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Traveling Waves in Epidemic Models: Non-monotone Diffusive Systems with Non-monotone Incidence Rates

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Abstract

We study the existence and nonexistence of traveling waves of diffusive epidemic models with general incidence rates. The model systems are non-monotone because of the intrinsic predator–prey interaction between the susceptible and infective compartments in epidemic systems. Moreover, the incidence rate may not be monotone in the infected population because social behaviors and collective activities may change in response to the prevalence of disease. To find positive traveling solutions of the non-monotone system with a non-monotone incidence function, we will construct a suitable convex set in a weighted function space, and then apply Schauder fixed point theorem. It turns out that the basic reproduction number of the corresponding ordinary differential equations plays an important role in the existence theory of traveling waves. Moreover, the critical wave speed can be explicitly obtained in terms of the diffusion coefficient, recovery rate and death rate for the infected group, and partial derivative of incidence function at the disease-free equilibrium. Finally, we prove that the positive traveling wave solution does not exist if the basic reproduction number is no more than one, or the wave speed is less than the critical value.

Keywords Traveling waves · Diffusive epidemic models · Schauder fixed point theorem · Non-monotonicity

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1 Introduction

Traveling waves in monotone diffusive systems have been investigated extensively in the literature, and there exists a general theory of monotone dynamical systems to deal with this problem; see [5,17] and references therein. Unfortunately, for non-monotone dynamical systems, especially those arising from diffusive epidemic models, the corresponding traveling wave solutions can be only studied case by case. So far as we know, there is no traveling wave theory for non-monotone systems as general and complete as that for monotone systems. Recently, Huang [14] used a geometric shooting method to study a non-monotone diffusive system with two compartments and relatively general reaction terms, where the incidence rate (or predation function) was assumed to be monotone in the infected group (or predators). A different approach via Schauder fixed point theorem on a similar problem was developed in [27], where the model system had only two compartments and the incidence rate was monotone. The objective of this paper is to extend the aforementioned results to the diffusive disease models with three compartments and non-monotone incidence rates. To be specific, we will consider the following reaction-diffusion system:

$$\partial_t S = d_1 \partial_{xx} S - \varphi(S, I, R), \tag{1.1}$$

$$\partial_t I = d_2 \partial_{xx} I + \varphi(S, I, R) - \gamma I - \delta I, \tag{1.2}$$

$$\partial_t R = d_3 \partial_{xx} R + \gamma I. \tag{1.3}$$

Here, S, I, R denote the susceptible, infected, and recovered populations which are time and spatial dependent. For sake of simplicity, we assume the spatial variable x is of one dimension. Let d_1, d_2, d_3 be diffusion coefficients for the three groups, respectively. We denote by γ the recovery rate and δ the disease-induced death rate. The nonlinear incidence function $\varphi(S, I, R) \in C^{1,2,1}(\mathbb{R}^3_+, \mathbb{R}_+)$ satisfies the following conditions. For any $S, I, R > 0$,

$$\varphi(0, I, R) = \varphi(S, 0, R) = 0, \tag{1.4}$$

$$\partial_S \varphi(S, I, R) \geq 0, \ \ \partial_R \varphi(S, I, R) \leq 0, \tag{1.5}$$

$$\frac{\varphi(S, I, 0)}{I} \leq \partial_I \varphi(S, 0, 0), \tag{1.6}$$

and for some $\theta > 0$ and any $S > 0$,

$$\sup_{I \geq 0} \varphi(S, I, 0) < \infty, \quad \inf_{I \leq 1 \leq R} \frac{R^\theta \varphi(S, I, R)}{I} > 0, \tag{1.7}$$

and for any $S, I > 0$,

$$\sup_{R \geq 0} |\nabla \varphi(S, I, R)| < \infty. \tag{1.8}$$

Here, $\nabla \varphi(S, I, R)$ is the gradient of $\varphi(S, I, R)$ and $|\nabla \varphi(S, I, R)|$ is a vector norm in $\mathbb{R}^3$. The interpretations of the conditions (1.4)–(1.8) are: there is no disease transmission if either $S = 0$ or $I = 0$; the incidence rate is increasing in S and decreasing in R (but not necessarily monotone in I); the incidence rate is sub-homogeneous in the sense that it makes the fastest change at the initial time of an outbreak; the incidence rate is saturated as I increases; the incidence rate decays at most algebraically as R increases; the incidence rate does change rapidly with small perturbation in the population.

If the recovered group R does not influent the disease transmission, the incidence function is independent of R and the three dimensional system can be reduced to a two dimensional system. Examples include the standard incidence rate $\beta SI/(S + I)$ in [20,21], the incidence rates with media impact μSIe^{-mI} in [3] and $(\mu_1 - \mu_2 f(I))SI/(S+I)$ in [4]. Note that when media coverage takes effect, contact patterns will change and thus the incidence rate is not necessary monotone as a function of infected population [16]. If the recovered group R comes back to the community and thus inhibits the transmission of disease to the susceptible group, the model system becomes three dimensional. Examples include the standard incidence rate $\beta SI/(S+I+R)$ in [19], Michaelis–Menten type incidence rate $\beta SI/(1+b(S+I+R))$ in [2], and a general incidence rate $\beta f(S+I+R)SI/(S+I+R)$ in [10]. Due to the involvement of multiple compartments, it seems difficult, if not impossible, to apply the geometric shooting method developed in [14]. The traveling wave solutions for the standard incidence rate have been studied in [19] where a technical assumption on the diffusion coefficients (i.e., $d_3 < 2d_2$) was used to prove the existence result. In this paper, we will extend this result to a more general incidence function which may not be monotone in I. Moreover, we will also remove the technical assumption by some new ideas in the construction of convex subset.

The traveling wave solutions are useful illustrations in understanding disease propagation of diffusive epidemic models, and critical wave speed provides important information about propagation speed of the disease in a spatial environment [5,6]. On the other hand, disease models are typical examples of non-monotone dynamical systems and the analysis of traveling wave solutions for these systems is generally much more challenging than that of monotone dynamical systems [25,26]. There are two major methods in the study of traveling waves for diffusive epidemic models. The first one is called shooting method, which was introduced by Dunbar [8,9], later developed by Hosono and Ilyas [11,12], and by Huang [13,14]. Since the geometric structure of the invariant manifold in high dimensional Euclidean space is very complicated to analyze, the application of shooting method mainly restricts to reaction-diffusion systems with only two compartments. Another widely used method, which will be applied in this paper, is the Schauder fixed point theorem. As we shall see, finding a traveling wave is equivalent with the problem of finding a fixed point for a certain integral map. The key step of this method is to construct a convex set that is invariant under the integral map. This convex set is usually bounded by several super- and sub-solutions. Unlike the monotone systems where comparison principle guarantees that super-solutions are independent of sub-solutions, for non-monotone systems, the super-solutions and sub-solutions are fully coupled. For instance, to construct a convex set for an epidemic model with n compartments, one should find n super-solutions and n sub-solutions which satisfy $2n$ differential inequalities. Once the convex set has been constructed and proven to be invariant, we can verify the continuity and compactness of the integral map by using Arzela–Ascoli theorem and a standard diagonal process, and thus conclude the existence of traveling wave from Schauder fixed point theorem. The nonexistence result can obtained by analyticity of Laplace transform and a contradiction argument. It is also a standard process to find the asymptotic behavior of traveling wave solution at infinity. Through this routine, the main difficult is in the construction of invariant convex set to overcome the non-monotonicity of diffusive systems. Note that non-monotonicity of incidence rate makes the task even more challenging, because it is more difficult to verify the inequalities satisfied by the super- and sub-solutions.

The traveling wave of (1.1)–(1.3) takes the form $(S(x+ct), I(x+ct), R(x+ct))$, and it connects an equilibrium $(S_{-\infty}, 0, 0)$ to another equilibrium $(S_\infty, 0, R_\infty)$. By introducing a new variable $\xi = x + ct$, we obtain

$$cS'(\xi) = d_1 S''(\xi) - \varphi(S(\xi), I(\xi), R(\xi)), \tag{1.9}$$

$$cI'(\xi) = d_2 I''(\xi) + \varphi(S(\xi), I(\xi), R(\xi)) - (\gamma + \delta)I(\xi), \tag{1.10}$$

$$cR'(\xi) = d_3 R''(\xi) + \gamma I(\xi). \tag{1.11}$$

For any $S_{-\infty} > 0$, we define the linearized transmission rate

$$\beta := \partial_I \varphi(S_{-\infty}, 0, 0), \tag{1.12}$$

and the critical wave speed (when $\beta > \gamma + \delta$)

$$c^* := 2\sqrt{d_2(\beta - \gamma - \delta)}. \tag{1.13}$$

Following [7,18], the basic reproduction number for the ordinary differential system without diffusion is given by

$$R_0 := \frac{\beta}{\gamma + \delta}. \tag{1.14}$$

Our main theorem is stated as below.

Theorem 1.1 *For any $S_{-\infty} > 0$, $R_0 > 1$ and $c > c^*$, there exist $S_\infty < S_{-\infty}$ and a traveling wave solution for (1.1)–(1.3) such that $S(-\infty) = S_{-\infty}$, $S(\infty) = S_\infty$, $I(\pm\infty) = 0$, $R(-\infty) = 0$, and $R(\infty) = \gamma(S_{-\infty} - S_\infty)/(\gamma + \delta)$. Furthermore, $S'(x) < 0$, $I(x) \leq S_{-\infty} - S_\infty$, $R'(x) > 0$, and*

$$\int_{-\infty}^{\infty} (\gamma + \delta) I(x) dx = \int_{-\infty}^{\infty} \varphi(S(x), I(x), R(x)) dx = c(S_{-\infty} - S_\infty). \tag{1.15}$$

On the other hand, if $c < c^*$ or $R_0 \leq 1$, then there does not exist a non-trivial and non-negative traveling wave solution for (1.1)–(1.3) such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$, and $R(-\infty) = 0$.

The paper is organized as follows. In Sect. 2, we will define some linear second-order differential operators and a linear integral map associated with traveling wave equations. A convex set will be constructed in Sect. 3 and we shall prove the invariance of this set under the integral map. In Sect. 4, we will prove continuity and compactness of the integral map; while in Sect. 5, we will prove the existence of traveling wave solutions under the conditions that $R_0 > 1$ and $c > c^*$. Throughout Sects. 2, 3, 4 and 5, we will assume these two conditions are satisfied. Section 6 is devoted to the non-existence result when the existence conditions are violated. Finally, we give conclusion and discussion in Sect. 7.

2 Differential Operators and Integral Map

We first assume that $R_0 > 1$ and $c > c^*$. Linearizing the Eq. (1.10) for I at the point $(S_{-\infty}, 0, 0)$ gives the characteristic function

$$f_0(\lambda) := -d_2\lambda^2 + c\lambda - (\beta - \gamma - \delta). \tag{2.1}$$

We denote the smallest positive root of the characteristic function $f_0(\lambda)$ by

$$\lambda_0 := \frac{c - \sqrt{c^2 - 4d_2(\beta - \gamma - \delta)}}{2d_2} > 0. \tag{2.2}$$

Let α_1, α_2 and α_3 be three sufficiently large constants to be determined later. For each $i = 1, 2, 3$, we define the second-order differential operator $\Delta_i h := -d_i h'' + ch' + \alpha_i h$ for any $h \in C^2(\mathbb{R})$. The characteristic roots for the differential operator Δ_i are calculated as

$$\lambda_i^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_i\alpha_i}}{2d_i}. \tag{2.3}$$

Denote

$$\rho_i := d_i(\lambda_i^+ - \lambda_i^-) = \sqrt{c^2 + 4d_i\alpha_i}. \tag{2.4}$$

The inverse operator Δ_i^{-1} has the following integral representation

$$(\Delta_i^{-1}h)(x) := \frac{1}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{1}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy \tag{2.5}$$

for $h \in C_{\mu^-,\mu^+}(\mathbb{R})$ with $\mu^- > \lambda_i^-$ and $\mu^+ < \lambda_i^+$, where

$$C_{\mu^-,\mu^+}(\mathbb{R}) := \{h \in C(\mathbb{R}) : \sup_{x\le 0}|h(x)e^{-\mu^- x}| + \sup_{x\ge 0}|h(x)e^{-\mu^+ x}| < \infty\}.$$

Given $h \in C_{\mu^-,\mu^+}(\mathbb{R})$, the derivatives of $\Delta_i^{-1}h$ are given by

$$(\Delta_i^{-1}h)'(x) = \frac{\lambda_i^-}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{\lambda_i^+}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy; \tag{2.6}$$

$$(\Delta_i^{-1}h)''(x) = \frac{(\lambda_i^-)^2}{\rho_i}\int_{-\infty}^{x} e^{\lambda_i^-(x-y)}h(y)dy + \frac{(\lambda_i^+)^2}{\rho_i}\int_{x}^{\infty} e^{\lambda_i^+(x-y)}h(y)dy - \frac{h(x)}{d_i}. \tag{2.7}$$

The following lemma provides some properties on the composition of the integral operator Δ_i^{-1} with the differential operator Δ_i.

Lemma 2.1 *Let $h \in C_{\mu^-,\mu^+}(\mathbb{R})$ be a piecewise function defined as $h(y) = h_j(y)$ for $y \in [y_j, y_{j+1}]$ with $j = 0, 1, \dots, m$, where $h_j(y) \in C^2[y_j, y_{j+1}]$, and for convenience, we denote $y_0 = -\infty$ and $y_{m+1} = \infty$. If $h'_{j-1}(y_j) \le h'_j(y_j)$ for $1 \le j \le m$, then $\Delta_i^{-1}(\Delta_i h) \ge h$. Similarly, if $h'_{j-1}(y_j) \ge h'_j(y_j)$ for $1 \le j \le m$, then $\Delta_i^{-1}(\Delta_i h) \le h$. Especially, if $h' \in C(\mathbb{R})$, then $\Delta_i^{-1}(\Delta_i h) = h$.*

Proof For any $x \in \mathbb{R}$, there exists l such that $x \in [y_l, y_{l+1}]$. It follows from (2.5) that

$$\begin{aligned}\rho_i[\Delta_i^{-1}(\Delta_i h)](x) = &\sum_{j=0}^{l-1}\int_{y_j}^{y_{j+1}} e^{\lambda_i^-(x-y)}\Delta_i h_j(y)dy + \int_{y_l}^{x} e^{\lambda_i^-(x-y)}\Delta_i h_l(y)dy \\ &+ \int_{x}^{y_{l+1}} e^{\lambda_i^+(x-y)}\Delta_i h_l(y)dy + \sum_{j=l+1}^{m}\int_{y_j}^{y_{j+1}} e^{\lambda_i^+(x-y)}\Delta_i h_j(y)dy.\end{aligned}$$

On account of (2.3), we take integration by parts twice and obtain

$$\begin{aligned}\rho_i\left[\Delta_i^{-1}(\Delta_i h)\right](x) = &\sum_{j=0}^{l-1}\left[-d_i h'_j(y) + ch_j(y) - d_i\lambda_i^- h_j(y)\right]e^{\lambda_i^-(x-y)}\Big|_{y_j}^{y_{j+1}} \\ &+ \left[-d_i h'_l(y) + ch_l(y) - d_i\lambda_i^- h_l(y)\right]e^{\lambda_i^-(x-y)}\Big|_{y_l}^{x} \\ &+ \left[-d_i h'_l(y) + ch_l(y) - d_i\lambda_i^+ h_l(y)\right]e^{\lambda_i^+(x-y)}\Big|_{x}^{y_{l+1}} \\ &+ \sum_{j=l+1}^{m}\left[-d_i h'_j(y) + ch_j(y) - d_i\lambda_i^+ h_j(y)\right]e^{\lambda_i^+(x-y)}\Big|_{y_j}^{y_{j+1}}.\end{aligned}$$

Since $h \in C_{\mu^-,\mu^+}(\mathbb{R})$, we observe

$$\lim_{y\to-\infty}\left[-d_i h_0'(y)+ch_0(y)-d_i\lambda_i^- h_0(y)\right]e^{\lambda_i^-(x-y)}=0,$$
$$\lim_{y\to\infty}\left[-d_i h_m'(y)+ch_m(y)-d_i\lambda_i^+ h_m(y)\right]e^{\lambda_i^+(x-y)}=0.$$

The continuity of h implies that $h_{j-1}(y_j)=h_j(y_j)$ for $1\le j\le m$. Moreover, since $h_{j-1}'(y_j)\le h_j'(y_j)$ for $1\le j\le m$, we have

$$\begin{aligned}\rho_i\left[\Delta_i^{-1}(\Delta_i h)\right](x) &\ge \left[-d_i h_l'(x)+ch_l(x)-d_i\lambda_i^- h_l(x)\right]\\ &\quad-\left[-d_i h_l'(x)+ch_l(x)-d_i\lambda_i^+ h_l(x)\right]\\ &=\rho_i h_l(x)=\rho_i h(x).\end{aligned}$$

Here, we have made use of $d_i(\lambda_i^+-\lambda_i^-)=\rho_i$; see (2.4).

On the other hand, if $h_{j-1}'(y_j)\ge h_j'(y_j)$ for $1\le j\le m$, then $-h_{j-1}'(y_j)\le -h_j'(y_j)$, which implies that $\Delta_i^{-1}[\Delta_i(-h)]\ge -h$; namely, $\Delta_i^{-1}(\Delta_i h)\le h$. This completes the proof. □

By (1.12), we have $\varphi(S_{-\infty},I,0)/I\to\beta>\gamma+\delta$ as $I\to 0^+$. By (1.7), we have $\varphi(S_{-\infty},I,0)/I\to 0<\gamma+\delta$ as $I\to\infty$. Hence, there exists $I_0>0$ such that

$$\varphi(S_{-\infty},I_0,0)=(\gamma+\delta)I_0. \tag{2.8}$$

Now, we choose $\alpha_1,\ \alpha_2$ sufficiently large and $\alpha_3>0$ such that

$$\alpha_1\ge\partial_S\varphi(S,I,R),\quad \alpha_2\ge\gamma+\delta-\partial_I\varphi(S,I,R)$$

for all $0\le S\le S_{-\infty}$, $0\le I\le I_0$, $R\ge 0$, and $|\lambda_i^-|=-\lambda_i^->\sigma$ for $i=1,2,3$, where $I_0>0$ is a solution to (2.8) and $\sigma>0$ is a small constant such that $\sigma<\lambda_0$, $\sigma<c/d_1$, $\sigma<c/d_3$, and $f_0(\lambda_0+\sigma)>0$. Find a $\mu>0$ such that $\sigma<\mu<-\lambda_i^-<\lambda_i^+$ for all $i=1,2,3$. We then have $\lambda_i^-<-\mu<\mu<\lambda_i^+$. Introduce a Banach space

$$B_\mu(\mathbb{R},\mathbb{R}^n):=\underbrace{C_{-\mu,\mu}(\mathbb{R})\times\cdots\times C_{-\mu,\mu}(\mathbb{R})}_{n}$$

equipped with the norm

$$|u|_\mu:=\max_{1\le i\le n}\sup_{x\in\mathbb{R}}e^{-\mu|x|}|u_i(x)|, \tag{2.9}$$

where $u=(u_1,\dots,u_n)^T\in B_\mu(\mathbb{R},\mathbb{R}^n)$ and n is a positive integer. Now, we define an integral map $F=(F_1,F_2,F_3)^T$ on the space $B_\mu(\mathbb{R},\mathbb{R}^3)$ as follows:

$$F(u)=\begin{pmatrix}F_1(u)\\F_2(u)\\F_3(u)\end{pmatrix}:=\begin{pmatrix}\Delta_1^{-1}[\alpha_1u_1-\varphi(u_1,u_2,u_3)]\\ \Delta_2^{-1}[\alpha_2u_2+\varphi(u_1,u_2,u_3)-(\gamma+\delta)u_2]\\ \Delta_3^{-1}(\alpha_3u_3+\gamma u_2)\end{pmatrix}, \tag{2.10}$$

for $u=(u_1,u_2,u_3)^T\in B_\mu(\mathbb{R},\mathbb{R}^3)$.

Lemma 2.2 *Let $u=(S,I,R)^T\in B_\mu(\mathbb{R},\mathbb{R}^3)$ be a fixed point of the map F, then (S,I,R) satisfy the traveling wave Eqs.* (1.9)–(1.11).

Proof Set $h_1 := \alpha_1 S - \varphi(S, I, R)$. It follows from (2.4) to (2.7) and the fact that $\lambda_1^{\pm}$ are the roots of $f_1(\lambda) = -d_1\lambda^2 + c + \alpha_1$ that

$$-d_1(\Delta_1^{-1}h_1)'' + c(\Delta_1^{-1}h_1)' + \alpha_1(\Delta_1^{-1}h_1) = h_1.$$

Since (S, I, R) is a fixed point of F, it follows that $\Delta_1^{-1}h_1 = S$. Thus, the above equation is the same as (1.9). Similarly, we can show that the other two Eqs. (1.10) and (1.11) are also satisfied. □

3 Convex Set

For $x \in \mathbb{R}$, we define super-solutions and sub-solutions as follows:

$$S_+(x) := \begin{cases} S_{-\infty}, & x \le x_0, \\ S_{-\infty} - \varepsilon + \varepsilon e^{-\lambda_1(x-x_0)}, & x \ge x_0, \end{cases} \tag{3.1}$$

$$S_-(x) := \begin{cases} S_{-\infty}\left(1 - e^{\sigma(x-x_1)}\right), & x \le x_1, \\ 0, & x \ge x_1, \end{cases} \tag{3.2}$$

$$I_+(x) := \begin{cases} e^{\lambda_0 x}, & x \le x_0, \\ e^{\lambda_0 x_0}, & x \ge x_0, \end{cases} \tag{3.3}$$

$$I_-(x) := \begin{cases} e^{\lambda_0 x}\left[1 - \frac{\lambda_0+\lambda_2}{\lambda_0+\lambda_2+\sigma} e^{\sigma(x-x_2)}\right], & x \le x_2, \\ \frac{\sigma e^{\lambda_0 x_2}}{\lambda_0+\lambda_2+\sigma} e^{-\lambda_2(x-x_2)}, & x \ge x_2, \end{cases} \tag{3.4}$$

$$R_+(x) := Me^{\sigma x}, \tag{3.5}$$

$$R_-(x) := 0, \tag{3.6}$$

where $x_0 = (\ln I_0)/\lambda_0$ with $I_0 > 0$ being a positive solution to (2.8); $\sigma > 0$ is a small constant such that $\sigma < \lambda_0$, $\sigma < c/d_1$, $\sigma < c/d_3$, and $f_0(\lambda_0 + \sigma) > 0$; x_1 is chosen such that $x_1 < x_0$ and $\beta e^{\lambda_0 x_1} \le S_{-\infty}(c\sigma - d_1\sigma^2)$; $M = \gamma e^{(\lambda_0-\sigma)x_0}/(c\sigma - d_3\sigma^2)$; $\lambda_2 > 0$ is the unique positive solution to the equation $d_2\lambda_2^2 + c\lambda_2 - (\gamma + \delta) = 0$; $\lambda_1 = \lambda_2 + \sigma\theta$ with θ given in (1.7); $x_2 \in \mathbb{R}$ and $\varepsilon > 0$ are constants to be determined in the following lemma.

Lemma 3.1 *By appropriate choices of* $\lambda_1 > 0, \lambda_2 > 0, \sigma > 0, \varepsilon > 0, x_2 < x_1 < x_0$, *we have*

$$\varphi(S_+, I, R_+) \ge d_1 S_+'' - cS_+', \ x \ne x_0, \tag{3.7}$$

$$\varphi(S_-, I, R_-) \le d_1 S_-'' - cS_-', \ x \ne x_0, \tag{3.8}$$

$$\varphi(S_+, I_+, R_-) - (\gamma + \delta)I_+ \le -d_2 I_+'' + cI_+', \ x \ne x_1, \tag{3.9}$$

$$\varphi(S_-, I_-, R_+) - (\gamma + \delta)I_- \ge -d_2 I_-'' + cI_-', \ x \ne x_1, \tag{3.10}$$

$$\gamma I_+ \le -d_3 R_+'' + cR_+', \ x \ne x_2, \tag{3.11}$$

$$\gamma I_- \ge -d_3 R_-'' + cR_-', \ x \ne x_2, \tag{3.12}$$

where $I(x)$ *is any continuous function such that* $I_- \le I \le I_+$.

Proof The last inequality (3.12) follows immediately from (3.4) to (3.6). We will prove the remaining five inequalities in the following order.

First, we prove (3.9). If $x < x_0$, then $I_+(x) = e^{\lambda_0 x}$. It follows from (1.6) to (1.12) that $\varphi(S, I, R) \le \beta I$. This together with (2.2) implies that

$$\varphi(S_+, I_+, R_-) - (\gamma+\delta)I_+ \le (\beta-\gamma-\delta)I_+ = -d_2 I_+'' + cI_+'.$$

If $x > x_0$, then $I_+(x) = e^{\lambda_0 x_0} = I_0$, where we have chosen $x_0 = (\ln I_0)/\lambda_0$. It can be obtained from (1.5) to (2.8) that

$$\varphi(S_+, I_+, R_-) - (\gamma+\delta)I_+ \le \varphi(S_{-\infty}, I_0, 0) - (\gamma+\delta)I_0 = 0 = -d_2 I_+'' + cI_+'.$$

This proves (3.9).

Then, we prove (3.8). If $x > x_1$, then $S_- = 0$ and the inequality is satisfied. If $x < x_1 < x_0$, we have $S_- = S_{-\infty}(1 - e^{\sigma(x-x_1)})$ and

$$d_1 S_-'' - cS_-' = S_{-\infty}(c\sigma - d_1\sigma^2)e^{\sigma(x-x_1)}.$$

Moreover, it follows from (1.6) to (1.12) that

$$\varphi(S_-, I, R_-) \le \beta I \le \beta I_+ = \beta e^{\lambda_0 x}.$$

If $\sigma < c/d_1$, $\sigma < \lambda_0$ and $\beta e^{\lambda_0 x_1} \le S_{-\infty}(c\sigma - d_1\sigma^2)$, we then have

$$\beta e^{\lambda_0 x} \le S_{-\infty}(c\sigma - d_1\sigma^2)e^{\sigma(x-x_1)}$$

for all $x < x_1$. This implies that

$$\varphi(S_-, I, R_-) \le d_1 S_-'' - cS_-'.$$

Thus, (3.8) is proved.

Next, we prove (3.11). If $x < x_0$, the inequality is the same as

$$\gamma e^{(\lambda_0-\sigma)x} \le M(c\sigma - d_3\sigma^2).$$

We may choose $\sigma < \lambda_0$, $\sigma < c/d_3$ and $M = \gamma e^{(\lambda_0-\sigma)x_0}/(c\sigma - d_3\sigma^2)$. The above inequality is satisfied. If $x > x_0$, by the same choice of M, we have

$$\gamma I_+ = \gamma e^{\lambda_0 x_0} = M(c\sigma - d_3\sigma^2)e^{\sigma x_0} \le M(c\sigma - d_3\sigma^2)e^{\sigma x} = -d_3 R_+'' + cR_+'.$$

This gives (3.11).

Now, we prove (3.10). We choose $\lambda_2 > 0$ such that $d_2\lambda_2^2 + c\lambda_2 - (\gamma+\delta) = 0$ (it is obvious that such λ_2 exists and is unique). If $x > x_2$, then (3.4) gives

$$\varphi(S_-, I_-, R_+) - (\gamma+\delta)I_- \ge -(\gamma+\delta)I_- = -d_2 I_-'' + cI_-'.$$

If $x < x_2 < x_1$, it follows from (2.2) to (3.4) that

$$d_2 I_-'' - cI_-' + (\beta-\gamma-\delta)I_- = \frac{\lambda_0+\lambda_2}{\lambda_0+\lambda_2+\sigma} e^{\lambda_0 x + \sigma(x-x_2)} f_0(\lambda_0+\sigma).$$

As $x \to -\infty$, we have $S_- \to S_{-\infty}$, $I_- \to 0$, $R_+ \to 0$, and $\varphi(S_-, I_-, R_+)/I_- \to \beta$. Since $\varphi \in C^{1,2,1}(\mathbb{R}_+^3, \mathbb{R}_+)$, there exist $X < x_1$ and $A > 0$ such that

$$\begin{aligned}|\beta I_- - \varphi(S_-, I_-, R_+)| &\le A(S_{-\infty} - S_- + I_- + R_+)I_- \\ &\le A[S_{-\infty}e^{\sigma(x-x_1)} + e^{\lambda_0 x} + Me^{\sigma x}]e^{\lambda_0 x}\end{aligned}$$

for all $x \le X$. We shall choose $x_2 < X$ such that

$$A\left[S_{-\infty}e^{\sigma(x-x_1)} + e^{\lambda_0 x} + Me^{\sigma x}\right]e^{\lambda_0 x} \le \frac{\lambda_0+\lambda_2}{\lambda_0+\lambda_2+\sigma} e^{\lambda_0 x + \sigma(x-x_2)} f_0(\lambda_0+\sigma)$$

for all $x < x_2$. The above inequality is equivalent with

$$S_{-\infty}e^{\sigma(x_2-x_1)} + e^{(\lambda_0-\sigma)x+\sigma x_2} + Me^{\sigma x_2} \le \frac{\lambda_0+\lambda_2}{A(\lambda_0+\lambda_2+\sigma)} f_0(\lambda_0+\sigma).$$

Note that when $\sigma < \lambda_0$ and $x < x_2$, the left-hand side is bounded above by

$$S_{-\infty}e^{\sigma(x_2-x_1)} + e^{\lambda_0 x_2} + Me^{\sigma x_2}$$

which can be arbitrarily small as long as x_2 is negative large. Hence, there exists x_2 such that $x_2 < X < x_1$ and

$$\beta I_- - \varphi(S_-, I_-, R_+) \le d_2 I_-'' - cI_-' + (\beta - \gamma - \delta)I_-$$

for all $x < x_2$. This proves (3.10).

Finally, we prove (3.7). If $x < x_0$, then $S_+ = S_{-\infty}$ and

$$\varphi(S_+, I, R_+) \ge 0 = d_1 S_+'' - cS_+'.$$

If $x > x_0$, then $S_+ = S_{-\infty} - \varepsilon + \varepsilon e^{-\lambda_1(x-x_0)}$ and

$$d_1 S_+'' - cS_+' = \varepsilon(d_1\lambda_1^2 + c\lambda_1)e^{-\lambda_1(x-x_0)}.$$

We set $\varepsilon < S_{-\infty}/2$ such that $S_+ > S_{-\infty}/2$. It follows from (1.5) to (1.7) that

$$\varphi(S_+, I, R_+) \ge \varphi(S_{-\infty}/2, I, R_+) \ge \frac{I}{KR_+^\theta} \ge \frac{I_-}{KR_+^\theta} = \frac{\sigma e^{\lambda_0 x_2 - \lambda_2(x-x_2) - \theta\sigma x}}{M^\theta K(\lambda_0+\lambda_2+\sigma)}$$

for some constant $K > 0$. We choose $\lambda_1 = \lambda_2 + \sigma\theta$ and $\varepsilon > 0$ small such that $\varepsilon < S_{-\infty}/2$ and

$$\varepsilon < \frac{\sigma e^{\lambda_0 x_2 + \lambda_2 x_2 - \lambda_1 x_0}}{M^\theta K(\lambda_0+\lambda_2+\sigma)(d_1\lambda_1^2 + c\lambda_1)}.$$

It is readily seen that

$$\varphi(S_+, I, R_+) \ge d_1 S_+'' - cS_+'$$

for all $x > x_0$. This proves (3.7). □

Remark 3.2 Note that (3.7) and (3.8) are stronger results than the inequalities

$$\varphi(S_+, I_+, R_+) \ge d_1 S_+'' - cS_+', \ \varphi(S_-, I_-, R_-) \le d_1 S_-'' - cS_-'.$$

Since $\varphi(S, I, R)$ is not monotone in I (for instance, $\partial_I \varphi$ may be positive for small I but negative for large I), we will later need (3.7) and (3.8), instead of the above weaker inequalities, to prove the invariance of the convex set; see Lemma 3.4.

Corollary 3.3 *We have*

$$\Delta_1^{-1}(\Delta_1 S_+) \le S_+, \ \Delta_2^{-1}(\Delta_2 I_+) \le I_+, \ \Delta_3^{-1}(\Delta_3 R_+) = R_+, \tag{3.13}$$

$$\Delta_1^{-1}(\Delta_1 S_-) \ge S_-, \ \Delta_2^{-1}(\Delta_2 I_-) = I_-, \ \Delta_3^{-1}(\Delta_3 R_-) = R_-. \tag{3.14}$$

Proof Note from (3.4) that

$$\lim_{x\to x_2^-} I_-(x) = \frac{\sigma e^{\lambda_0 x_2}}{\lambda_0+\lambda_2+\sigma} = \lim_{x\to x_2^+} I_-(x),$$
$$\lim_{x\to x_2^-} I_-'(x) = \frac{-\lambda_2\sigma e^{\lambda_0 x_2}}{\lambda_0+\lambda_2+\sigma} = \lim_{x\to x_2^+} I_-'(x).$$

Thus, it follows from Lemma 2.1 that $\Delta_2^{-1}(\Delta_2 I_-) = I_-$. The other equalities and inequalities can be obtained in a similar manner. □

With the aid of the super-solutions and sub-solutions in (3.1)–(3.6), we define a convex set Γ as

$$\Gamma := \{(S, I, R) \in B_\mu(\mathbb{R}, \mathbb{R}^3) : \; S_- \le S \le S_+, \; I_- \le I \le I_+, \; R_- \le R \le R_+\}. \quad (3.15)$$

Since $\mu > \sigma > 0$, it is easily seen that Γ is uniformly bounded with respect to the norm $|\cdot|_\mu$ defined in (2.9). Now, we are ready to show that the convex set Γ defined in (3.15) is invariant under the integral map $F = (F_1, F_2, F_3)^T$ defined in (2.10).

Lemma 3.4 *The convex set Γ is invariant under the integral map F; that is, for any $(S, I, R)^T \in B_\mu(\mathbb{R}, \mathbb{R}^3)$ such that $S_- \le S \le S_+$, $I_- \le I \le I_+$ and $R_- \le R \le R_+$, we have $S_- \le F_1(S, I, R) \le S_+$, $I_- \le F_2(S, I, R) \le I_+$, and $R_- \le F_3(S, I, R) \le R_+$.*

Proof Since $\alpha_1 \ge \partial_S\varphi(S, I, R)$ for all $0 \le S \le S_{-\infty}, 0 \le I \le I_0, R \ge 0$, it follows from (1.5) that the function $\alpha_1 S - \varphi(S, I, R)$ is increasing in S and R. Consequently, we obtain from (3.7) to (3.8) that

$$\Delta_1 S_- \le \alpha_1 S_- - \varphi(S_-, I, R_-) \le \alpha_1 S - \varphi(S, I, R) \le \alpha_1 S_+ - \varphi(S_+, I, R_+) \le \Delta_1 S_+.$$

This together with Corollary 3.3 implies that

$$S_- \le \Delta_1^{-1}(\Delta_1 S_-) \le F_1(S, I, R) \le \Delta_1^{-1}(\Delta_1 S_+) \le S_+.$$

Since $\alpha_2 \ge \gamma + \delta - \partial_I\varphi(S, I, R)$ for all $0 \le S \le S_{-\infty}, 0 \le I \le I_0, R \ge 0$, it follows from (1.5) that the function $(\alpha_2-\gamma-\delta)I+\varphi(S, I, R)$ is increasing in S and I, and decreasing in R. Consequently, we obtain from (3.9) to (3.10) that

$$\begin{aligned} \Delta_2 I_- &\le (\alpha_2 - \gamma - \delta)I_- + \varphi(S_-, I_-, R_+) \\ &\le (\alpha_2 - \gamma - \delta)I + \varphi(S, I, R) \\ &\le (\alpha_2 - \gamma - \delta)I_+ + \varphi(S_+, I_+, R_-) \le \Delta_2 I_+. \end{aligned}$$

This together with Corollary 3.3 implies that

$$I_- = \Delta_2^{-1}(\Delta_2 I_-) \le F_2(S, I, R) \le \Delta_2^{-1}(\Delta_2 I_+) \le I_+.$$

Finally, we observe from (3.11) to (3.12) that

$$\Delta_3 R_- \le \alpha_3 R_- + \gamma I_- \le \alpha_3 R + \gamma I \le \alpha_3 R_+ + I_+ \le \Delta_3 R_+.$$

This together with Corollary 3.3 implies that

$$R_- = \Delta_3^{-1}(\Delta_3 R_-) \le F_3(S, I, R) \le \Delta_3^{-1}(\Delta_3 R_+) \le R_+.$$

Hence, the proof is complete. □

4 Continuity and Compactness of the Integral Map

To apply Schauder fixed point theorem, we need the following result.

Lemma 4.1 *The integral map F is continuous and compact on Γ with respect to the norm $|\cdot|_\mu$.*

Proof By (1.8), the gradient of $\varphi(u_1, u_2, u_3)$ is uniformly bounded for $0 \le u_1 \le S_{-\infty}$, $0 \le u_2 \le I_0$, and $u_3 \ge 0$. There exists $L > 0$ such that for any $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in Γ,

$$|F_1(u)(x) - F_1(v)(x)|e^{-\mu|x|} \le Lg(x)|u - v|_\mu,$$

where

$$g(x) := e^{-\mu|x|}[\int_{-\infty}^{x} e^{\lambda_1^-(x-y)+\mu|y|}dy + \int_{x}^{\infty} e^{\lambda_1^+(x-y)+\mu|y|}dy].$$

Recall that $\lambda_1^- < -\mu < \mu < \lambda_1^+$. A simple application of L'Hôpital's rule gives

$$\lim_{x\to-\infty} g(x) = \frac{1}{\mu+\lambda_1^+} - \frac{1}{\mu+\lambda_1^-}, \quad \lim_{x\to\infty} g(x) = \frac{1}{\lambda_1^+-\mu} + \frac{1}{\mu-\lambda_1^-}.$$

Consequently, $g(x)$ is uniformly bounded on $\mathbb{R}$, which implies that F_1 is continuous on Γ. Similarly, one can show that F_2 and F_3 are also continuous on Γ.

Now, we want to prove that F is compact on Γ; namely, for any bounded sequence $\{u^{(n)}\} \in \Gamma$, the sequence $\{Fu^{(n)}\}$ has a convergent subsequence in Γ with respect to the norm $|\cdot|_\mu$. We first show that for any $k \in \mathbb{N}$, there exists a convergent subsequence of $\{Fu^{(n)}\}$, denoted by $\{Fu^{(n,k)}\}$, on the Banach space $C([-k, k], \mathbb{R}^3)$ equipped with the maximum norm. To see this, we note that $\{Fu^{(n)}\}$ is uniformly bounded because $\{Fu^{(n)}\}$ are bounded by $S_\pm$, $I_\pm$, $R_\pm$ and these functions are uniformly bounded in $C([-k, k], \mathbb{R}^3)$. Furthermore, for any $u = (u_1, u_2, u_3)^T \in \Gamma$, one can estimate the derivative of each component of Fu as below:

$$\begin{aligned}|[F_1(u)]'(x)| &\le \frac{-\lambda_1^-\alpha_1 S_{-\infty}}{\rho_1}\int_{-\infty}^{x} e^{\lambda_1^-(x-y)}dy + \frac{\lambda_1^+\alpha_1 S_{-\infty}}{\rho_1}\int_{x}^{\infty} e^{\lambda_1^+(x-y)}dy\\ &= \frac{2\alpha_1 S_{-\infty}}{\rho_1},\end{aligned}$$

and

$$\begin{aligned}|[F_2(u)]'(x)| &\le \frac{-\lambda_2^-(\alpha_2+\beta-\gamma-\delta)e^{\lambda_0 x_0}}{\rho_2}\int_{-\infty}^{x} e^{\lambda_2^-(x-y)}dy\\ &\quad + \frac{\lambda_2^+(\alpha_2+\beta-\gamma-\delta)e^{\lambda_0 x_0}}{\rho_2}\int_{x}^{\infty} e^{\lambda_2^+(x-y)}dy\\ &= \frac{2(\alpha_2+\beta-\gamma-\delta)e^{\lambda_0 x_0}}{\rho_2},\end{aligned}$$

and

$$\begin{aligned}|[F_3(u)]'(x)| &\leq \frac{-\lambda_3^-}{\rho_3}\int_{-\infty}^{x} e^{\lambda_3^-(x-y)}(\gamma e^{\lambda_0 x_0}+\alpha_3 M e^{\sigma y})dy\\ &\quad+\frac{\lambda_3^+}{\rho_3}\int_{x}^{\infty} e^{\lambda_3^+(x-y)}(\gamma e^{\lambda_0 x_0}+\alpha_3 M e^{\sigma y})dy\\ &=\frac{2\gamma e^{\lambda_0 x_0}}{\rho_3}+\frac{\alpha_3 M e^{\sigma x}}{\rho_3}\left(\frac{-\lambda_3^-}{\sigma-\lambda_3^-}+\frac{\lambda_3^+}{\lambda_3^+-\sigma}\right).\end{aligned}$$

Thus, the sequence $\{Fu^{(n)}\}$ is equi-continuous in $C([-k,k],\mathbb{R}^3)$. By Arzela–Ascoli theorem, there exists a convergent subsequence of $\{Fu^{(n)}\}$, denoted by $\{Fu^{(n,k)}\}$, on $C([-k,k],\mathbb{R}^3)$, for any $k\in\mathbb{N}$. We now extract a sequence of the diagonal terms: $v^{(n)}=Fu^{(n,n)}$. It follows that $v^{(n)}$ converges to a function $v\in C([-k,k],\mathbb{R}^3)$ for any $k\in\mathbb{N}$. Actually, v is continuous on $\mathbb{R}$. Since $\{v^{(n)}\}$ are bounded by $S_\pm$, $I_\pm$, $R_\pm$, so is the limit v. Thus, $v\in\Gamma$. Moreover, for any $\mu>\sigma$ and $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that

$$e^{-\mu|x|}|v^{(n)}(x)-v(x)|<\varepsilon,\quad |x|\geq K,\ n\in\mathbb{N}.$$

On the other hand, for any $|x|\leq K$, there exists $N\in\mathbb{N}$ such that

$$e^{-\mu|x|}|v^{(n)}(x)-v(x)|<\varepsilon,\quad n\geq N.$$

From the above two inequalities, we conclude that $v^{(n)}\to v$ in Γ with respect to the norm $|\cdot|_\mu$, which implies that F is compact. This completes the proof. □

5 Existence Result

The following proposition gives the first part of our main theorem.

Proposition 5.1 *The map F has a fixed point $u=(S,I,R)^T\in\Gamma$ which corresponds to a non-trivial traveling wave solution to the Eqs.* (1.9)–(1.11)*. The asymptotic behaviors of $u(x)$ are given as follows:*

$$u(x)\sim\begin{pmatrix}S_{-\infty}\\ e^{\lambda_0 x}\\ 0\end{pmatrix},\ u'(x)\to\begin{pmatrix}0\\0\\0\end{pmatrix},\ u''(x)\to\begin{pmatrix}0\\0\\0\end{pmatrix},\quad x\to-\infty, \tag{5.1}$$

$$u(x)\to\begin{pmatrix}S_{\infty}\\ 0\\ \frac{\gamma(S_{-\infty}-S_\infty)}{\gamma+\delta}\end{pmatrix},\ u'(x)\to\begin{pmatrix}0\\0\\0\end{pmatrix},\ u''(x)\to\begin{pmatrix}0\\0\\0\end{pmatrix},\quad x\to\infty. \tag{5.2}$$

Moreover, we have $S'(x)<0$, $I(x)\leq S_{-\infty}-S_\infty$, $R'(x)>0$, and

$$\int_{-\infty}^{\infty}(\gamma+\delta)I(x)dx=\int_{-\infty}^{\infty}\varphi(S(x),I(x),R(x))dx=c(S_{-\infty}-S_\infty). \tag{5.3}$$

Proof It follows from Lemma 3.4, Lemma 4.1 and Schauder fixed point theorem that the integral map has a fixed point $u=(S,I,R)^T\in\Gamma$. By Lemma 2.2, (S,I,R) satisfy the traveling wave Eqs. (1.9)–(1.11). By squeeze theorem, we observe

$$u(x)\sim\begin{pmatrix}S_{-\infty}\\ e^{\lambda_0 x}\\ 0\end{pmatrix},\quad x\to-\infty.$$

From the integral representation of the derivatives of the integral map (2.6) and L'Hôpital's rule, we obtain

$$u'(x) \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad x \to -\infty.$$

Substituting the above two formulas into the traveling wave equations gives

$$u''(x) \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad x \to -\infty.$$

Now, we integrate (1.9) from $-\infty$ to x:

$$d_1 S'(x) = c[S(x) - S_{-\infty}] + \int_{-\infty}^{x} \varphi(S(y), I(y), R(y))dy.$$

This implies that $\varphi(S(y), I(y), R(y))$ is integrable on $\mathbb{R}$, and $S'(x)$ is bounded on $\mathbb{R}$; otherwise, $S'(x) \to \infty$ as $x \to \infty$, which contradicts to the boundedness of $S(x)$. Consequently, $S''(x)$ is also bounded on $\mathbb{R}$. Note from (1.10) that

$$I(x) = \frac{1}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\varphi(S(y), I(y), R(y))dy + \frac{1}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\varphi(S(y), I(y), R(y))dy,$$

where

$$\lambda^{\pm} := \frac{c \pm \sqrt{c^2 + 4d_2(\gamma + \delta)}}{2d_2}, \quad \rho := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_2(\gamma + \delta)}.$$

Fubini's theorem gives

$$\int_{-\infty}^{\infty} I(x)dx = \frac{1}{\gamma + \delta}\int_{-\infty}^{\infty} \varphi(S(x), I(x), R(x))dx,$$

where the integrability of $\varphi(S(x), I(x), R(x))$ yields integrability of $I(x)$. On the other hand, we have

$$\begin{aligned} I'(x) = &\frac{\lambda^-}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\varphi(S(y), I(y), R(y))dy \\ &+ \frac{\lambda^+}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\varphi(S(y), I(y), R(y))dy, \end{aligned}$$

and thus

$$|I'(x)| \le \frac{\beta(\lambda^+ - \lambda^-)}{\rho}\int_{-\infty}^{\infty} I(x)dx.$$

This together with integrability and non-negativity of $I(x)$ implies that $I(x) \to 0$ as $x \to \infty$. Moreover, by L'Hôpital's rule, we have $I'(x) \to 0$ as $x \to \infty$. It then follows from (1.10) that $I''(x)$ also tends to zero as $x \to \infty$. Next, we multiply both sides of (1.9) by $e^{-c\xi/d_1}$ and integrate from x to ∞:

$$e^{-cx/d_1} S'(x) = -\int_{x}^{\infty} e^{-cy/d_1}\frac{\varphi(S(y), I(y), R(y))}{d_1}dy.$$

Hence, $S'(x) < 0$ for all $x \in \mathbb{R}$. This together with boundedness of $S''(x)$ gives $S'(x) \to 0$ and $S(x) \to S_\infty$ as $x \to \infty$, for some $S_\infty < S_{-\infty}$. It then follows from (1.9) that $S''(x)$ also tends to zero as $x \to \infty$. An integration of (1.9) yields

$$\int_{-\infty}^{\infty} \varphi(S(x), I(x), R(x))dx = c(S_{-\infty} - S_\infty).$$

From (1.11), we obtain

$$R(x) = \frac{\gamma}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma}{c}\int_{x}^{\infty} e^{(c/d_3)(x-y)} I(y)dy,$$

and

$$R'(x) = \frac{\gamma}{d_3}\int_{x}^{\infty} e^{(c/d_3)(x-y)} I(y)dy > 0.$$

By L'Hôpital's rule, we have

$$\lim_{x\to\infty} R(x) = \frac{\gamma}{c}\int_{-\infty}^{\infty} I(x)dx = \frac{\gamma}{\gamma+\delta}(S_{-\infty} - S_\infty),$$

and

$$\lim_{x\to\infty} R'(x) = 0.$$

Furthermore, it is easily seen from (1.11) that $R''(x) \to 0$ as $x \to \infty$. Finally, we introduce the function

$$J(x) := I(x) + \frac{\gamma+\delta}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma+\delta}{c}\int_{x}^{\infty} e^{(c/d_2)(x-y)} I(y)dy,$$

which satisfies the differential equation $-d_2 J'' + cJ' = \varphi(S, I, R)$ with boundary conditions $J(-\infty) = 0$ and $J(\infty) = S_{-\infty} - S_\infty$. Note that

$$J'(x) = \frac{1}{d_2}\int_{x}^{\infty} e^{(c/d_2)(x-y)} \varphi(S(y), I(y), R(y))dy > 0.$$

It is readily seen that $I(x) < J(x) < S_{-\infty} - S_\infty$ for any $x \in \mathbb{R}$. This completes the proof. □

6 Non-existence Result

If (S, I, R) is a traveling wave solution to the Eqs. (1.9) and (1.10) with boundary conditions $S(-\infty) = S_{-\infty},\ I(\pm\infty) = 0,\ R(-\infty) = 0$, then

$$I(x) = \frac{1}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\varphi(S(y), I(y), R(y))dy + \frac{1}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\varphi(S(y), I(y), R(y))dy,$$

where

$$\lambda^\pm := \frac{c \pm \sqrt{c^2 + 4d_2(\gamma+\delta)}}{2d_2}, \quad \rho := d_2(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_2(\gamma+\delta)}.$$

Moreover, we have

$$I'(x) = \frac{\lambda^-}{\rho}\int_{-\infty}^{x} e^{\lambda^-(x-y)}\varphi(S(y), I(y), R(y))dy + \frac{\lambda^+}{\rho}\int_{x}^{\infty} e^{\lambda^+(x-y)}\varphi(S(y), I(y), R(y))dy.$$

By L'Hôpital's rule, we have $I'(\pm\infty) = 0$. It then follows from (1.10) that $I''(\pm\infty) = 0$. Thus, we obtain

$$I(\pm\infty) = 0,\ \ I'(\pm\infty) = 0,\ \ I''(\pm\infty) = 0. \tag{6.1}$$

On the other hand, it follows from (1.9) that

$$S'(x) = -\frac{1}{d_1}\int_{x}^{\infty} e^{c/d_1(x-y)}\varphi(S(y), I(y), R(y))dy \le 0.$$

This implies that the limit of $S(x)$ as $x \to \infty$ exists. We denote the limit by S_∞. Again, by L'Hôpital's rule, we have $S'(\pm\infty) = 0$. On account of (1.9), we observe $S''(\pm\infty) = 0$. Thus,

$$S(-\infty) = S_{-\infty},\ \ S(\infty) = S_\infty,\ \ S'(\pm\infty) = 0,\ \ S''(\pm\infty) = 0. \tag{6.2}$$

The non-existence results are given in the following two propositions.

Proposition 6.1 *If $R_0 > 1$ and $c < c^*$, then there does not exist a non-trivial, non-negative and bounded traveling wave solution of* (1.9)–(1.11) *such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$.*

Proof Assume to the contrary that (S, I, R) is a traveling wave solution for (1.9)–(1.11). Note that $\lim_{x\to-\infty}\varphi(S(x), I(x), R(x))/I(x) = \beta > \gamma + \delta$. Let $\tau := (\beta - \gamma - \delta)/2 > 0$. We may find $\bar{x} \in \mathbb{R}$ such that

$$\varphi(S(x), I(x), R(x))/I(x) - \gamma - \delta > \tau,\ \ x < \bar{x}.$$

Substituting this into (1.10) gives

$$cI'(x) - d_2 I''(x) > \tau I(x),\ \ x < \bar{x}.$$

An integration of the above inequality yields

$$\int_{-\infty}^{x} I(y)dy < \frac{cI(x) - d_2 I'(x)}{\tau},\ \ x < \bar{x},$$

where we have made use of (6.1) and Lebesgue's dominated convergence theorem. For convenience, we denote the left-hand side of the above inequality as

$$K(x) := \int_{-\infty}^{x} I(y)dy.$$

Then, the inequality reads $K(x) < [cI(x) - d_2 I'(x)]/\tau$. An integration of this inequality gives

$$\int_{-\infty}^{x} K(y)dy \le \frac{cK(x)}{\tau},\ \ x < \bar{x}.$$

In view of nonegativeness of $I(x)$, the function $K(x)$ is non-decreasing. Consequently, for any $\eta > 0$,

$$\eta K(x-\eta) \le \int_{x-\eta}^{x} K(y)dy \le \frac{cK(x)}{\tau}, \quad x < \bar{x}.$$

We choose $\eta > 2c/\tau$ such that $K(x-\eta) < K(x)/2$ for all $x < \bar{x}$. Define $L(x) := e^{-\mu_0 x}K(x)$ with $\mu_0 := (\ln 2)/\eta > 0$. It is readily seen that $L(x-\eta) < L(x)$ for all $x < \bar{x}$. Thus, $L(x) = e^{-\mu_0 x}K(x)$ is bounded on $(-\infty, \bar{x})$. Recall that $cI'(x) - d_2 I''(x) > \tau I(x) \ge 0$ for $x < \bar{x}$. We have $cI(x) > d_2 I'(x)$ and $cK(x) > d_2 I(x)$ for $x < \bar{x}$. Consequently, the functions $e^{-\mu_0 x}I(x)$, $e^{-\mu_0 x}I'(x)$ and $e^{-\mu_0 x}I''(x)$ are bounded on $(-\infty, \bar{x})$.

Next, we integrate (1.9) and (1.10) from $-\infty$ to x, and use (6.1) and (6.2) to obtain

$$c[S(x) - S_{-\infty}] = d_1 S'(x) - \int_{-\infty}^{x} \varphi(S(y), I(y), R(y))dy,$$

and

$$cI(x) = d_2 I'(x) + \int_{-\infty}^{x} \varphi(S(y), I(y), R(y))dy - (\gamma + \delta)\int_{-\infty}^{x} I(y)dy.$$

For convenience, we denote

$$\Phi(x) := \int_{-\infty}^{x} \varphi(S(y), I(y), R(y))dy.$$

It follows that $\Phi(x) = O(e^{\mu_0 x})$ as $x \to -\infty$. Furthermore, we multiply the equation $c[S(x) - S_{-\infty}] - d_1 S'(x) = -\Phi(x)$ by e^{-cx/d_1} and integrate to obtain

$$[S(x) - S_{-\infty}]e^{-cx/d_1} = -\frac{1}{d_1}\int_{x}^{\infty} e^{-cy/d_1}\Phi(y)dy.$$

Choose any positive $\mu_1 < \min\{\mu_0, c/d_1, c/d_3\}$. If $e^{-cy/d_1}\Phi(y)$ is integrable on $\mathbb{R}$, then $S(x) - S_{-\infty} = O(e^{cx/d_1})$ and $e^{-\mu_1 x}[S(x) - S_{-\infty}] \to 0$ as $x \to -\infty$. Otherwise, we have from L'Hôpital's rule and $\Phi(x) = O(e^{\mu_0 x})$ that

$$\lim_{x\to-\infty} e^{-\mu_1 x}[S(x) - S_{-\infty}] = \lim_{x\to-\infty} \frac{-\int_x^\infty e^{-cy/d_1}\Phi(y)dy}{d_1 e^{(\mu_1 - c/d_1)x}} = \lim_{x\to-\infty} \frac{\Phi(x)e^{-\mu_1 x}}{d_1\mu_1 - c} = 0.$$

Solving the linear Eq. (1.11) gives

$$R(x) = \frac{\gamma}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma}{c}\int_{x}^{\infty} e^{c(x-y)/d_3} I(y)dy.$$

A similar argument shows that $e^{-\mu_1 x}R(x) \to 0$ as $x \to -\infty$. On account of (1.4) and (1.12), we have

$$\beta - \frac{\varphi(S(x), I(x), R(x))}{I(x)} = O(|S(x) - S_{-\infty}| + |I(x)| + |R(x)|),$$

as $x \to -\infty$. Consequently,

$$\lim_{x\to-\infty} e^{-\mu_1 x}[\beta - \frac{\varphi(S(x), I(x), R(x))}{I(x)}] = 0.$$

Especially, $e^{-\mu_1 x}[\beta - \varphi(S(x), I(x), R(x))/I(x)]$ is uniformly bounded on the whole real line. For any $\mu \in (0, \mu_1)$, we make a two-side Laplace transform on (1.10):

$$f_0(\mu) \int_{-\infty}^{\infty} e^{-\mu x} I(x)dx = - \int_{-\infty}^{\infty} e^{-\mu x} I(x)[\beta - \frac{\varphi(S(x), I(x), R(x))}{I(x)}]dx,$$

where f_0 is the characteristic function defined in (2.1). By analyticity of Laplace transform, the two integrals on both sides of the above equation can be analytically continued to all $\mu > 0$; see [1,22,23]. We rewrite the equation as

$$\int_{-\infty}^{\infty} e^{-\mu x} I(x)[f_0(\mu) + \beta - \frac{\varphi(S(x), I(x), R(x))}{I(x)}]dx = 0.$$

Note that $f_0(\mu) \to -\infty$ as $\mu \to \infty$, and $\beta - \varphi(S(x), I(x), R(x))/I(x)$ is bounded for all $x \in \mathbb{R}$. Thus, the integrand on the left-hand side of the above equation is always negative for large μ, which leads to a contradiction. This proves the non-existence of non-trivial and non-negative traveling wave solution. □

Proposition 6.2 *If $R_0 \leq 1$, then there does not exist a non-trivial and non-negative traveling wave solution of* (1.9)–(1.11) *such that $S(-\infty) = S_{-\infty}$, $S(\infty) < S_{-\infty}$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$.*

Proof Assume to the contrary that $(S(x), I(x), R(x))$ is a non-trivial and non-negative traveling wave solution. Since $R_0 \leq 1$, we have $\varphi(S(x), I(x), R(x)) \leq (\gamma + \delta)I(x)$ for all $x \in \mathbb{R}$. It follows from (1.10) that

$$\frac{d}{dx}[e^{-(c/d_2)x} \frac{d}{dx} I(x)] = -\frac{1}{d_2} e^{-(c/d_2)x} [\varphi(S(x), I(x), R(x)) - (\gamma + \delta)I(x)] \geq 0.$$

Hence, the function $e^{-(c/d_2)x} I'(x)$ is non-decreasing. This together with (6.1) implies that $I'(x) \leq 0$ for all $x \in \mathbb{R}$. Since $I(\pm\infty) = 0$, we have $I(x) = 0$ for all $x \in \mathbb{R}$, a contradiction. This completes the proof. □

7 Conclusion and Discussion

In this paper, we considered diffusive epidemic models with general incidence rates. By applying Schauder fixed point theorem, we proved that there exist non-trivial traveling wave solutions if the basic reproduction number is greater than one and the traveling speed exceeds a critical value, which is explicitly determined in terms of model parameters. Moreover, we used a contradiction argument and analytical properties of two-sided Laplace transform to show that the non-trivial traveling wave solution does not exist if either the basic reproduction number is not larger than one, or the traveling speed is less than the critical value. Our theorem generalizes and improves those obtained in [19,20], where the incidence function is restricted to the standard incidence rate. In particular, we relaxed the technical condition on diffusion coefficients in [19]. Our results are still valid if the incidence function is non-monotone, while this non-monotonicity will violate the conditions given in [14,27].

A perturbation technique was used in [26] to a different diffusive system. The main idea was to prove the existence of traveling wave solutions for perturbed systems and then let the perturbation parameter decrease to zero. However, it is not easy to apply this perturbation technique to our model system. First of all, if we use the super- and sub-solutions constructed in [26], a technical assumption similar to that in [19] shall be imposed in the verification of (3.11). Moreover, it is difficult to construct sub-solutions that are independent of the perturbation parameter, which makes it a challenging task to prove that the traveling wave solutions of perturbed systems converge to a positive solution during the limiting process. Eventually, one may need to apply and modify the super- and sub-solutions constructed in this paper so that they are independent of the perturbation parameter, and then prove the existence of traveling wave solutions for perturbed and unperturbed systems simultaneously. This means that the perturbation technique is applicable but not necessary for our model system.

A possible future project would be the study of traveling wave solutions for diffusive disease models with more refined compartmental structures to include susceptible, exposed, infectious, and recovered groups; i.e., the SEIR model [24], or the SEIRS model which takes the immunity loss of recovered individuals into consideration. Another possible extension of our results in this paper is to incorporate transmission delay in the disease models [15,22]. Finally, an open and challenging problem in diffusive disease model is to study the stability and uniqueness of traveling wave solution (if it exists); see [20].

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