0^p \ln \tilde{T}(\tilde{y};t)dt. \tag{4.13}$$

A combination of (3.15)–(3.17), (4.5), (4.7), (4.9) and (4.11) yields

$$\begin{aligned}\tilde{r}(\tilde{y}) &:= \lim_{m\to\infty} \bar{r}(\bar{y}) \\ &= \frac{1}{2}\ln\frac{\tilde{y}-\tilde{A}(p)-\tilde{C}(p)+\tilde{S}(\tilde{y};p)}{\tilde{y}-\tilde{A}(0)-\tilde{C}(0)+\tilde{S}(\tilde{y};0)} + \int_0^p \frac{\tilde{G}(\tilde{y};t)}{2\tilde{S}(\tilde{y};t)^2}dt - \int_0^p \frac{2tq(1-q)}{\tilde{S}(\tilde{y};t)\tilde{T}(\tilde{y};t)}dt.\end{aligned} \tag{4.14}$$

The last two integrals can be evaluated explicitly. Indeed, on account of (4.5), (4.8) and (4.9), they are equal to

$$\begin{aligned}&\int_0^p \frac{\tilde{y}-2\tilde{y}q+q-t}{2[t^2-2(\tilde{y}-2\tilde{y}q+q)t+(\tilde{y}-q)^2]}dt + \int_0^p \frac{1-2q}{2\sqrt{[t-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}dt \\ &= -\frac{1}{4}\ln\frac{p^2-2(\tilde{y}-2\tilde{y}q+q)p+(\tilde{y}-q)^2}{(\tilde{y}-q)^2} \\ &\quad + \frac{1-2q}{2}\ln\frac{p-(\tilde{y}-2\tilde{y}q+q)+\sqrt{[p-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}{2q(\tilde{y}-1)}\end{aligned}$$

and

$$\begin{aligned}&\int_0^p \frac{\tilde{y}-q-t(1-2q)-\sqrt{[t-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}{2(1-t)\sqrt{[t-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}dt \\ &= \frac{1}{2}\ln\frac{\tilde{y}^2+q^2-\tilde{y}-q-(\tilde{y}-2\tilde{y}q+q-1)p+(\tilde{y}+q-1)\sqrt{[p-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}{2\tilde{y}(\tilde{y}-1)} \\ &\quad + \frac{1-2q}{2}\ln\frac{p-(\tilde{y}-2\tilde{y}q+q)+\sqrt{[p-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}{2q(\tilde{y}-1)}\end{aligned}$$

respectively, where we have also made use of the following integral formulas [5]

$$\int \frac{1}{\sqrt{u^2+a}}du = \ln(u+\sqrt{u^2+a}),$$

$$\int \frac{1}{(b-u)\sqrt{u^2+a}}du = \frac{1}{\sqrt{a+b^2}}\ln\frac{2[a+bu+\sqrt{(a+b^2)(a+u^2)}]}{b-u}.$$

Finally, we substitute the above explicit formulas for the two integrals into (4.14). Upon simplification, we obtain

$$\tilde{r}(\tilde{y}) = \frac{1}{2}\ln\frac{\tilde{y}-q-(2\tilde{y}-1)p+\sqrt{[p-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}{2(1-p)\sqrt{[p-(\tilde{y}-2\tilde{y}q+q)]^2+4q(1-q)\tilde{y}(\tilde{y}-1)}}. \tag{4.15}$$

In view of (3.13) and (4.4), we have the following results.

Corollary 4.1. *Let $\tilde{\pi}_n(x)$ be the monic Krawtchouk polynomials as defined in (4.3). Assume that $n/N = p$ is a fixed number in $(0,1)$. Let $\tilde{y}_\pm$ be defined as in (4.12). Assume $\tilde{y}_+(t)$ is increasing and $\tilde{y}_-(t)$ is decreasing for $t \in [0,p]$. As $n \to \infty$, we have*

$$\tilde{\pi}_n(N\tilde{y}) = \left(\frac{N}{2}\right)^n e^{N\tilde{g}(\tilde{y})+\tilde{r}(\tilde{y})}\left[1+O\left(\frac{1}{n}\right)\right] \tag{4.16}$$

for $\tilde{y} \in \mathbf{C} \setminus [\tilde{y}_-(p), \tilde{y}_+(p)]$. Here, $\tilde{g}(\tilde{y})$ and $\tilde{r}(\tilde{y})$ are given by (4.13) and (4.15), respectively.

The asymptotic formula of monic Krawtchouk polynomials $\tilde{\pi}_n(N\tilde{y})$ in the oscillatory interval can again be obtained by either the matching method or taking limit in the corresponding formula of Hahn polynomials.

Corollary 4.2. *Assume that $n/N = p$ is a fixed number in $(0,1)$. Let $\tilde{y}_\pm(t)$ be defined as in (4.12). Assume $\tilde{y}_+(t)$ is increasing and $\tilde{y}_-(t)$ is decreasing for $t \in [0,p]$. Let $\tilde{S}^\pm(\tilde{y};t)$ and $\tilde{T}^\pm(\tilde{y};t)$ be defined similarly as in (2.23) and (2.25). As $n \to \infty$, we have*

$$\tilde{\pi}_n(N\tilde{y}) = \left(\frac{N}{2}\right)^n \left\{e^{N\tilde{g}^+(\tilde{y})+\tilde{r}^+(\tilde{y})}\left[1+O\left(\frac{1}{n}\right)\right] + e^{N\tilde{g}^-(\tilde{y})+\tilde{r}^-(\tilde{y})}\left[1+O\left(\frac{1}{n}\right)\right]\right\} \tag{4.17}$$

for $\tilde{y} \in (\tilde{y}_-(p), \tilde{A}(0)) \cup (\tilde{A}(0), \tilde{y}_+(p))$. Here,

$$\tilde{g}^\pm(\tilde{y}) = \int_0^p \ln \tilde{T}^\pm(\tilde{y};t)dt, \tag{4.18}$$

and

$$\tilde{r}^\pm(\tilde{y}) = \frac{1}{2}\ln\frac{\tilde{y} - q - (2\tilde{y}-1)p + \tilde{S}^\pm(\tilde{y};p)}{2(1-p)\tilde{S}^\pm(\tilde{y};p)}. \tag{4.19}$$

Remark 4.3. Our assumption that $\tilde{y}_+(t)$ is increasing and $\tilde{y}_-(t)$ is decreasing for $t \in [0,p]$ corresponds to the void-band-void (VBV) case considered in [4]. We have verified that the results in this section agree with those in [4] and [6], although the work is very involved.

5. Racah polynomials with fixed variable

In this section, we shall provide a simple formula for the Racah polynomials with fixed variable z. For simplicity, we assume $A(t) > C(t)$ for all $t \in [0,p]$. From (2.3), we have

$$\sqrt{y_\pm(t)} = \sqrt{A(t)} \pm \sqrt{C(t)}.$$

Since $0 < y_-(t) < y_+(t)$, we obtain from (2.11) and the above equation

$$S(0;t) = -\sqrt{y_-(t)}\sqrt{y_+(t)} = -[A(t) - C(t)] < 0.$$

Here, we remark that the square roots $\sqrt{y - y_-(t)}$ and $\sqrt{y - y_+(t)}$ in (2.11) are taking their principal values; thus $S(y;t) < 0$ for all real $y < y_-(t) < y_+(t)$. Consequently, we have

$$S(z/N^2;t) = S(0;t) + O(1/N^2) = C(t) - A(t) + O(1/N^2). \tag{5.1}$$

Coupling the above equation with (2.12) gives

$$T(z/N^2;t) = -2A(t) + O(1/N^2). \tag{5.2}$$

In view of $n = Np$, we substitute (5.2) into (2.19) to obtain

$$\exp[Ng(z/N^2)] = (-2)^n \exp\left[N\int_0^p \ln A(t)dt\right][1 + O(1/N)]. \tag{5.3}$$

Next, we intend to estimate the correction function $r(z/N^2)$. Recall from (2.20) that

$$r(z/N^2) = \frac{1}{2}\ln\frac{T(z/N^2;p)}{T(z/N^2;0)} + \Omega_0(z/N^2) + \Omega_1(z/N^2), \tag{5.4}$$

where Ω_0 and Ω_1 are the two integrals defined in (2.17) and (2.14), respectively; namely,

$$\Omega_0(z/N^2) = \int_0^p \left[\frac{A(t) - C(t)}{\mathcal{D}(t)S(z/N^2;t)} - \frac{2A(t)C(t)\mathcal{E}(t)}{S(z/N^2;t)T(z/N^2;t)}\right] dt, \tag{5.5}$$

$$\Omega_1(z/N^2) = \int_0^p \frac{G(z/N^2;t)}{2S(z/N^2;t)^2} dt. \tag{5.6}$$

Substituting (5.1) and (5.2) into (5.5) yields

$$\Omega_0(z/N^2) = \int_0^p \left[\frac{-1}{\mathcal{D}(t)} + \frac{C(t)\mathcal{E}(t)}{C(t) - A(t)}\right] dt + O(1/N^2), \tag{5.7}$$

and substituting (5.2) into (2.13) gives

$$\begin{aligned} G(z/N^2;t) &= -2A(t)[A^*(t) + C^*(t)] + 2[A(t)C^*(t) + A^*(t)C(t)] + O(1/N^2) \\ &= 2A^*(t)[C(t) - A(t)] + O(1/N^2). \end{aligned} \tag{5.8}$$

From (5.1), (5.6) and (5.8), it then follows that

$$\Omega_1(z/N^2) = \int_0^p \frac{A^*(t)}{C(t) - A(t)} dt + O(1/N^2) = \int_0^p \frac{-A(t)\mathcal{E}(t)}{C(t) - A(t)} dt + O(1/N^2), \tag{5.9}$$

where we have used the identity $A^*(t) = -A(t)\mathcal{E}(t)$, which can be obtained via (2.2), (2.8), (2.9) and (2.16). Coupling (5.7) and (5.9), we obtain

$$\begin{aligned} &\Omega_0(z/N^2) + \Omega_1(z/N^2) \\ &= \int_0^p \left[\frac{-1}{\mathcal{D}(t)} + \mathcal{E}(t)\right] dt + O(1/N^2) \\ &= \int_0^p \left[\frac{3}{2t + a + b} - \frac{1}{t + a} - \frac{1}{t + a + b} - \frac{1}{t + b + d} - \frac{1}{t + c}\right] dt + O(1/N^2) \\ &= \frac{3}{2}\ln\frac{2p + a + b}{a + b} + \ln\frac{ac(a + b)(b + d)}{(p + a)(p + c)(p + a + b)(p + b + d)} + O(1/N^2). \end{aligned} \tag{5.10}$$

Here, we have made use of the definitions of $\mathcal{D}(t)$ and $\mathcal{E}(t)$ given in (2.9) and (2.16), respectively. On the other hand, we have from (2.2) and (5.2)

$$\ln \frac{T(z/N^2;p)}{T(z/N^2;0)} = \ln \frac{(p+a)(p+c)(p+a+b)(p+b+d)(a+b)}{ac(b+d)(2p+a+b)^2} + O(1/N^2). \tag{5.11}$$

Applying (5.10) and (5.11) to (5.4) yields

$$r(z/N^2) = \frac{1}{2} \ln \frac{ac(b+d)(2p+a+b)}{(p+a)(p+c)(p+b+d)(p+a+b)} + O(1/N^2). \tag{5.12}$$

By substituting (5.3) and (5.12) into (2.18), we obtain the following result.

Corollary 5.1. *Let $\pi_n(z)$ be the monic Racah polynomials satisfying the recurrence relation (1.3). Assume $n/N = p$ is fixed in $(0,1)$. Let $A(t)$, $C(t)$ and $y_\pm(t)$ be defined as in (2.2) and (2.3). We further assume that $A(t) > C(t)$, $y_+(t)$ is increasing and $y_-(t)$ is decreasing for $t \in [0,p]$. Then as $n \to \infty$, we have for any fixed z,*

$$\pi_n(z) = (-N^2)^n \exp\left[N \int_0^p \ln A(t)dt\right] \sqrt{\frac{ac(b+d)(2p+a+b)}{(p+a)(p+c)(p+b+d)(p+a+b)}} \times [1 + O(1/N)]\,. \tag{5.13}$$

Acknowledgments

We would like to thank an anonymous referee for checking the derivations of our main theorems and for providing valuable suggestions which have helped to improve the presentation of this paper. XSW is partially supported by the GRFC (grant number 103417) and Summer Research Grant from Southeast Missouri State University.

References

[1] R. Askey, J. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols, SIAM J. Math. Anal. 10 (1979) 1008–1016.
[2] L.-C. Chen, M.E.H. Ismail, P. Simeonov, Asymptotics of Racah coefficients and polynomials, J. Phys. A 32 (3) (1999) 537–553.
[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Math. Appl., vol. 13, Gordon and Breach Science Publishers, New York, London, Paris, 1978.
[4] D. Dai, R. Wong, Global asymptotics of Krawtchouk polynomials – a Riemann–Hilbert approach, Chin. Ann. Math. Ser. B 28 (1) (2007) 1–34.
[5] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, sixth edition, Academic Press, Inc., San Diego, CA, 2000.
[6] M.E.H. Ismail, P. Simeonov, Strong asymptotics for Krawtchouk polynomials, J. Comput. Appl. Math. 100 (1998) 121–144.
[7] R. Koekoek, P.A. Lesky, R.F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer Monogr. Math., Springer-Verlag, Berlin, 2010.
[8] X.-S. Wang, Plancherel–Rotach asymptotics of second-order difference equations with linear coefficients, J. Approx. Theory 188 (2014) 1–18.
[9] Z. Wang, R. Wong, Asymptotic expansions for second-order linear difference equations with a turning point, Numer. Math. 94 (1) (2003) 147–194.
[10] X.-S. Wang, R. Wong, Asymptotics of orthogonal polynomials via recurrence relations, Anal. Appl. (Singap.) 10 (2012) 215–235.
[11] J.A. Wilson, Hypergeometric series recurrence relations and some new orthogonal functions, PhD thesis, Univ. Wisconsin, Madison, 1978.