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Pulsating waves of a partially degenerate reaction–diffusion system in a periodic habitat ☆

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Abstract

We investigate a partially degenerate reaction–diffusion system in a periodic habitat and prove the existence and stability of pulsating waves. More specifically, we show that if the wave speed is greater than the spreading speed, then there exists a pulsating wave connecting the stable positive periodic steady state to the unstable trivial one. Further, this pulsating wave attracts exponentially in time all solutions with initial functions in its bounded neighborhood with respect to a weighted maximum norm.

MSC: 35K57; 92D25

Keywords: Pulsating waves; Partially degenerate; Reaction–diffusion system; Periodic habitat

1. Introduction

Pulsating waves (also called spatially periodic traveling waves in the literature) were first introduced in [11] to study the invasion of a new migrating species in a heterogeneous environment. The definitions of pulsating waves vary but are equivalent in previous studies (see, e.g., [3,10,18]). Here we adopt the following generalized definition.

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Definition 1.1. For a reaction–diffusion system of general type:

$$\partial_t u = \mathcal{A}(x)u + \mathcal{F}(x,u),\ t \geq 0,\ \ x \in \mathbb{R}^n, u = (u_1, \cdots, u_m), \tag{1.1}$$

where $\mathcal{A} = (\mathcal{A}_1, \cdots, \mathcal{A}_m)$ is a linear diffusion operator (involving local differential operators or/and nonlocal integral operators) and $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_m)$ is a nonlinear operator, assume that $\mathcal{A}$ and $\mathcal{F}$ are periodic in x with the same period. A solution $u(t,x)$ is called a *pulsating wave* connecting two periodic steady states $p_-(x)$ and $p_+(x)$ with speed c and direction e provided that

(i) $u(t,x)$ has the special form $u(t,x) = \Psi(x \cdot e - ct, x)$, where e is a unit vector in $\mathbb{R}^n$ and $\Psi(s,x)$ is periodic in the second variable x with the same period as $\mathcal{A}$ and $\mathcal{F}$.
(ii) As $s \to \pm\infty$, $\Psi(s,x)$ tends to $p_\pm(x)$, respectively, uniformly in $x \in \mathbb{R}^n$.

Note that when $m = 1$ (i.e., the scalar case) and the diffusion term $\mathcal{A}(x)u$ has the divergent form $\nabla \cdot (A(x)\nabla u)$, system (1.1) was investigated by Berestycki, Hamel and Roques [1,2], where they proved the existence and uniqueness of stationary solution and analyzed asymptotic behavior of solutions. Furthermore, they obtained the existence of pulsating waves and a variational characterization of the minimal wave speed.

A nonlocal and time-delayed population model in a periodic habitat was proposed by Weng and Zhao [16], and the authors studied the spatial dynamics of the model system, the global attractiveness of spatially periodic steady state, and the existence of spreading speeds and pulsating waves. This work was further extended by Ouyang and Ou [8] to obtain the stability and convergence rate of pulsating waves.

For the general case ($m \geq 1$), if the solution maps associated with (1.1) are compact with respect to the compact open topology, or more weakly, if the solution maps are α-contractions with respect to the Kuratowski measure of non-compactness, then one may use the abstract theory developed by Weinberger [14,15] and Liang and Zhao [5,6] to show that a pulsating wave exists if and only if the wave speed c is no less than the spreading speed. A natural question is whether this result (i.e., the spreading speed coincides with the minimal wave speed) can be extended to the non-compact case. For the special case of scalar equations ($m = 1$) with a nonlocal integral diffusion operator, the authors of [3,10] gave an affirmative answer. Our purpose is to study a class of partially-degenerate systems, namely, $m > 1$ and some components of the diffusion operator $\mathcal{A} = (\mathcal{A}_1, \cdots, \mathcal{A}_m)$ vanish. For simplicity, we consider the following partially degenerate reaction–diffusion system in a periodic and one-dimensional habitat:

$$\begin{aligned} &\partial_t u_1(t,x) = D_1(x)\partial_{xx}u_1 + D_2(x)\partial_x u_1 + f(x,u_1,u_2), \\ &\partial_t u_2(t,x) = g(x,u_1,u_2). \end{aligned} \tag{1.2}$$

This system is motivated by the benthic–pelagic population model proposed by Lutscher, Lewis and McCauley [7], where u_1 and u_2 are the densities of individuals in the pelagic and benthic zones, respectively. Note that the benthic individuals u_2 do not have any diffusion term, which leads to the non-compactness of solution maps. So we may not expect to apply the abstract results in [5,6,15] to study the spatial dynamics of system (1.2). Recently, Wu, Xiao and Zhao [17] established the existence of spreading speeds by combining the theory of monotone dynamical systems and the ideas in [9]. As a consequence, we can easily conclude that (1.2) admits no pulsating wave if the wave speed is less than the spreading speed. It remains an open problem whether the pulsating waves exist for the case where the wave speed is larger than the spreading speed. In this paper, we will give an assertive answer by constructing suitable upper and lower solutions and appealing to the comparison principle developed by Thieme [12]. We should point out that this method of upper and lower solutions was used earlier in [10] to prove the existence of pulsating waves for a scalar nonlocal dispersal equation.

Another main contribution of this paper is the stability of pulsating waves. To be specific, we will show that if the initial function is within a bounded distance from a certain pulsating wave with respect to a weighted maximum norm, then the solution will converge to the pulsating wave exponentially in time. A similar result for the scalar nonlocal dispersal equation was obtained in [8,10]. Although the case of scalar equations has been studied intensively in the existing literature (see, e.g., [1–3,8,10,16] and references therein), not much is known about pulsating waves of partially degenerate parabolic systems. The main difficulties come from the loss of compactness for solution maps and interaction between different components in a higher dimensional system. Our work provides insights on how to obtain the existence and stability of pulsating waves for the general high-dimensional reaction–diffusion systems without the compactness condition.

The rest of this paper is organized as follows. In Section 2, we present some basic assumptions and preliminary results. Our main theorems on the existence and stability of pulsating waves are stated and proved in Section 3.

2. Preliminaries

Throughout this paper, we make the following assumptions:

(A1) The functions D_1, D_2, f and g are periodic in x with the same period and Hölder continuous in x of order $\nu \in (0,1)$. Moreover, $D_1(x) > 0$ for all $x \in \mathbb{R}$, which implies that the differential operator $D_1(x)\partial_{xx} + D_2(x)\partial_x$ is uniformly elliptic.

(A2) The functions f and g are second-order differentiable with respect to u_1 and u_2, $f(x,0,0) \equiv 0$, and $g(x,0,0) \equiv 0$.

(A3) There exists a positive vector $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2) \in \mathbb{R}^2$ such that $f(x,\mathcal{M}) \le 0$ and $g(x,\mathcal{M}) \le 0$ for all $x \in \mathbb{R}$.

(A4) $\partial_{u_2} f(x,u_1,u_2) > 0$ and $\partial_{u_1} g(x,u_1,u_2) > 0$ for all $x \in \mathbb{R}$ and $u \in [0,\mathcal{M}]$, where $[0,\mathcal{M}] := [0,\mathcal{M}_1] \times [0,\mathcal{M}_2]$.

(A5) The reaction term $\mathcal{F}(x,u) := (f(x,u), g(x,u))$ is strictly subhomogeneous on $[0,\mathcal{M}]$ in the sense that $\mathcal{F}(x,\nu u) > \nu\mathcal{F}(x,u)$ for all $x \in \mathbb{R}$, $\nu \in (0,1)$ and $u \in (0,\mathcal{M}_1] \times (0,\mathcal{M}_2]$.

Without loss of generality, we may scale the period of functions D_1, D_2, f and g to be 1. In the followings, we may not mention it each time but all the periodic functions should have the same period 1.

Definition 2.1. A solution $u(t,x)$ of (1.2) is called a (rightward) pulsating wave connecting a positive periodic state $p(x)$ to the trivial solution 0 with speed $c > 0$ provided that

(i) $u(t,x) = \Psi(x-ct,x)$ and $\Psi(s,x)$ is periodic in the second variable x, namely, $\Psi(s,x+1) = \Psi(s,x)$.

(ii) $\lim_{s\to-\infty}(\Psi(s,x) - p(x)) = 0$ and $\lim_{s\to\infty}\Psi(s,x) = 0$ uniformly in $x \in \mathbb{R}$.

If the functions D_1, D_2, f and g are independent of x (i.e., in the homogeneous case), then the pulsating wave $u(t, x) = \Psi(x - ct)$ is reduced to the classical traveling wave and the system (1.2) admits the property of translation invariance, namely, $u(t, x + z)$ is also a solution of the system (1.2) for any $z \in \mathbb{R}$. However, in the heterogeneous case, we shall consider the following shifted system (with $z \in \mathbb{R}$):

$$\begin{aligned}\partial_t u_1(t,x) &= D_1(x+z)\partial_{xx}u_1 + D_2(x+z)\partial_x u_1 + f(x+z,u_1,u_2),\\ \partial_t u_2(t,x) &= g(x+z,u_1,u_2).\end{aligned} \tag{2.1}$$

Note that if $u(t, x)$ is a solution of (1.2), then $u(t, x + z)$ is a solution of (2.1). We introduce the Green function $\mathcal{G}(t, x, y)$ (cf. [17]) associated with the operator $D_1(x)\partial_{xx} + D_2(x)\partial_x - K$ on the whole real line, where $K > 0$ is a sufficiently large constant such that $K + \partial_{u_1} f(x, u_1, u_2)$ and $K + \partial_{u_2} g(x, u_1, u_2)$ are positive for any $x \in \mathbb{R}$ and $u \in [0, \mathcal{M}]$. Therefore, the solution map of the linear equation

$$\partial_t u_1 = D_1(x+z)\partial_{xx}u_1 + D_2(x+z)\partial_x u_1 - Ku_1 \tag{2.2}$$

has the following integral representation

$$[l_1(t)u_1^0](x;z) = \int_{\mathbb{R}} \mathcal{G}(t, x+z, y)u_1^0(y)dy. \tag{2.3}$$

For convenience, we denote

$$\begin{aligned}\tilde{f}(x,u_1,u_2) &:= f(x,u_1,u_2) + Ku_1,\\ \tilde{g}(x,u_1,u_2) &:= g(x,u_1,u_2) + Ku_2.\end{aligned} \tag{2.4}$$

Then system (2.1) can be transformed to the following integral form:

$$\begin{aligned}u_1(t,x;u^0,z) &= [l_1(t)u_1^0](x;z)\\ &\quad + \int_0^t [l_1(t-s)\tilde{f}(\cdot+z, u_1(s,\cdot;u^0,z), u_2(s,\cdot;u^0,z))](x;z)ds,\\ u_2(t,x;u^0,z) &= e^{-Kt}u_2^0(x)\\ &\quad + \int_0^t e^{-K(t-s)}\tilde{g}(x+z, u_1(s,x;u^0,z), u_2(s,x;u^0,z))ds.\end{aligned} \tag{2.5}$$

Given any Borel measurable and bounded function $u^0 = (u_1^0, u_2^0)$, the functions $l_1(t)u_1^0$ and $e^{-Kt}u_2^0$ are also Borel measurable and bounded. By Banach's fixed point theorem, there exists a unique Borel measurable solution $u = u(t, x; u^0, z)$ of the above integral equation on $[0, \infty)$ such that u is bounded on $[0, T]$ for any $T \geq 0$ (see [12, Theorem 2.2]). Such u is called a *mild solution* of (2.1). Furthermore, if u^0 is continuous in x, then u becomes a classical solution of (2.1). For simplicity, we denote $u(t, x; u^0) := u(t, x; u^0, 0)$ when $z = 0$.

Given a bounded Borel measurable function u^0, we can define the lower (upper) solution of the integral equation (2.5) if the equalities are replaced by inequalities $\le$ ($\ge$). By the argument similar to [12, Comparison Lemma 3.2], we can establish the following comparison principle.

Lemma 2.2. *Let w be a lower solution and v be an upper solution of (2.5) corresponding to a bounded Borel measurable function u^0. If $w(0,x) \le v(0,x)$ for all $x \in \mathbb{R}$, then $w(t,x) \le v(t,x)$ for all $t \ge 0$ and $x \in \mathbb{R}$.*

A weaker version of the above lemma can be stated as follows. Let $w(t,x)$ be a lower solution of (2.5) with initial value $w(0,x)$, and $v(t,x)$ be an upper solution of (2.5) with initial value $v(0,x)$. If $w(0,x) \le v(0,x)$ for all $x \in \mathbb{R}$, then $w(t,x) \le v(t,x)$ for all $t \ge 0$ and $x \in \mathbb{R}$. To prove this, we only need to choose $u^0(x)$ to be either $w(0,x)$ or $v(0,x)$ and then apply the above lemma. Since this weak version will be frequently used in the following context, it is convenient to simply say that $u(t,x)$ is a lower (or upper) solution without specifying the initial value if the initial value is the same as $u(0,x)$.

For convenience, let

$$f_i(x) = \partial_{u_i} f(x,0,0), \quad g_i(x) = \partial_{u_i} g(x,0,0), \quad i = 1,2.$$

We then consider the following periodic eigenvalue problem:

$$\begin{aligned} \lambda\phi_1(x) &= D_1(x)\phi_1''(x) + [D_2(x) - 2\mu D_1(x)]\phi_1'(x) \\ &\qquad + [\mu^2 D_1(x) - \mu D_2(x) + f_1(x)]\phi_1(x) + f_2(x)\phi_2(x), \\ \lambda\phi_2(x) &= g_1(x)\phi_1(x) + g_2(x)\phi_2(x). \end{aligned} \tag{2.6}$$

The following two results come from [17, Theorems 2.2–2.4].

Proposition 2.3. *Assume that either of the following conditions is satisfied:*

(i) *There exist $x_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that $g_2(x) = \max_{x\in\mathbb{R}} g_2(x)$ for all $x \in (x_0 - \delta_0, x_0 + \delta_0)$.*
(ii) *$\max_{x\in\mathbb{R}} g_2(x) < \lambda^*$, where λ^* is the principal eigenvalue of the eigenvalue problem with the Dirichlet boundary condition:*

$$\begin{aligned} &D_1(x)\xi_1''(x) + D_0(x)\xi_1'(x) + f_1(x)\xi_1(x) = \lambda\xi_1(x), \ x \in (0,1), \\ &\xi_1(0) = \xi_1(1) = 0. \end{aligned}$$

Then for any parameter $\mu \in \mathbb{R}$, the periodic eigenvalue problem (2.6) has a geometrically simple eigenvalue $\lambda(\mu)$ and a strongly positive and periodic eigenfunction $\phi(x) = (\phi_1(x), \phi_2(x))$.

To obtain a threshold type result for system (1.2), we need the following assumption:

(A6) For any parameter $\mu \in \mathbb{R}$, the periodic eigenvalue problem (2.6) has a geometrically simple eigenvalue $\lambda(\mu)$ and a strongly positive and periodic eigenfunction $\phi(x) = (\phi_1(x), \phi_2(x))$.

Proposition 2.4. *Assume that (A6) holds. Then the following statements are valid:*

(i) *If* $\lambda(0) < 0$, *then for any periodic initial value* $u^0 \in [0, \mathcal{M}]$, *we have* $\lim_{t\to\infty} u(t,x;u^0) = 0$ *uniformly for* $x \in \mathbb{R}$.

(ii) *If* $\lambda(0) > 0$, *then there exists a unique positive periodic stationary solution* $p \in [0, \mathcal{M}]$ *such that for any nonzero periodic initial value* $u^0 \in [0, \mathcal{M}]$, *we have* $\lim_{t\to\infty} u(t,x;u^0) = p(x)$ *uniformly for* $x \in \mathbb{R}$.

From now on, we will always assume that (A6) and the following assumption are satisfied:

(A7) The trivial solution 0 is linearly unstable, namely, $\lambda(0) > 0$.

Define

$$\begin{aligned} \mathcal{C}_+^* := \{u = (u_1, u_2) \in C(\mathbb{R}, \mathbb{R}^2) : 0 \le u_i(x) < \min_{x\in\mathbb{R}} p_i(x), \\ \liminf_{x\to-\infty} u_i(x) > 0,\ u_i(x) = 0,\ \forall\, x > x_1 \text{ for some } x_1 \in \mathbb{R}\}. \end{aligned}$$

According to [17], the rightward spreading speed is given by

$$\begin{aligned} c_+^* &= \sup\{c : \forall u^0 \in \mathcal{C}_+^*, \lim_{t\to\infty, x\le ct} (u(t,x;u^0) - p(x)) = 0\} \\ &= \inf\{c : \forall u^0 \in \mathcal{C}_+^*, \lim_{t\to\infty, x\ge ct} u(t,x;u^0) = 0\} \\ &= \inf_{\mu>0} \frac{\lambda(\mu)}{\mu}, \end{aligned}$$

and the following statements hold true:

(a) If $c > c_+^*$, then for every $u^0 \in \mathcal{C}_+^*$, we have

$$\lim_{t\to\infty, x\ge ct} u(t,x;u^0,z) = 0 \text{ uniformly for } z \in \mathbb{R}.$$

(b) If $c < c_+^*$, then for every $u^0 \in \mathcal{C}_+^*$, we have

$$\lim_{t\to\infty, x\le ct} (u(t,x;u^0,z) - p(x+z)) = 0 \text{ uniformly for } z \in \mathbb{R}.$$

We remark that the rightward spreading speed c_+^* may not be necessarily positive due to the presence of advection term $D_2(x)$. However, if we define the leftward spreading speed c_-^* in a similar manner, the sum of two spreading speeds should be positive, namely, $c_+^* + c_-^* > 0$; see [17, Theorem 4.1]. It is also noted from [17, Theorem 2.5] that $\lambda(\mu)$ is convex, $\lambda(\mu)/\mu \to \infty$ as $\mu \to 0$ or $\mu \to \infty$, and hence, $c_+^* = \lambda(\mu^*)/\mu^*$ for some finite $\mu^* > 0$.

A straightforward application of the above spreading result is the non-existence of the pulsating wave with $c < c_+^*$.

Proposition 2.5. *If* $c < c_+^*$, *then there is no pulsating wave connecting* $p(x)$ *to* 0 *with speed* c.

Proof. We proceed by contradiction. Assume $u(t,x) = \Psi(x-ct,x)$ is a pulsating wave with speed $c < c_+^*$. Choose a nonzero number $c_1 \in (c, c_+^*)$ and $u^0 \in \mathcal{C}_+^*$ such that $u^0(x) < \Psi(x,x)$. By the spreading result, we obtain

$$\lim_{t\to\infty, x\le c_1 t}(u(t,x;u^0) - p(x)) = 0.$$

Especially, there exist $\delta > 0$ and $T > 0$ such that $u(t,x;u^0) > \delta$ for all $t > T$ and $x \le c_1 t$. Then the comparison principle implies that $\Psi(x-ct,x) > \delta$ for all $t > T$ and $x \le c_1 t$. Given any $x_0 \in \mathbb{R}$, we choose $x_n = x_0 + sign(c_1)n$ and $t_n = x_n/c_1$ for large integers $n > 0$. Note that Ψ is periodic with respect to the second variable. We have $\Psi(x_n - ct_n, x_0) = \Psi(x_n - ct_n, x_n) > \delta$. On the other hand, since $c < c_1$, it follows that as $n \to \infty$, $x_n - ct_n = (c_1 - c)t_n \to \infty$ and $\Psi(x_n - ct_n, x_0) \to 0$. This leads to a contradiction. $\square$

We denote by $p = (p_1, p_2)$ the unique positive stationary solution of (1.2) and study the linearized eigenvalue problem at $p(x)$:

$$\begin{aligned}
\tilde{\lambda}\psi_1(x) &= D_1(x)\psi_1''(x) + D_2(x)\psi_1'(x) + F_1(x)\psi_1(x) + F_2(x)\psi_2(x),\\
\tilde{\lambda}\psi_2(x) &= G_1(x)\psi_1(x) + G_2(x)\psi_2(x),
\end{aligned} \tag{2.7}$$

where

$$F_i := \partial_{u_i} f(x,u_1,u_2)\big|_{u=p}, \qquad G_i := \partial_{u_i} g(x,u_1,u_2)\big|_{u=p},\ i = 1,2.$$

The following proposition comes from [17, Theorems 2.2 and 2.3].

Proposition 2.6. *Assume that either of the following conditions is satisfied:*

(i) *There exist $x_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that $G_2(x) = \max_{x\in\mathbb{R}} G_2(x)$ for all $x \in (x_0 - \delta_0, x_0 + \delta_0)$.*
(ii) *$\max_{x\in\mathbb{R}} G_2(x) < \lambda^*$, where λ^* is the principal eigenvalue of the eigenvalue problem with the Dirichlet boundary condition:*

$$\begin{aligned}
&D_1(x)\xi_1''(x) + D_0(x)\xi_1'(x) + F_1(x)\xi_1(x) = \lambda\xi_1(x),\ x \in (0,1),\\
&\xi_1(0) = \xi_1(1) = 0.
\end{aligned}$$

Then the periodic eigenvalue problem (2.7) has a geometrically simple eigenvalue $\tilde{\lambda}$ and a strongly positive and periodic eigenfunction $\psi(x) = (\psi_1(x), \psi_2(x))$.

To obtain the local asymptotic stability of the periodic solution $p(x)$, we need the following assumptions:

(A8) The periodic eigenvalue problem (2.7) has a geometrically simple eigenvalue $\tilde{\lambda} \in \mathbb{R}$ and a strongly positive and periodic eigenfunction $\psi(x) = (\psi_1(x), \psi_2(x))$.
(A9) Let $p = (p_1, p_2)$ be the unique positive periodic stationary solution of (1.2). Assume that

$$f(x, p_1(x), p_2(x)) \geq F_1(x)p_1(x) + F_2(x)p_2(x),$$
$$g(x, p_1(x), p_2(x)) \geq G_1(x)p_1(x) + G_2(x)p_2(x),$$

where $F_i := \partial_{u_i} f(x, u_1, u_2)\big|_{u=p}$ and $G_i := \partial_{u_i} g(x, u_1, u_2)\big|_{u=p}$, $i = 1, 2$, and one of these two inequalities is strict for some $x \in \mathbb{R}$.

Proposition 2.7. *Assume that (A8) and (A9) hold. Then the principle eigenvalue of the linearized problem (2.7) is negative, namely,* $\tilde{\lambda} < 0$.

Proof. Note that $p = (p_1, p_2)$ is a stationary positive solution of (1.2), namely,

$$0 = D_1(x)p_1''(x) + D_2(x)p_1'(x) + f(x, p_1(x), p_2(x)),$$
$$0 = g(x, p_1(x), p_2(x)).$$

The second equation together with (A9) implies that $0 \geq G_1 p_1 + G_2 p_2$. Especially, $G_2 < 0$ on account of the positivity of G_1, p_1 and p_2. The first equation implies that 0 is the principle eigenvalue (with p_1 being the eigenfunction) of the following problem:

$$\lambda\varphi = D_1(x)\varphi''(x) + D_2(x)\varphi'(x) + \frac{f(x, p_1(x), p_2(x))}{p_1(x)}\varphi(x).$$

In view of (2.7), it follows that $\tilde{\lambda}$ is the principle eigenvalue (with ψ_1 being the eigenfunction) of the following problem:

$$\lambda\varphi(x) = D_1(x)\varphi''(x) + D_2(x)\varphi'(x) + \left(F_1(x) + \frac{F_2(x)G_1(x)}{\tilde{\lambda} - G_2(x)}\right)\varphi(x).$$

We further claim that $\tilde{\lambda} < 0$. Assuming $\tilde{\lambda} \geq 0$, we see from $0 \geq G_1 p_1 + G_2 p_2$ and (A9) that

$$F_1(x) + \frac{F_2(x)G_1(x)}{\tilde{\lambda} - G_2(x)} \leq F_1(x) + \frac{F_2(x)p_2(x)}{p_1(x)} \leq \frac{f(x, p_1(x), p_2(x))}{p_1(x)},$$

and one of the above two inequalities is strict for some $x \in \mathbb{R}$. By the monotonicity of the principal eigenvalue with respect to the weight function, we then obtain that $\tilde{\lambda} < 0$, a contradiction. Therefore, we have $\tilde{\lambda} < 0$. □

3. Main results

In this section, we always assume that (A1)–(A9) are satisfied. Recall that the rightward spreading speed is $c_+^* = \inf_{\mu>0} \lambda(\mu)/\mu$. We can choose $\mu^* > 0$ such that $c_+^* = \lambda(\mu^*)/\mu^*$. Given $c > c_+^*$, since $\lambda(0) > 0$, there exists $0 < \mu < \mu^*$ such that $c = \lambda(\mu)/\mu$. We use $\phi = (\phi_1, \phi_2)$ to denote the strongly positive and periodic eigenfunction associated with the eigenvalue $\lambda = \lambda(\mu)$.

Next, we choose $\mu^\varepsilon = \mu + \varepsilon > \mu$ sufficiently close to μ such that $\mu^\varepsilon < \min\{2\mu, \mu^*\}$. Let $\lambda^\varepsilon = \lambda(\mu^\varepsilon)$ and $\phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ be the principal eigenvalue and eigenfunction of the problem (2.6) with parameter μ^ε. Since $\lambda(\mu)$ is convex in μ (see [17, Theorem 2.5]), we obtain $\mu^\varepsilon c - \lambda^\varepsilon > 0$.

Moreover, we denote by $p = (p_1, p_2)$ the unique positive stationary solution of (1.2). Without loss of generality, we assume that $\phi < p$.

Motivated by [10], we define

$$\bar{u}(t, x; T, z) := \min\{e^{-\mu(x+cT-ct)}\phi(x+z), p(x+z)\},$$

$$\underline{u}(t, x; T, z) := \begin{cases} \max\{e^{-\mu(x+cT-ct)}\phi(x+z) - Me^{-\mu^\varepsilon(x+cT-ct)}\phi^\varepsilon(x+z), \sigma\phi^0(x+z)\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad x + cT - ct < K_M, \\ e^{-\mu(x+cT-ct)}\phi(x+z) - Me^{-\mu^\varepsilon(x+cT-ct)}\phi^\varepsilon(x+z), \\ \qquad\qquad\qquad\qquad\qquad\qquad x + cT - ct \geq K_M, \end{cases}$$

where $M > 0$, $K_M > 0$ and $\sigma > 0$ are to be determined, and ϕ^0 is the principal eigenfunction corresponding to the problem (2.6) with parameter $\mu = 0$. Furthermore, we require

$$\lambda(0) > \max\{\frac{f_1(x)\sigma\phi_1^0(x) + f_2(x)\sigma\phi_2^0(x) - f(x, \sigma\phi_1^0(x), \sigma\phi_2^0(x))}{\sigma\phi_1^0(x)},$$
$$\frac{g_1(x)\sigma\phi_1^0(x) + g_2(x)\sigma\phi_2^0(x) - g(x, \sigma\phi_1^0(x), \sigma\phi_2^0(x))}{\sigma\phi_2^0(x)}\}, \ \forall x \in \mathbb{R}.$$

Note that $\lambda(0) > 0$ and, by Taylor's expansion, the right-hand side converges to zero uniformly for $x \in \mathbb{R}$ as $\sigma \to 0^+$. Thus, the above inequality is satisfied as long as $\sigma > 0$ is small enough. In the following, we verify that for sufficiently large $M > 0$, $K_M > 0$ and sufficiently small $\sigma > 0$, $\bar{u}$ and $\underline{u}$ formulate an ordered pair of upper and lower solutions of the shifted system (2.1).

Proposition 3.1. *Let C_f (resp. C_g) be the maximum of the second derivatives of f (resp. g) with respect to $u = (u_1, u_2)$ for all $x \in \mathbb{R}$, $0 \leq u_1 \leq \max \phi_1$ and $0 \leq u_2 \leq \max \phi_2$. We first choose $M > 0$ such that*

$$M > \max_{x\in\mathbb{R}} \frac{\phi_1(x)}{\phi_1^\varepsilon(x)} + \max_{x\in\mathbb{R}} \frac{C_f[\phi_1^2(x) + \phi_2^2(x)]}{(\mu^\varepsilon c - \lambda^\varepsilon)\phi_1^\varepsilon(x)}$$
$$+ \max_{x\in\mathbb{R}} \frac{\phi_2(x)}{\phi_2^\varepsilon(x)} + \max_{x\in\mathbb{R}} \frac{C_g[\phi_1^2(x) + \phi_2^2(x)]}{(\mu^\varepsilon c - \lambda^\varepsilon)\phi_2^\varepsilon(x)}.$$

When $M > 0$ is determined, we choose $K_M > 0$ so large that

$$e^{-\mu K_M} \min_{x\in\mathbb{R}} \phi(x) - Me^{-\mu^\varepsilon K_M} \max_{x\in\mathbb{R}} \phi^\varepsilon(x) > 0.$$

Finally, we choose $\sigma > 0$ to be sufficiently small such that

$$\sigma \max_{x\in\mathbb{R}} \phi^0(x) < e^{-\mu K_M} \min_{x\in\mathbb{R}} \phi(x) - Me^{-\mu^\varepsilon K_M} \max_{x\in\mathbb{R}} \phi(x),$$

and

$$\sigma \max_{x\in\mathbb{R}} \phi^0(x) < \min_{x\in\mathbb{R}} p(x).$$

With such choices of M, K_M and σ, the functions $\bar{u}$ and $\underline{u}$ formulate an ordered pair (i.e., $\bar{u} > \underline{u}$) of upper and lower solutions of the shifted system (2.1).

Proof. For convenience, we denote

$$\varphi(t,x;T,z) := e^{-\mu(x+cT-ct)}\phi(x+z),$$
$$\varphi^{\varepsilon}(t,x;T,z) := e^{-\mu^{\varepsilon}(x+cT-ct)}\phi^{\varepsilon}(x+z),$$

and $\tilde{\varphi} := \varphi - M\varphi^{\varepsilon}$. We then have $\bar{u}(t,x;T,z) = \min\{\varphi(t,x;T,z), p(x+z)\}$ and

$$\underline{u}(t,x;T,z) = \begin{cases} \max\{\tilde{\varphi}(t,x;T,z), \sigma\phi^0(x+z)\}, & x+cT-ct < K_M, \\ \tilde{\varphi}(t,x;T,z), & x+cT-ct \geq K_M. \end{cases}$$

By the choice of M, we have $\phi(x) - M\phi^{\varepsilon}(x) < 0$ for all $x \in \mathbb{R}$. It then follows that $\tilde{\varphi}(t,x;T,z) < 0$ and $\underline{u}(t,x;T,z) = \sigma\phi^0(x+z)$ for all $x+cT-ct \leq 0$. Since $\sigma\phi^0(x) < e^{-\mu K_M}\phi(x)$ for all $x \in \mathbb{R}$, we obtain $\sigma\phi^0(x+z) < \varphi(t,x;T,z)$ for all $x+cT-ct < K_M$. From the choice of K_M, we see that

$$e^{-\mu K_M}\min_{x\in\mathbb{R}}\phi(x) - Me^{-\mu^{\varepsilon}K_M}\max_{x\in\mathbb{R}}\phi^{\varepsilon}(x) > 0.$$

Thus, $\tilde{\varphi}(t,x;T,z) > 0$ for all $x+cT-ct \geq K_M$. By the choice of σ, we have

$$\sigma\max_{x\in\mathbb{R}}\phi^0(x) < e^{-\mu K_M}\min_{x\in\mathbb{R}}\phi(x) - Me^{-\mu^{\varepsilon}K_M}\max_{x\in\mathbb{R}}\phi^{\varepsilon}(x).$$

Thus, for $x+cT-ct$ near K_M, $\tilde{\varphi}(t,x;T,z) > \sigma\phi^0(x+z)$, and consequently, $\underline{u}(t,x;T,z) = \tilde{\varphi}(t,x;T,z)$. Especially, $\underline{u}$ is continuous at $x+cT-ct = K_M$. Since the maximum function of two continuous function is still continuous, it follows that $\underline{u}$ is continuous everywhere. Moreover, for any $x, z, T \in \mathbb{R}$, $\underline{u}$ is differentiable in t except at finitely many points. We intend to show that for any t where $\underline{u}$ is differentiable,

$$\partial_t\underline{u}_1 - (D_1\partial_{xx}\underline{u}_1 + D_2\partial_x\underline{u}_1) \leq f(x,\underline{u}_1,\underline{u}_2),$$
$$\partial_t\underline{u}_2 \leq g(x,\underline{u}_1,\underline{u}_2).$$

By the definition, we have either (i) $\underline{u} = \tilde{\varphi}$; or (ii) $\underline{u} = \sigma\phi^0$; or (iii) $\underline{u} = (\underline{u}_1,\underline{u}_2) = (\tilde{\varphi}_1, \sigma\phi^0_2)$; or (iv) $\underline{u} = (\underline{u}_1,\underline{u}_2) = (\sigma\phi^0_1, \tilde{\varphi}_2)$.

(i) If $\underline{u} = \tilde{\varphi} > 0$, then $x+cT-ct > 0$ and $0 < \tilde{\varphi} < \varphi$. Since ϕ and ϕ^{ε} are the principal eigenfunctions corresponding to the problem (2.6) with parameters μ and μ^{ε}, respectively, we have

$$\partial_t\varphi_1 - (D_1\partial_{xx}\varphi_1 + D_2\partial_x\varphi_1) = f_1\varphi_1 + f_2\varphi_2,$$
$$\partial_t\varphi_2 = g_1\varphi_1 + g_2\varphi_2,$$

and

$$\begin{aligned}\partial_t\tilde{\varphi}_1 - (D_1\partial_{xx}\tilde{\varphi}_1 + D_2\partial_x\tilde{\varphi}_1) &= -M(\mu^\varepsilon c - \lambda^\varepsilon)\varphi_1^\varepsilon + f_1\tilde{\varphi}_1 + f_2\tilde{\varphi}_2,\\ \partial_t\tilde{\varphi}_2 &= -M(\mu^\varepsilon c - \lambda^\varepsilon)\varphi_2^\varepsilon + g_1\tilde{\varphi}_1 + g_2\tilde{\varphi}_2.\end{aligned}$$

In view of $\mu^\varepsilon < 2\mu$, $x + cT - ct > 0$, $0 < \tilde{\varphi} < \varphi$ and the choice of M, we obtain

$$\begin{aligned}f_1\tilde{\varphi}_1 + f_2\tilde{\varphi}_2 - f(x,\tilde{\varphi}_1,\tilde{\varphi}_2) \le C_f(\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2) \le C_f(\varphi_1^2 + \varphi_2^2) \le M(\mu^\varepsilon c - \lambda^\varepsilon)\varphi_1^\varepsilon,\\ g_1\tilde{\varphi}_1 + g_2\tilde{\varphi}_2 - g(x,\tilde{\varphi}_1,\tilde{\varphi}_2) \le C_g(\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2) \le C_g(\varphi_1^2 + \varphi_2^2) \le M(\mu^\varepsilon c - \lambda^\varepsilon)\varphi_2^\varepsilon.\end{aligned}$$

It then follows that

$$\begin{aligned}\partial_t\tilde{\varphi}_1 - (D_1\partial_{xx}\tilde{\varphi}_1 + D_2\partial_x\tilde{\varphi}_1) &\le f(x,\tilde{\varphi}_1,\tilde{\varphi}_2),\\ \partial_t\tilde{\varphi}_2 &\le g(x,\tilde{\varphi}_1,\tilde{\varphi}_2).\end{aligned}$$

Replacing $\tilde{\varphi}$ by $\underline{u}$ in the above inequalities gives our desired inequalities for the first case where $\underline{u} = \tilde{\varphi}$.

(ii) If $\underline{u} = \sigma\phi^0$, since ϕ^0 is the principal eigenfunction corresponding to the problem (2.6) with parameter $\mu = 0$, we have

$$\begin{aligned}\partial_t(\sigma\phi_1^0) - [D_1\partial_{xx}(\sigma\phi_1^0) + D_2\partial_x(\sigma\phi_1^0)] &= [f_1 - \lambda(0)](\sigma\phi_1^0) + f_2(\sigma\phi_2^0)\\ &\le f(x,\sigma\phi_1^0,\sigma\phi_2^0),\\ \partial_t(\sigma\phi_2^0) &= g_1(\sigma\phi_1^0) + [g_2 - \lambda(0)](\sigma\phi_2^0)\\ &\le g(x,\sigma\phi_1^0,\sigma\phi_2^0).\end{aligned}$$

Recall that we have chosen $\sigma > 0$ so small that the above two inequalities are satisfied. Replacing $\sigma\phi^0$ by $\underline{u}$ in the above inequalities gives our desired inequalities for the second case where $\underline{u} = \sigma\phi^0$.

(iii) If $\underline{u} = (\underline{u}_1, \underline{u}_2) = (\tilde{\varphi}_1, \sigma\phi_2^0)$, we have $\tilde{\varphi}_2 < \sigma\phi_2^0$, $0 < \tilde{\varphi}_1 < \varphi_1$, $0 < x + cT - ct < K_M$, and $\sigma\phi^0 < \varphi$. By (A4), we obtain $f_2 > 0$ and $f_2\tilde{\varphi}_2 < f_2\sigma\phi_2^0$. It follows from $0 < \sigma\phi_2^0 < \varphi_2$, $0 < \tilde{\varphi}_1 < \varphi_1$, $\mu^\varepsilon < 2\mu$, $x + cT - ct > 0$ and the choice of M that

$$f_1\tilde{\varphi}_1 + f_2\tilde{\varphi}_2 - f(x,\tilde{\varphi}_1,\sigma\phi_2^0) \le C_f[\tilde{\varphi}_1^2 + (\sigma\phi_2^0)^2] \le C_f(\varphi_1^2 + \varphi_2^2) \le M(\mu^\varepsilon c - \lambda^\varepsilon)\varphi_1^\varepsilon.$$

Consequently,

$$\partial_t\tilde{\varphi}_1 - (D_1\partial_{xx}\tilde{\varphi}_1 + D_2\partial_x\tilde{\varphi}_1) \le f(x,\tilde{\varphi}_1,\sigma\phi_2^0).$$

Moreover, since $\partial_{u_1}g(x,u_1,u_2) > 0$ by (A4), it follows that

$$\partial_t(\sigma\phi_2^0) \le g(x,\sigma\phi_1^0,\sigma\phi_2^0) \le g(x,\tilde{\varphi}_1,\sigma\phi_2^0).$$

Replacing $\tilde{\varphi}_1$ by $\underline{u}_1$ and $\sigma\phi_2^0$ by $\underline{u}_2$ in the above inequalities gives our desired inequalities for the third case where $\underline{u} = (\underline{u}_1, \underline{u}_2) = (\tilde{\varphi}_1, \sigma\phi_2^0)$.

(iv) If $\underline{u} = (\underline{u}_1, \underline{u}_2) = (\sigma\phi_1^0, \tilde{\varphi}_2)$, we have $\tilde{\varphi}_1 < \sigma\phi_1^0$, $0 < \tilde{\varphi}_2 < \varphi_2$, $0 < x + cT - ct < K_M$, and $\sigma\phi^0 < \varphi$. By (A4), we obtain $g_1 > 0$ and $g_1\tilde{\varphi}_1 < g_1\sigma\phi_1^0$. It follows from $0 < \sigma\phi_1^0 < \varphi_1$, $0 < \tilde{\varphi}_2 < \varphi_2$, $\mu^\varepsilon < 2\mu$, $x + cT - ct > 0$ and the choice of M that

$$g_1\tilde{\varphi}_1 + g_2\tilde{\varphi}_2 - g(x, \sigma\phi_1^0, \tilde{\varphi}_2) \le C_g[(\sigma\phi_1^0)^2 + \tilde{\varphi}_2^2] \le C_g(\varphi_1^2 + \varphi_2^2) \le M(\mu^\varepsilon c - \lambda^\varepsilon)\varphi_2^\varepsilon.$$

Consequently,

$$\partial_t\tilde{\varphi}_2 \le g(x, \sigma\phi_1^0, \tilde{\varphi}_2).$$

Moreover, since $\partial_{u_2} f(x, u_1, u_2) > 0$ by (A4), it follows that

$$\partial_t(\sigma\phi_1^0) - [D_1\partial_{xx}(\sigma\phi_1^0) + D_2\partial_x(\sigma\phi_1^0)] \le f(x, \sigma\phi_1^0, \sigma\phi_2^0) \le f(x, \sigma\phi_1^0, \tilde{\varphi}_2).$$

Replacing $\sigma\phi_1^0$ by $\underline{u}_1$ and $\tilde{\varphi}_2$ by $\underline{u}_2$ in the above inequalities gives our desired inequalities for the fourth case where $\underline{u} = (\underline{u}_1, \underline{u}_2) = (\sigma\phi_1^0, \tilde{\varphi}_2)$.

Based on the above arguments, we see that for any t where $\underline{u}$ is differentiable,

$$\begin{aligned} \partial_t\underline{u}_1 - (D_1\partial_{xx}\underline{u}_1 + D_2\partial_x\underline{u}_1) &\le f(x, \underline{u}_1, \underline{u}_2), \\ \partial_t\underline{u}_2 &\le g(x, \underline{u}_1, \underline{u}_2). \end{aligned}$$

At the point t where $\underline{u}$ is not differentiable, the left-hand derivative of $\underline{u}$ is less than the right-hand derivative of $\underline{u}$ because $\underline{u}$ is defined as the maximum of two differentiable functions. It is readily seen from (2.2)–(2.5) that $\underline{u}$ is a lower solution of (2.1).

On the other hand, since $\overline{u} = \min\{\varphi, p\}$, $\partial_{u_2} f > 0$, $\partial_{u_1} g > 0$ and $\mathcal{F} = (f, g)$ is strictly subhomogeneous, we have $f_1\varphi_1 + f_2\varphi_2 - f(x, \overline{u}_1, \overline{u}_2) \ge 0$ and $g_1\varphi_1 + g_2\varphi_2 - g(x, \overline{u}_1, \overline{u}_2) \ge 0$. Therefore, it follows from (2.2)–(2.5) that $\overline{u}$ is an upper solution of (2.1).

Finally, we prove that $\underline{u} < \overline{u}$. By the choice of $M > 0$, we obtain

$$\phi(x) - M\phi^\varepsilon(x) < 0, \quad x \in \mathbb{R}.$$

Consequently, for all $x + cT - ct \le 0$, $\tilde{\varphi}(t, x; T, z) < 0$ and $\underline{u} = \sigma\phi^0$. Since $\underline{u} = \tilde{\varphi}$ for $x + cT - ct \ge K_M$ and $K_M > 0$, we only need to prove that $\tilde{\varphi} \le \overline{u}$ for $x + cT - ct > 0$ and $\sigma\phi^0 \le \overline{u}$ for $x + cT - ct < K_M$.

If $x + cT - ct < K_M$, then the choice of σ implies that

$$\sigma\phi^0(x + z) \le \sigma \max_{x\in\mathbb{R}} \phi^0(x) < \min_{x\in\mathbb{R}} p(x) \le p(x + z),$$

and

$$\sigma\phi^0(x + z) \le \sigma \max_{x\in\mathbb{R}} \phi^0(x) < e^{-\mu K_M} \min_{x\in\mathbb{R}} \phi(x) \le e^{-\mu(x+cT-ct)}\phi(x + z).$$

Thus, $\sigma\phi^0(x + z) < \overline{u}(t, x; T, z)$.

If $x + cT - ct > 0$, then we have

$$\tilde{\varphi}(t, x; T, z) < \varphi(t, x; T, z) < \phi(x + z) < p(x + z).$$

Here we have used the fact that $\phi < p$. Thus, $\tilde{\varphi}(t, x; T, z) < \overline{u}(t, x; T, z)$.

Combining the above two cases, we obtain $\underline{u}(t, x; T, z) < \bar{u}(t, x; T, z)$ for all $t, x, T, z \in \mathbb{R}$. $\square$

Recall that $u(t, x; u^0, z)$ is the unique mild solution of the shifted system (2.1) with a given bounded Borel measurable function $u^0(x)$, namely, u satisfies the integral equation (2.5). The following result shows that $u(t + \tau, x; \underline{u}(0, \cdot; \tau, z), z)$ is increasing in $\tau > 0$ and $u(t + \tau, x; \bar{u}(0, \cdot; \tau, z), z)$ is decreasing in $\tau > 0$.

Lemma 3.2. *Given any $z \in \mathbb{R}$ and $\tau_2 > \tau_1 > 0$, we have for any $t > -\tau_1$ and $x \in \mathbb{R}$,*

$$u(t + \tau_2, x; \underline{u}(0, \cdot; \tau_2, z), z) \geq u(t + \tau_1, x; \underline{u}(0, \cdot; \tau_1, z), z),$$
$$u(t + \tau_2, x; \bar{u}(0, \cdot; \tau_2, z), z) \leq u(t + \tau_1, x; \bar{u}(0, \cdot; \tau_1, z), z).$$

Proof. By the comparison principle, we have

$$u(\tau_2 - \tau_1, x; \underline{u}(0, \cdot; \tau_2, z), z) \geq \underline{u}(\tau_2 - \tau_1, x; \tau_2, z) = \underline{u}(0, x; \tau_1, z).$$

A further application of the comparison principle yields

$$\begin{aligned} u(t + \tau_2, x; \underline{u}(0, \cdot; \tau_2, z), z) &= u(t + \tau_1, x; u(\tau_2 - \tau_1, \cdot; \underline{u}(0, \cdot; \tau_2, z)), z) \\ &\geq u(t + \tau_1, x; \underline{u}(0, \cdot; \tau_1, z), z). \end{aligned}$$

This gives the first inequality. Similarly, we obtain by the comparison principle

$$u(\tau_2 - \tau_1, x; \bar{u}(0, \cdot; \tau_2, z), z) \leq \bar{u}(\tau_2 - \tau_1, x; \tau_2, z) = \bar{u}(0, x; \tau_1, z).$$

A further application of the comparison principle yields

$$\begin{aligned} u(t + \tau_2, x; \bar{u}(0, \cdot; \tau_2, z), z) &= u(t + \tau_1, x; u(\tau_2 - \tau_1, \cdot; \bar{u}(0, \cdot; \tau_2, z)), z) \\ &\leq u(t + \tau_1, x; \bar{u}(0, \cdot; \tau_1, z), z). \end{aligned}$$

This gives the second inequality. $\square$

By monotonicity, we can define

$$\overline{U}(t, x; z) := \lim_{\tau \to \infty} u(t + \tau, x; \bar{u}(0, \cdot; \tau, z), z), \tag{3.1}$$
$$\underline{U}(t, x; z) := \lim_{\tau \to \infty} u(t + \tau, x; \underline{u}(0, \cdot; \tau, z), z), \tag{3.2}$$

and

$$\overline{\Phi}(x, z) := \overline{U}(0, x; z) = \lim_{\tau \to \infty} u(\tau, x; \bar{u}(0, \cdot; \tau, z), z), \tag{3.3}$$
$$\underline{\Phi}(x, z) := \underline{U}(0, x; z) = \lim_{\tau \to \infty} u(\tau, x; \underline{u}(0, \cdot; \tau, z), z). \tag{3.4}$$

It is easy to see that $\overline{\Phi}(x,z)$ and $\underline{\Phi}(x,z)$ are periodic in z. In the following proposition, we prove that $\overline{U}(t,x;z)$ (resp. $\underline{U}(t,x;z)$) is a mild solution of (2.1) with initial value $\overline{\Phi}(x,z)$ (resp. $\underline{\Phi}(x,z)$).

Proposition 3.3. *Let $\overline{U}$, $\underline{U}$, $\overline{\Phi}$, and $\underline{\Phi}$ be defined as in (3.1)–(3.4). Then we have $\overline{U}(t,x;z) = u(t,x;\overline{\Phi}(\cdot,z),z)$ and $\underline{U}(t,x;z) = u(t,x;\underline{\Phi}(\cdot,z),z)$.*

Proof. Note that

$$u_1(t+\tau,x;\bar{u}(0,\cdot;\tau,z),z) = [l_1(t+\tau)\bar{u}(0,\cdot;\tau,z)](x;z) + \int\limits_0^{t+\tau} [l_1(t+\tau-s) \\ \tilde{f}(\cdot+z,u_1(s,\cdot;\bar{u}(0,\cdot;\tau,z),z),u_2(s,\cdot;\bar{u}(0,\cdot;\tau,z),z))](x;z)ds,$$

and

$$u_1(\tau,x;\bar{u}(0,\cdot;\tau,z),z) = [l_1(\tau)\bar{u}(0,\cdot;\tau,z)](x;z) \\ + \int\limits_0^{\tau} [l_1(\tau-s)\tilde{f}(\cdot+z,u_1(s,\cdot;\bar{u}(0,\cdot;\tau,z),z),u_2(s,\cdot;\bar{u}(0,\cdot;\tau,z),z))](x;z)ds.$$

It follows from the semigroup property of $l_1(t)$ that

$$u_1(t+\tau,x;\bar{u}(0,\cdot;\tau,z),z) = [l_1(t)u_1(\tau,\cdot;\bar{u}(0,\cdot;\tau,z),z)](x;z) + \int\limits_0^{t} [l_1(t-s) \\ \tilde{f}(\cdot+z,u_1(s+\tau,\cdot;\bar{u}(0,\cdot;\tau,z),z),u_2(s+\tau,\cdot;\bar{u}(0,\cdot;\tau,z),z))](x;z)ds.$$

Taking $\tau\to\infty$ and making use of the dominated convergence theorem, we obtain

$$\overline{U}_1(t,x;z) = [l_1(t)\overline{U}_1(0,\cdot;z)](x;z) \\ + \int\limits_0^{t} [l_1(t-s)\tilde{f}(\cdot+z,\overline{U}_1(s,\cdot;z),\overline{U}_2(s,\cdot;z))](x;z)ds.$$

Similarly, we have

$$\overline{U}_2(t,x;z) = e^{-Kt}\overline{U}_2(0,x;z) + \int\limits_0^{t} e^{-K(t-s)}\tilde{g}(x,\overline{U}_1(s,\cdot;z),\overline{U}_2(s,\cdot;z))ds.$$

This proves that $\overline{U}(t,x;z)$ is the solution of integral equation (2.5) with initial value $\overline{\Phi}(x,z)$. The argument for $\underline{U}(t,x;z)$ can be proceeded similarly. □

Now we are in a position to prove that $\overline{U}(t,x;0)$ and $\underline{U}(t,x;0)$ are pulsating waves of (1.2) connecting $p(x)$ to 0 with speed $c>c_+^*$.

Theorem 3.4. *Let $c > c_+^*$ be given, $\mu \in (0, \mu^*)$ be chosen such that $c = \lambda(\mu)/\mu$, and ϕ be the positive and periodic eigenfunction of (2.6) associated with the principal eigenvalue $\lambda(\mu)$. Then we have*

$$\overline{U}(t, x; z) = u(t, x; \overline{\Phi}(\cdot, z), z) = \overline{\Phi}(x - ct, z + ct)$$

and

$$\lim_{x-ct\to\infty} \frac{\overline{U}_i(t, x; z)}{e^{-\mu(x-ct)}\phi_i(x + z)} = 1, \ i = 1, 2,$$
$$\lim_{x-ct\to-\infty} (\overline{U}(t, x; z) - p(x + z)) = 0$$

uniformly in $z \in \mathbb{R}$. Especially, if we define $\Psi(s, x) := \overline{\Phi}(s, x - s)$, then $\overline{U}(t, x; 0) = \overline{\Phi}(x - ct, ct) = \Psi(x - ct, x)$ is a pulsating wave of system (1.2). Similar results hold for $\underline{U}(t, x; z)$.

Proof. The equality $\overline{U}(t, x; z) = u(t, x; \overline{\Phi}(\cdot, z), z)$ was already proved in the previous proposition. We further claim $u(t, x; \overline{\Phi}(\cdot, z), z) = \overline{\Phi}(x - ct, z + ct)$. In view of (2.1) and the definition of $\bar{u}$, we see that for any $T \geq 0$,

$$u(t, x; \bar{u}(0, \cdot; \tau, z), z) = u(t, x - cT; \bar{u}(0, \cdot; \tau + T, z + cT), z + cT).$$

It follows that for any $t \geq 0$,

$$\begin{aligned} u(t, x; \overline{\Phi}(\cdot, z), z) &= \lim_{\tau\to\infty} u(t, x; u(\tau, \cdot; \bar{u}(0, \cdot; \tau, z), z), z) \\ &= \lim_{\tau\to\infty} u(t + \tau, x; \bar{u}(0, \cdot; \tau, z), z) \\ &= \lim_{\tau\to\infty} u(t + \tau, x - ct; \bar{u}(0, \cdot; \tau + t, z + ct), z + ct) \\ &= \overline{\Phi}(x - ct, z + ct). \end{aligned}$$

Similarly, we can prove $u(t, x; \underline{\Phi}(\cdot, z), z) = \underline{\Phi}(x - ct, z + ct)$.

Next, by the comparison principle, we have for any $\tau > 0$,

$$\begin{aligned} \bar{u}(t + \tau, x; \tau, z) &\geq u(t + \tau, x; \bar{u}(0, \cdot; \tau, z), z) \\ &\geq u(t + \tau, x; \underline{u}(0, \cdot; \tau, z), z) \geq \underline{u}(t + \tau, x; \tau, z). \end{aligned}$$

On account of Lemma 3.2 and the definitions of $\overline{U}$ and $\underline{U}$, we obtain for any $\tau > 0$,

$$\bar{u}(t + \tau, x; \tau, z) \geq \overline{U}(t, x; z) \geq \underline{U}(t, x; z) \geq \underline{u}(t + \tau, x; \tau, z).$$

By the definition of $\bar{u}$ and $\underline{u}$, we have for large $x - ct$,

$$\begin{aligned} \bar{u}(t + \tau, x; \tau, z) &= e^{-\mu(x-ct)}\phi(x + z), \\ \underline{u}(t + \tau, x; \tau, z) &= e^{-\mu(x-ct)}\phi(x + z) - Me^{-\mu^\varepsilon(x-ct)}\phi^\varepsilon(x + z). \end{aligned}$$

Therefore, the squeezing argument gives (noting that $\mu^\varepsilon > \mu > 0$)

$$\lim_{x-ct\to\infty} \frac{\overline{U}_i(t,x;z)}{e^{-\mu(x-ct)}\phi_i(x+z)} = 1, \qquad \lim_{x-ct\to\infty} \frac{\underline{U}_i(t,x;z)}{e^{-\mu(x-ct)}\phi_i(x+z)} = 1, \ i = 1,2.$$

Finally, since c_+^* is the rightward spreading speed, it follows that for any $c_1 < c_+^*$ and any $u^0 \in \mathcal{C}_+^*$,

$$\lim_{t\to\infty, x\le c_1 t} (u(t,x;u^0,z) - p(x+z)) = 0$$

uniformly in $z \in \mathbb{R}$. First, we choose $u^0 \in \mathcal{C}_+^*$ such that $u^0(x) \equiv \sigma \min_{y\in\mathbb{R}} \phi^0(y)$ for $x \le K_M - 1$ and $u^0(x) \equiv 0$ for $x \ge K_M$ and $u^0(x)$ is continuous and non-increasing for $K_M - 1 \le x \le K_M$. It is easily seen that $u^0(x) \le \underline{u}(0,x;0,z) \le \underline{\Phi}(x,z) \le \overline{\Phi}(x,z)$ for any $x, z \in \mathbb{R}$. Next, we obtain from the above formula and the comparison principle that for any $\varepsilon > 0$, there exists $T > 0$ such that for any $x \le c_1 T$ and $z \in \mathbb{R}$,

$$|u(T,x;\overline{\Phi}(\cdot,z),z) - p(x+z)| < \varepsilon.$$

Recall that $u(T,x;\overline{\Phi}(\cdot,z),z) = \overline{U}(T,x,z) = \overline{\Phi}(x-cT, z+cT)$. It is readily seen that for any $x < c_1 T$ and $z \in \mathbb{R}$,

$$|\overline{\Phi}(x-cT, z+cT) - p(x+z)| < \varepsilon.$$

Let $x' = x - cT$ and $z' = z + cT$. It follows that for any $x' < (c_1 - c)T$ and $z' \in \mathbb{R}$,

$$|\overline{\Phi}(x',z') - p(x'+z')| < \varepsilon.$$

Note that the choice of $T > 0$ depends on the value of $\varepsilon > 0$. In view of $c_1 < c$, the above inequality implies that

$$\lim_{x'\to-\infty} (\overline{\Phi}(x',z') - p(x'+z')) = 0$$

uniformly in $z' \in \mathbb{R}$. Now we make use of $\overline{U}(t,x;z) = \overline{\Phi}(x-ct, z+ct)$ again with $x' = x - ct$ and $z' = z + ct$ to obtain

$$\lim_{x-ct\to-\infty} (\overline{U}(t,x;z) - p(x+z)) = 0$$

uniformly in $z \in \mathbb{R}$. A similar result holds for $\underline{U}(t,x;z)$. $\square$

Recall from Proposition 2.7 that $\psi = (\psi_1, \psi_2)$ and $\tilde{\lambda} < 0$ are the principal eigenfunction and eigenvalue of the problem (2.7). Choose $0 < \bar{\varepsilon} < -\tilde{\lambda}$ and $\varepsilon_1 > 0$ such that

$$\varepsilon_1(\psi_1(x) + \psi_2(x)) < \bar{\varepsilon} \min\{\psi_1(x), \psi_2(x)\}, \ \forall x \in \mathbb{R}.$$

Next, we choose a small $\delta > 0$ such that for any two vector-valued functions $v = (v_1, v_2)$ and $\bar{v} = (\bar{v}_1, \bar{v}_2)$ satisfying $|v_i(x) - p_i(x)| < \delta$, $|\bar{v}_i(x) - p_i(x)| < \delta$ and $v_i(x) \le \bar{v}_i(x)$ for all $x \in \mathbb{R}$, we have for any $x \in \mathbb{R}$,

$$\begin{aligned} |f(x, \bar{v}) - f(x, v)| &< (F_1 + \varepsilon_1)[\bar{v}_1(x) - v_1(x)] + (F_2 + \varepsilon_1)[\bar{v}_2(x) - v_2(x)], \\ |g(x, \bar{v}) - g(x, v)| &< (G_1 + \varepsilon_1)[\bar{v}_1(x) - v_1(x)] + (G_2 + \varepsilon_1)[\bar{v}_2(x) - v_2(x)], \end{aligned} \tag{3.5}$$

where

$$F_i := \partial_{u_i} f(x, u_1, u_2)\big|_{u=p}, \qquad G_i := \partial_{u_i} g(x, u_1, u_2)\big|_{u=p}, \ i = 1, 2.$$

Then we have the following result on the stability of pulsating waves.

Theorem 3.5. *Let $U(t, x)$ be a rightward pulsating wave of (1.2) connecting $p(x)$ and 0 with speed $c > c_+^*$. For a sufficiently small $\varepsilon > 0$, we denote $\mu^\varepsilon = \mu + \varepsilon$ and introduce the weight function*

$$w^\varepsilon(\xi) := \begin{cases} e^{\mu^\varepsilon(\xi - \xi_0)}, & \xi \ge \xi_0, \\ 1, & \xi < \xi_0, \end{cases}$$

where $\xi_0 \in \mathbb{R}$ is chosen such that $|U_i(t, x) - p_i(x)| < \delta$ for all $x - ct < \xi_0$ and $i = 1, 2$. Then there exists a real number $\varepsilon_0 > 0$ such that for any given initial value $u^0(x)$ with $0 \le u^0(x) \le p(x)$ and

$$[u^0(\cdot) - U(0, \cdot)]w^\varepsilon(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R}^2),$$

we have

$$\sup_{x \in \mathbb{R}} |u_i(t, x; u^0) - U_i(t, x)| \le Ce^{-\varepsilon_0 t}, \ i = 1, 2, \ t \ge 0,$$

for some $C > 0$.

Proof. Let $U^\pm(t, x)$ be the mild solutions of (1.2) with initial values

$$U^+(0, x) := \max\{u^0(x), U(0, x)\}, \qquad U^-(0, x) := \min\{u^0(x), U(0, x)\},$$

respectively. A simple application of the comparison principle yields

$$\begin{aligned} 0 \le U^-(t, x) &\le \min\{u(t, x; u^0), U(t, x)\} \\ &\le \max\{u(t, x; u^0), U(t, x)\} \le U^+(t, x) \le p(x). \end{aligned}$$

We aim to show that $U^\pm(t, x)$ converges to $U(t, x)$ exponentially in time. If this is done, then the desired stability result follows from the above inequality and the squeezing argument. By symmetry, we only need to prove $V(t, x) := U^+(t, x) - U(t, x)$ vanishes exponentially in time. In view of $0 \le V(0, x) \le |u^0(x) - U(0, x)|$, we obtain that $V(0, x)w^\varepsilon(x)$ is uniformly bounded on $\mathbb{R}$. We consider two cases $x - ct \ge \xi_0$ and $x - ct < \xi_0$, respectively.

Case I. $x - ct \geq \xi_0$.

By the subhomogeneity of $f(t,u)$ in $u = (u_1, u_2)$, we obtain

$$f(x, U_1 + V_1, U_1 + V_2) - f(x, U_1, U_2) \leq f_1 V_1 + f_2 V_2$$

and hence,

$$V_1(t,x) \leq [l_1(t)V_1(0,\cdot)](x) + \int_0^t l_1(t-s)[\tilde{f}_1(\cdot)V_1(s,\cdot) + f_2(\cdot)V_2(s,\cdot)](x)ds,$$

where $\tilde{f}_1(x) := f_1(x) + K$. Similarly, define $\tilde{g}_2(x) := g_2(x) + K$. We obtain from the subhomogeneity of $g(t,u)$ in $u = (u_1, u_2)$ that

$$V_2(t,x) \leq e^{-Kt}V_2(0,x) + \int_0^t e^{-K(t-s)}[g_1(x)V_1(s,x) + \tilde{g}_2(x)V_2(s,x)]ds.$$

Next, we define

$$\bar{V}_i(t,x) := C_1\phi_i^\varepsilon(x)e^{-\mu^\varepsilon(x-\xi_0)+\lambda^\varepsilon t},\ i = 1,2,$$

where $\phi^\varepsilon = (\phi_1^\varepsilon, \phi_2^\varepsilon)$ is the eigenfunction of (2.6) associated with the eigenvalue λ^ε and parameter $\mu^\varepsilon = \mu + \varepsilon$. The constant $C_1 > 0$ is chosen sufficiently large such that $V(0,x) \leq \bar{V}(0,x)$. This can be done since $V(0,x)w^\varepsilon(x)$ is uniformly bounded on $\mathbb{R}$. A straightforward calculation gives

$$\begin{aligned}
\partial_t \bar{V}_1(t,x) &= D_1(x)\partial_{xx}\bar{V}_1(t,x) + D_2(x)\partial_x\bar{V}_1(t,x) \\
&\quad + f_1(x)\bar{V}_1(t,x) + f_2(x)\bar{V}_2(t,x), \\
\partial_t \bar{V}_2(t,x) &= g_1(x)\bar{V}_1(t,x) + g_2(x)\bar{V}_2(t,x).
\end{aligned}$$

Therefore, we have

$$\bar{V}_1(t,x) \geq [l_1(t)\bar{V}_1(0,\cdot)](x) + \int_0^t l_1(t-s)[\tilde{f}_1(\cdot)\bar{V}_1(s,\cdot) + f_2(\cdot)\bar{V}_2(s,\cdot)](x)ds,$$

$$\bar{V}_2(t,x) \geq e^{-Kt}\bar{V}_2(0,x) + \int_0^t e^{-K(t-s)}[g_1(x)\bar{V}_1(s,x) + \tilde{g}_2(x)\bar{V}_2(s,x)]ds.$$

The inequalities are actually equalities but we choose to use the inequalities to emphasize that $\bar{V}$ is an upper solution. By using the comparison principle, we obtain $V(t,x) \leq \bar{V}(t,x)$, which implies

$$V_i(t,x) \leq \bar{V}_i(t,x) = C_1\phi_i^\varepsilon(x)e^{-\mu^\varepsilon(x-ct-\xi_0)}e^{-(c\mu^\varepsilon-\lambda^\varepsilon)t} \leq \tilde{C}_1 e^{-(c\mu^\varepsilon-\lambda^\varepsilon)t},$$

where in the last inequality we have made use of the fact that $x - ct \geq \xi_0$.

Case II. $x - ct < \xi_0$.

Note that $U(t,x)$ is very close to $p(x)$ (i.e., $|U_i(t,x) - p_i(x)| < \delta$) for all $x - ct < \xi_0$ and $i = 1, 2$. Since $U(t,x) \leq U(t,x) + V(t,x) = U^+(t,x) \leq p(x)$, we obtain from (3.5) that for $x - ct < \xi_0$,

$$f(x, U^+) - f(x, U) \leq (F_1 + \varepsilon_1)V_1 + (F_2 + \varepsilon_1)V_2,$$

and

$$g(x, U^+) - g(x, U) \leq (G_1 + \varepsilon_1)V_1 + (G_2 + \varepsilon_1)V_2,$$

where $F_i := \partial_{u_i} f(x, u_1, u_2)\big|_{u=p}$, $G_i := \partial_{u_i} g(x, u_1, u_2)\big|_{u=p}$, $i = 1, 2$. Consequently, $V(t,x)$ satisfies

$$\begin{aligned}
\partial_t V_1(t,x) &\leq D_1(x)\partial_{xx} V_1(t,x) + D_2(x)\partial_x V_1(t,x) \\
&\qquad + [F_1(x) + \varepsilon_1]V_1(t,x) + [F_2(x) + \varepsilon_1]V_2(t,x), \\
\partial_t V_2(t,x) &\leq [G_1(x) + \varepsilon_1]V_1(t,x) + [G_2(x) + \varepsilon_1]V_2(t,x),
\end{aligned}$$

for all (t,x) in the domain

$$D := \{(t,x) : \ t > 0, \ x - ct < \xi_0\}.$$

Next, we choose $\varepsilon_0 = \min\{c\mu^\varepsilon - \lambda^\varepsilon, -\tilde{\lambda} - \bar{\varepsilon}\} > 0$ and define

$$\tilde{V}_i(t,x) := C_2 \psi_i(x) e^{-\varepsilon_0 t}, \ i = 1, 2,$$

where $\psi = (\psi_1, \psi_2)$ and $\tilde{\lambda} < 0$ are the principal eigenfunction and eigenvalue of the linearized eigenvalue problem (2.7). The constant $C_2 > 0$ is chosen sufficiently large such that $V(t,x) \leq \tilde{V}(t,x)$ on the boundary

$$\partial D = \{(t,x) : \ t = 0, x \leq \xi_0\} \cup \{(t,x) : \ t \geq 0, x - ct = \xi_0\}.$$

This can be done due to the result obtained from case I and the inequality $\varepsilon_0 \leq c\mu^\varepsilon - \lambda^\varepsilon$. Now, we verify that the function $\tilde{V}(t,x)$ is an upper solution of the following linear system in the domain D:

$$\begin{aligned}
\partial_t V_1(t,x) &= D_1(x)\partial_{xx} V_1(t,x) + D_2(x)\partial_x V_1(t,x) \\
&\qquad + [F_1(x) + \varepsilon_1]V_1(t,x) + [F_2(x) + \varepsilon_1]V_2(t,x), \\
\partial_t V_2(t,x) &= [G_1(x) + \varepsilon_1]V_1(t,x) + [G_2(x) + \varepsilon_1]V_2(t,x).
\end{aligned}$$

Since $\varepsilon_0 \leq -\tilde{\lambda} - \bar{\varepsilon}$, we have

$$\begin{aligned}\partial_t \tilde{V}_1(t,x) &= -\varepsilon_0 \tilde{V}_1(t,x)\\ &\geq (\tilde{\lambda}+\bar{\varepsilon})\tilde{V}_1(t,x)\\ &= D_1(x)\partial_{xx}\tilde{V}_1(t,x) + D_2(x)\partial_x \tilde{V}_1(t,x)\\ &\quad + [F_1(x)+\bar{\varepsilon}]\tilde{V}_1(t,x) + F_2(x)\tilde{V}_2(t,x).\end{aligned}$$

Note that $\varepsilon_1(\psi_1+\psi_2) < \bar{\varepsilon}\min\{\psi_1,\psi_2\}$. It then follows that $\varepsilon_1(\tilde{V}_1+\tilde{V}_2) < \bar{\varepsilon}\tilde{V}_1$, and hence,

$$\begin{aligned}\partial_t \tilde{V}_1(t,x) \geq\ & D_1(x)\partial_{xx}\tilde{V}_1(t,x) + D_2(x)\partial_x \tilde{V}_1(t,x)\\ & + [F_1(x)+\varepsilon_1]\tilde{V}_1(t,x) + [F_2(x)+\varepsilon_1]\tilde{V}_2(t,x).\end{aligned}$$

Similarly, we obtain

$$\partial_t \tilde{V}_2(t,x) \geq [G_1(x)+\varepsilon_1]\tilde{V}_1(t,x) + [G_2(x)+\varepsilon_1]\tilde{V}_2(t,x).$$

Thus, by the comparison principle in the domain D (see, e.g., [13, Theorem 4.6]), we have $V(t,x) \leq \tilde{V}(t,x)$, and hence,

$$V_i(t,x) \leq \tilde{V}_i(t,x) = C_2\psi_i(x)e^{-\varepsilon_0 t} \leq \tilde{C}_2 e^{-\varepsilon_0 t}$$

for all $t \geq 0$ and $x - ct < \xi_0$.

Combining cases I and II above, we then see that

$$V_i(t,x) \leq Ce^{-\varepsilon_0 t},\ i=1,2,\ t\geq 0,\ x\in\mathbb{R},$$

where $C > 0$ denotes a large fixed constant. □

As a consequence of the above stability theorem, we are able to show that two pulsating waves $\overline{U}(t,x;0)$ and $\underline{U}(t,x;0)$ obtained from the upper and lower solutions are indeed identical.

Corollary 3.6. *If $U(t,x) = \Psi(x-ct,x)$ and $\tilde{U}(t,x) = \tilde{\Psi}(x-ct,x)$ are two pulsating waves of (1.2) connecting $p(x)$ to 0 with speed $c > c_+^*$ and satisfy*

$$\lim_{x-ct\to\infty}\frac{U_1(t,x)}{e^{-\mu(x-ct)}\phi_1(x)} = \lim_{x-ct\to\infty}\frac{U_2(t,x)}{e^{-\mu(x-ct)}\phi_2(x)} = A > 0$$

and

$$\lim_{x-ct\to\infty}\frac{\tilde{U}_1(t,x)}{e^{-\mu(x-ct)}\phi_1(x)} = \lim_{x-ct\to\infty}\frac{\tilde{U}_2(t,x)}{e^{-\mu(x-ct)}\phi_2(x)} = B > 0$$

for some positive constants A and B, then $U(t,x) = \tilde{U}(t+t^*,x)$ for some real number t^*. Especially, $\overline{U}(t,x;0) = \underline{U}(t,x;0)$ and it is a classical solution of (1.2).

Proof. Upon a linear translation of the variable t, we may assume both $U(t,x)$ and $\tilde{U}(t,x)$ satisfy the same asymptotic conditions:

$$\lim_{x-ct\to\infty}\frac{U_i(t,x)}{e^{-\mu(x-ct)}\phi_i(x)}=\lim_{x-ct\to\infty}\frac{\tilde{U}_i(t,x)}{e^{-\mu(x-ct)}\phi_i(x)}=1,\ i=1,2,$$
$$\lim_{x-ct\to-\infty}(U(t,x)-p(x))=\lim_{x-ct\to-\infty}(\tilde{U}(t,x)-p(x))=0.$$

It remains to show that $U(t,x)=\tilde{U}(t,x)$ for all $t\ge 0$ and $x\in\mathbb{R}$. Since $U(t,x)$ and $\tilde{U}(t,x)$ are two solutions of (1.2) with the same leading term as $x-ct\to\infty$, we obtain from the equation satisfied by the difference $U(t,x)-\tilde{U}(t,x)$ and a simple asymptotic analysis near $x-ct\to\infty$ that $[U(t,x)-\tilde{U}(t,x)]e^{\mu^\varepsilon(x-ct)}$ is uniformly bounded for some small $\varepsilon=\mu^\varepsilon-\mu>0$. Especially, $[U(0,\cdot)-\tilde{U}(0,\cdot)]w^\varepsilon(\cdot)\in L^\infty(\mathbb{R},\mathbb{R}^2)$. It follows from Theorem 3.5 that

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|U_i(t,x)-\tilde{U}_i(t,x)|=0,\ i=1,2.$$

Note that $U(t,x)=\Psi(x-ct,x)$ and $\tilde{U}(t,x)=\tilde{\Psi}(x-ct,x)$; and $\Psi(\xi,x)$ and $\tilde{\Psi}(\xi,x)$ are periodic in the second variable x. We conclude that $U(t,x)=\tilde{U}(t,x)$ for all $t\ge 0$ and $x\in\mathbb{R}$. Especially, we have $\overline{U}(t,x)=\underline{U}(t,x)$. Finally, we observe from the definition (3.1) that $\overline{U}(t,x)$ is upper semi-continuous in x since it is the limit of a decreasing sequence of continuous functions. Similarly, it follows from (3.2) that $\underline{U}(t,x)$ is lower semi-continuous in x. Thus, $\overline{U}(t,x)=\underline{U}(t,x)$ is actually continuous in x. Recall that $\overline{U}(t,x)$ is the solution of (1.2) with initial function $\overline{U}(0,x)$. The continuity of $\overline{U}(0,x)$ implies that $\overline{U}(t,x)$ is a classical solution. □

Inspired by the results in [4] for the case of scalar equations, we may expect that the exponential decay condition as imposed in Corollary 3.6 is automatically satisfied by any pulsating wave $u(t,x)=\Psi(x-ct,x)$ connecting 0 and $p(x)$ with speed $c>c_+^*$. It is also worthy to point out that the existence and stability of the critical pulsating wave with $c=c_+^*$ have not been addressed in this paper. We leave these challenging problems for future investigations.

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