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Asymptotic expansion of orthogonal polynomials via difference equations

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Abstract

This paper aims to develop a simple and unified technique in finding asymptotic expansion of orthogonal polynomials from their difference equations. By preserving the symmetry in the difference equation, we are able to express the higher-order terms in the asymptotic expansion as an integral whose integrand can be explicitly obtained by a recurrence relation, while the integration constant is to be determined by a matching condition that relates to the initial conditions and coefficients in the difference equation.

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1. Introduction

The objective of this paper is to study asymptotic solution to the following difference equation

$$P_{n+1}(x) + P_{n-1}(x) = (A_n x - B_n)P_n(x), \quad n \geq 1, \tag{1.1}$$

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together with the initial conditions $P_0(x) = 1$ and $P_1(x) = A_0x - B_0$. If we define $P_{-1}(x) = 0$, then the above difference equation is still true for $n = 0$. We assume that $A_n > 0$, B_n are real, and as $n \to \infty$, they have the asymptotic expansions:

$$A_n \sim n^{-\theta} \sum_{k=0}^{\infty} \frac{\alpha_k}{n^k}, \quad B_n \sim \sum_{k=0}^{\infty} \frac{\beta_k}{n^k}, \tag{1.2}$$

where $\theta > 0$ and $\alpha_0 > 0$. It is obvious that the exact solutions to the difference equation are polynomials. Furthermore, these polynomials are orthogonal with respect to a positive-definite moment functional [8, Theorem 4.4], and consequently, their zeros are all real, simple, and located in the interior of the supporting interval of the corresponding moment functional [8, Theorem 5.2].

Even though it takes a simple form with only two coefficients, the difference Eq. (1.1) is general in the sense that many orthogonal polynomials within Askey scheme [16], such as Hermite, Laguerre, Krawtchouk, Meixner, Hahn, Racah and Wilson polynomials satisfy this equation with different choices of A_n and B_n. Moreover, the difference equation is the most explicit and possibly the only way to define orthogonal polynomials related to birth and death process [15], and orthogonal polynomials with indeterminate moment problems; see for example, Letessier–Valent polynomials [17], Concrad–Flajolet polynomials [21], Chen–Ismail polynomials [7], and so on.

Asymptotic analysis of orthogonal polynomials is a classical problem and plenty of methods have been developed in the literature. For example, the Laplace's method and steepest-descent method [26] can be applied to deal with the integral representation of the orthogonal polynomials; the WKB method [18] is very powerful and easy to implement if the orthogonal polynomials satisfy a second-order linear differential equation; in the past two decades, Riemann–Hilbert approach and Deift–Zhou nonlinear steepest-descent method [2,11,12] have been playing an important role in studying asymptotic behavior of orthogonal polynomials from their orthogonality relation and the corresponding weight function. However, not much has been done in asymptotic analysis of orthogonal polynomials via difference equations.

There are some early works on asymptotic analysis of difference equations [1,3–5]. However, their papers were so complicated that even an expert in asymptotics may not be able to understand them [27]. It was only by the end of 20th century that other researchers start to pick up this long-time challenging task. Geronimo and his collaborators [13,14,20] obtained asymptotic formulas for the orthogonal polynomials in the outer region, while Wong and Li [29] studied asymptotic solutions to the difference equations in the oscillatory region. At the beginning of this century, there was a big breakthrough by Wang and Wong [22–24], who developed a general theory of uniform asymptotic expansions for difference equations near the turning-point. This work was further extended and completed by Cao and Li [6]. Recently, there have been various applications of difference technique in the study of coherent state polynomials [9] and birth-death type orthogonal polynomials [10]. We refer to [28] for an overview of asymptotic theory on linear difference equations.

As a complement of the pioneer works in [22–24] and in [6], this paper is dedicated to providing an asymptotic expansion for the solution to (1.1) away from the turning points. We will use a different approach as those given in [13,14,20] and develop a general and relatively simple technique in finding general asymptotic solutions to the difference Eq. (1.1). The key idea is to take advantage of the symmetry in the difference equation, and propose a logarithmic-type asymptotic technique. We will also generalize the method of asymptotic matching introduced in [25] to derive asymptotic expansion of orthogonal polynomials in the outer region and oscillatory region, respectively.

We shall introduce a scale $x = n^\theta y$ so as to balance the order of the coefficient A_n. By [6,22–24], the linear difference Eq. (1.1) has two linearly independent solutions with asymptotic expansion given in terms of a special function (Airy or Bessel functions). On account of asymptotic expansion of the special function, we can rearrange the asymptotic expansion in the following form:

$$\Phi_n(n^\theta y) \sim \exp\left\{\sum_{k=-1}^{\infty} \frac{\phi_k(y)}{n^k}\right\}, \quad n \to \infty. \tag{1.3}$$

It is remarked that the asymptotic expansion lies inside the exponent, which helps us to preserve the symmetry of the difference equation in $n \pm 1$. As we shall see, $\phi_k(y)$ satisfies a simple first-order differential equation: $-k\phi_k(y) - \theta y \phi_k'(y) = \psi_{k+1}(y)$, where the inhomogeneous term $\psi_{k+1}(y)$ can be explicitly given and it is very close to a rational function; see details later in Section 2. Solving the above equation gives an integration constant, which will be determined by the initial conditions and the principle of asymptotic matching; see details later in Section 3.

The paper is organized as follows. In Section 2, we will derive an explicit asymptotic expansion up to any order for the two linearly independent solutions to the difference Eq. (1.1) without initial conditions. In Section 3, we will connect the asymptotic expansion with the initial condition by asymptotic matching, and then derive asymptotic expansions for the orthogonal polynomials in the outer and oscillatory regions, respectively. In Section 4, we will use the Hermite polynomials and continuous dual Hahn polynomials as illustrative examples to demonstrate the feasibility of finding higher-order approximations by our general formula. Numerical simulation will also be conducted to confirm the order of accuracy in the asymptotic expansion. Finally, we will give a brief discussion in Section 5.

2. Asymptotic expansion

Let $\Phi_n(n^\theta y)$ be a solution to (1.1) with asymptotic expansion given as in (1.3). We have

$$\Phi_{n\pm1}(n^\theta y) = \Phi_{n\pm1}((n\pm1)^\theta y_\pm) \sim \exp\left\{\sum_{k=-1}^{\infty} \frac{\phi_k(y_\pm)}{(n\pm1)^k}\right\}, \quad n \to \infty, \tag{2.1}$$

where $y_\pm = (1 \pm 1/n)^{-\theta} y$. Substituting (1.2), (1.3) and the above expansion into (1.1) gives

$$\exp\left\{\sum_{k=-1}^{\infty}\left[\frac{\phi_k(y_+)}{(n+1)^k} - \frac{\phi_k(y)}{n^k}\right]\right\} + \exp\left\{\sum_{k=-1}^{\infty}\left[\frac{\phi_k(y_-)}{(n-1)^k} - \frac{\phi_k(y)}{n^k}\right]\right\}$$
$$\sim \sum_{s=0}^{\infty} \frac{\alpha_s y - \beta_s}{n^s}. \tag{2.2}$$

To make comparison with the asymptotic expansion on the right-hand side, we have to find the coefficient of n^{-s} in the asymptotic expansion of the left-hand side. Actually, the coefficient can be explicitly given in terms of $\phi_k(y)$ with $k = -1, \ldots, s-1$, and their derivatives. To see this, we will need the following lemmas.

Lemma 2.1. *Let $F_{k,y}(s) = s^{-k}\phi_k(s^{-\theta} y)$. The mth derivative of $F_{k,y}(s)$ is given by*

$$F_{k,y}^{(m)}(s) = \sum_{j=0}^{m} c_{m,j}(k) s^{-m-k-j\theta}(-\theta y)^j \phi_k^{(j)}(s^{-\theta} y), \tag{2.3}$$

where the coefficients satisfy the recurrence relation:

$$c_{m+1,j}(k) = c_{m,j-1}(k) - (m+k+j\theta)c_{m,j}(k), \tag{2.4}$$

together with initial conditions $c_{0,0}(k) = 1$. *Here, for convenience, we set* $c_{m,j}(k) = 0$ *if* $j < 0$ *or* $j > m$.

Proof. We will prove the result by induction. Let $c_{m,j}(k)$ be defined as in (2.4). It is obvious that (2.3) is true for $m = 0$. If (2.3) is true for mth derivative of $F_{k,y}(s)$, we take derivative on both sides of (2.3) and obtain

$$\begin{aligned} F_{k,y}^{(m+1)}(s) &= \sum_{j=0}^{m} c_{m,j}(k)(-m-k-j\theta)s^{-m-1-k-j\theta}(-\theta y)^j \phi_k^{(j)}(s^{-\theta}y) \\ &+ \sum_{j=0}^{m} c_{m,j}(k)s^{-m-1-k-(j+1)\theta}(-\theta y)^{j+1}\phi_k^{(j+1)}(s^{-\theta}y) \\ &= -(m+k+j\theta)c_{m,0}(k)s^{-m-1-k}\phi_k(s^{-\theta}y) \\ &+ c_{m,m}(k)s^{-m-1-k-(m+1)\theta}(-\theta y)^{m+1}\phi_k^{(m+1)}(s^{-\theta}y) \\ &+ \sum_{j=1}^{m}[c_{m,j-1}(k) - (m+k+j\theta)c_{m,j}(k)]s^{-m-1-k-j\theta}(-\theta y)^j\phi_k^{(j)}(s^{-\theta}y) \\ &= \sum_{j=0}^{m+1} c_{m+1,j}(k)s^{-m-1-k-j\theta}(-\theta y)^j\phi_k^{(j)}(s^{-\theta}y), \end{aligned}$$

where we have made use of (2.4) in the last equality. Thus, (2.3) is also true for $(m+1)$th derivative. This completes the proof. □

It can be easily calculated from the recurrence relation (2.4) that

$$\begin{aligned} &c_{0,0}(k) = 1,\ c_{1,0}(k) = -k,\ c_{1,1}(k) = 1, \\ &c_{2,0}(k) = k(k+1),\ c_{2,1}(k) = -(2k+\theta+1),\ c_{2,2}(k) = 1, \\ &c_{3,0}(k) = -k(k+1)(k+2),\ c_{3,1}(0) = 3k^2 + (3\theta+6)k + \theta^2 + 3\theta + 2, \\ &c_{3,2}(k) = -(3k+3\theta+3),\ c_{3,3}(k) = 1. \end{aligned}$$

Moreover, $c_{m,0}(k) = (-1)^m(k)_m$ and $c_{m,m}(k) = 1$.

Lemma 2.2. *Let* $F_{k,y}(s) = s^{-k}\phi_k(s^{-\theta}y)$ *and* $\psi_{k+1}(y) = F'_{k,y}(1)$. *Then we have*

$$-k\phi_k(y) - \theta y\phi'_k(y) = \psi_{k+1}(y), \tag{2.5}$$

and

$$F_{k,y}^{(m)}(1) = \sum_{j=0}^{m-1} d_{m-1,j}(k)(-\theta y)^j \psi_{k+1}^{(j)}(y), \tag{2.6}$$

where

$$d_{m,j}(k) = \sum_{l=0}^{m-j}(-m)_l c_{m-l,j}(k). \tag{2.7}$$

For convenience, we define $d_{m,j}(k) = 0$ *when* $j < 0$ *or* $j > m$.

Proof. It follows from (2.3) that

$$\psi_{k+1}(y) = F'_{k,y}(1) = c_{1,0}(k)\phi_k(y) + c_{1,1}(k)(-\theta y)\phi'_k(y) = -k\phi_k(y) - \theta y\phi'_k(y).$$

Moreover, we have

$$\psi^{(j)}_{k+1}(y) = -(k+j\theta)\phi^{(j)}_k(y) - \theta y\phi^{(j+1)}_k(y).$$

For $m \geq 1$, we obtain from (2.4) that

$$\begin{aligned}
&d_{m-1,j-1}(k) - (k+j\theta)d_{m-1,j}(k)\\
=&\sum_{l=0}^{m-j-1}(-m+1)_l[c_{m-l-1,j-1}(k) - (k+j\theta)c_{m-l-1,j}(k)] + (-m+1)_{m-j}\\
=&\sum_{l=0}^{m-j-1}(-m+1)_l[c_{m-l,j}(k) - (l-m+1)c_{m-l-1,j}(k)] + (-m+1)_{m-j}\\
=&\sum_{l=0}^{m-j-1}(-m+1)_l c_{m-l,j}(k) - \sum_{l=0}^{m-j-1}(-m+1)_{l+1}c_{m-l-1,j}(k) + (-m+1)_{m-j}\\
=&c_{m,j}(k).
\end{aligned}$$

Substituting the above two equations into (2.3) gives

$$\begin{aligned}
F^{(m)}_{k,y}(1) &= \sum_{j=0}^{m}[d_{m-1,j-1}(k) - (k+j\theta)d_{m-1,j}(k)](-\theta y)^j\phi^{(j)}_k(y)\\
&= \sum_{j=0}^{m-1} d_{m-1,j}(k)(-\theta y)^{j+1}\phi^{(j+1)}_k(y) - \sum_{j=0}^{m-1}(k+j\theta)d_{m-1,j}(k)(-\theta y)^j\phi^{(j)}_k(y)\\
&= \sum_{j=0}^{m-1} d_{m-1,j}(k)(-\theta y)^j\psi^{(j)}_{k+1}(y),
\end{aligned}$$

which completes the proof. □

It can be easily calculated from (2.7) and (2.4) that

$$\begin{aligned}
&d_{0,0}(k) = 1,\ d_{1,0}(k) = -(k+1),\ d_{1,1}(k) = 1,\\
&d_{2,0}(k) = k^2 + 3k + 2,\ d_{2,1}(k) = -(2k+\theta+3),\ d_{2,2}(k) = 1,\\
&d_{3,0}(k) = -(k+1)(k+2)(k+3),\ d_{3,1}(k) = 3k^2 + (3\theta+12)k + \theta^2 + 6\theta + 11,\\
&d_{3,2}(k) = -(3k+3\theta+6),\ d_{3,3}(k) = 1.
\end{aligned}$$

Moreover, $d_{m,0}(k) = (-1)^m(k+1)_m$ and $d_{m,m}(k) = 1$.

Lemma 2.3. *Let $y_\pm = (1 \pm 1/n)^{-\theta}y$. We have*

$$\sum_{k=-1}^{\infty}\left[\frac{\phi_k(y_\pm)}{(n\pm 1)^k} - \frac{\phi_k(y)}{n^k}\right] \sim \sum_{s=0}^{\infty}\frac{g_{s,\pm}(y)}{n^s}, \tag{2.8}$$

where

$$g_{s,\pm}(y) = \sum_{l=0}^{s} \frac{(\pm 1)^{l+1}}{(l+1)!} \sum_{j=0}^{l} d_{l,j}(s-l-1)(-\theta y)^j \psi_{s-l}^{(j)}(y) \tag{2.9}$$

with $\psi_{k+1}(y) = -k\phi_k(y) - \theta y \phi_k'(y)$.

Proof. Let $F_{k,y}(s) = s^{-k}\phi_k(s^{-\theta} y)$. In view of (2.6), the Taylor expansion of $F_{k,y}(1+h)$ about $h = 0$ is given by

$$F_{k,y}(1+h) = (1+h)^{-k}\phi_k((1+h)^{-\theta} y) = \sum_{m=0}^{\infty} \frac{h^m}{m!} \sum_{j=0}^{m-1} d_{m-1,j}(k)(-\theta y)^j \psi_{k+1}^{(j)}(y).$$

By choosing $h = \pm 1/n$, subtracting $F_{k,y}(1)$ on both sides, multiplying by n^{-k}, and then adding along $k = -1, 0, \ldots$, we obtain

$$\begin{aligned}\sum_{k=-1}^{\infty} \left[\frac{\phi_k(y_\pm)}{(n\pm 1)^k} - \frac{\phi_k(y)}{n^k} \right] &\sim \sum_{m=1}^{\infty} \sum_{k=-1}^{\infty} \frac{(\pm 1)^m}{m! n^{m+k}} \sum_{j=0}^{m-1} d_{m-1,j}(k)(-\theta y)^j \psi_{k+1}^{(j)}(y) \\ &\sim \sum_{s=0}^{\infty} \sum_{k=-1}^{s-1} \frac{(\pm 1)^{s-k}}{(s-k)! n^s} \sum_{j=0}^{s-k-1} d_{s-k-1,j}(k)(-\theta y)^j \psi_{k+1}^{(j)}(y) \\ &\sim \sum_{s=0}^{\infty} \frac{1}{n^s} \sum_{l=0}^{s} \frac{(\pm 1)^{l+1}}{(l+1)!} \sum_{j=0}^{l} d_{l,j}(s-l-1)(-\theta y)^j \psi_{s-l}^{(j)}(y),\end{aligned}$$

where we set $s = m + k$ and then $l = s - k - 1$. This completes the proof. □

Substituting (2.8) into (2.2) gives

$$\sum_{s=0}^{\infty} \frac{\alpha_s y - \beta_s}{n^s} \sim \exp\{\sum_{l=0}^{\infty} \frac{g_{l,+}(y)}{n^l}\} + \exp\{\sum_{l=0}^{\infty} \frac{g_{l,-}(y)}{n^l}\},$$

which implies that

$$\alpha_s y - \beta_s = \sum_{m=0}^{s} \sum_{\substack{i_1+\cdots+i_s=m \\ 1i_1+\cdots+si_s=s}} \left\{ e^{g_{0,+}(y)} \prod_{l=1}^{s} \frac{[g_{l,+}(y)]^{i_l}}{i_l!} + e^{g_{0,-}(y)} \prod_{l=1}^{s} \frac{[g_{l,-}(y)]^{i_l}}{i_l!} \right\} \tag{2.10}$$

for $s \geq 0$.

Theorem 2.4. *Let* $\psi_{k+1}(y) = -k\phi_k(y) - \theta y \phi_k'(y)$ *and* $g_{s,\pm}(y)$ *be defined as in* (2.9)*. Define two functions*

$$L(y) = \cosh[\psi_0(y)], \tag{2.11}$$

$$R(y) = \sinh[\psi_0(y)]. \tag{2.12}$$

We then have

$$L(y) = (\alpha_0 y - \beta_0)/2, \;\; R(y)^2 = L(y)^2 - 1, \;\; e^{\pm\psi_0(y)} = L(y) \pm R(y), \tag{2.13}$$

and for $s \geq 1$,

$$\psi_s(y) = \frac{\alpha_s y - \beta_s}{2R(y)} - \sum_{1\leq k\leq s/2} \frac{1}{(2k+1)!} \sum_{j=0}^{2k} d_{2k,j}(s-2k-1)(-\theta y)^j \psi_{s-2k}^{(j)}(y)$$
$$- \frac{L(y)}{R(y)} \sum_{1\leq k\leq (s+1)/2} \frac{1}{(2k)!} \sum_{j=0}^{2k-1} d_{2k-1,j}(s-2k)(-\theta y)^j \psi_{s-2k+1}^{(j)}(y)$$
$$- \sum_{m=2}^{s} \sum_{\substack{i_1+\cdots+i_{s-1}=m \\ 1i_1+\cdots+(s-1)i_{s-1}=s}} \left\{ \frac{L(y)+R(y)}{2R(y)} \prod_{l=1}^{s-1} \frac{[g_{l,+}(y)]^{i_l}}{i_l!} \right.$$
$$\left. + \frac{L(y)-R(y)}{2R(y)} \prod_{l=1}^{s-1} \frac{[g_{l,-}(y)]^{i_l}}{i_l!} \right\}. \tag{2.14}$$

Proof. When $s = 0$, it follows from (2.9) that $g_{0,\pm}(y) = \pm\psi_0(y)$. Substituting this into (2.10) yields

$$\alpha_0 y - \beta_0 = e^{\psi_0(y)} + e^{-\psi_0(y)}.$$

Coupling this with (2.11) implies $L(y) = (\alpha_0 y - \beta_0)/2$. Furthermore, it is readily seen from (2.11) and (2.12) that $R(y)^2 = L(y)^2 - 1$ and $e^{\pm\psi_0(y)} = L(y) \pm R(y)$. This proves (2.13). When $s \geq 1$, it follows from (2.9) and $d_{0,0}(s-1) = 1$ that

$$g_{s,\pm}(y) = \sum_{k=1}^{s} \frac{(\pm 1)^{k+1}}{(k+1)!} \sum_{j=0}^{k} d_{k,j}(s-k-1)(-\theta y)^j \psi_{s-k}^{(j)}(y) \pm \psi_s(y).$$

Thus,

$$[L(y)+R(y)]g_{s,+}(y) + [L(y)-R(y)]g_{s,-}(y)$$
$$=2L(y) \sum_{1\leq k\leq (s+1)/2} \frac{1}{(2k)!} \sum_{j=0}^{2k-1} d_{2k-1,j}(s-2k)(-\theta y)^j \psi_{s-2k+1}^{(j)}(y)$$
$$+2R(y) \sum_{1\leq k\leq s/2} \frac{1}{(2k+1)!} \sum_{j=0}^{2k} d_{2k,j}(s-2k-1)(-\theta y)^j \psi_{s-2k}^{(j)}(y) + 2R(y)\psi_s(y).$$

Substituting this into (2.10) yields

$$\alpha_s y - \beta_s = \sum_{m=2}^{s} \sum_{\substack{i_1+\cdots+i_{s-1}=m \\ 1i_1+\cdots+(s-1)i_{s-1}=s}} \left\{ [L(y)+R(y)] \prod_{l=1}^{s-1} \frac{[g_{l,+}(y)]^{i_l}}{i_l!} \right.$$
$$\left. + [L(y)-R(y)] \prod_{l=1}^{s-1} \frac{[g_{l,-}(y)]^{i_l}}{i_l!} \right\}$$
$$+ 2L(y) \sum_{1\leq k\leq (s+1)/2} \frac{1}{(2k)!} \sum_{j=0}^{2k-1} d_{2k-1,j}(s-2k)(-\theta y)^j \psi_{s-2k+1}^{(j)}(y)$$

$$+ 2R(y) \sum_{1\le k\le s/2} \frac{1}{(2k+1)!} \sum_{j=0}^{2k} d_{2k,j}(s-2k-1)(-\theta y)^j \psi_{s-2k}^{(j)}(y)$$
$$+ 2R(y)\psi_s(y),$$

where we have made use of $g_{0,\pm}(y) = \pm\psi_0(y)$ and (2.13). We then divide both sides by $2R(y)$ and isolate $\psi_s(y)$ to obtain (2.14). This completes the proof. □

Remark 2.5. From (2.11) and (2.13), we observe that $L(y)$ is a linear function in y, while $R(y)$ is radical function composited with a quadratic function. There are two choices of $R(y)$ by taking square root of $L(y)^2 - 1 = (\alpha_0 y - \beta_0)^2/4 - 1$, which correspond to two linearly independent solutions for the difference Eq. (1.1) without initial conditions.

Remark 2.6. Note that when $j = 0$ and $s = l > 0$, we have

$$d_{l,j}(s-l-1) = d_{l,0}(-1) = (-1)^l(-1)_l = 0,$$

and hence, the term $\psi_0(y)$ does not appear in the expression of $g_{s,\pm}(y)$ if $s \ge 1$. Due to a similar reason, the term $\psi_0(y)$ does not appear in the expression of $\psi_s(y)$ if $s \ge 1$.

In the following corollary, we shall provide explicit expressions for $\psi_1(y)$ and $\psi_2(y)$.

Corollary 2.7. *Let $R(y)$ be given as in (2.12). We have*

$$\psi_1(y) = \frac{\alpha_1 y - \beta_1}{2R(y)} + \frac{\theta\alpha_0 y L(y)}{4R(y)^2}, \tag{2.15}$$
$$\psi_2(y) = \frac{\psi_{2,1}(y)}{[R(y)]^4} + \frac{\psi_{2,2}(y)}{[R(y)]^5}, \tag{2.16}$$

where

$$\psi_{2,1}(y) = \frac{2L(y)[(\theta+1)\alpha_1 y - \beta_1][R(y)]^2 - \theta\alpha_0 y(\alpha_1 y - \beta_1)\{[R(y)]^2 + 2\}}{8},$$
$$\psi_{2,2}(y) = \frac{[6(\alpha_2 y - \beta_2) - \theta(\theta+1)\alpha_0 y][R(y)]^4}{12} + \frac{(\theta\alpha_0 y)^2 L(y)[R(y)]^2}{24}$$
$$+ \frac{\theta\alpha_0 y[L(y)R(y)]^2}{8}$$
$$- \frac{2\theta^2\alpha_0 y\beta_0 L(y)[R(y)]^2 + 5(\theta\alpha_0 y)^2 L(y) + 4(\alpha_1 y - \beta_1)^2 L(y)[R(y)]^2}{32}.$$

Proof. First, we note from (2.11)–(2.13) that

$$\psi_0'(y) = \frac{\alpha_0}{2R(y)}, \quad \psi_0''(y) = -\frac{\alpha_0^2(\alpha_0 y - \beta_0)}{8[R(y)]^3}. \tag{2.17}$$

When $s = 1$, it follows from $d_{1,0}(-1) = 0$, $d_{1,1}(-1) = 1$ and (2.14) that,

$$\psi_1(y) = \frac{\alpha_1 y - \beta_1}{2R(y)} - \frac{L(y)}{2R(y)}[d_{1,0}(-1)\psi_0(y) + d_{1,1}(-1)(-\theta y)\psi_0'(y)]$$
$$= \frac{\alpha_1 y - \beta_1}{2R(y)} + \frac{\theta y L(y)}{2R(y)}\psi_0'(y).$$

Substituting (2.17) into the above equation gives (2.15). Taking derivative of (2.15) yields

$$\begin{aligned}\psi_1'(y) &= \frac{2\alpha_1[R(y)]^2-(\alpha_1 y-\beta_1)\alpha_0 L(y)}{4R(y)^3}\\ &\quad+\frac{(2\theta\alpha_0^2 y-\theta\alpha_0\beta_0)[R(y)]^2-2\theta\alpha_0 yL(y)\alpha_0 L(y)}{8R(y)^4}\\ &=\frac{2\alpha_1[R(y)]^2-(\alpha_1 y-\beta_1)\alpha_0 L(y)}{4R(y)^3}-\frac{\theta\alpha_0\beta_0[R(y)]^2+2\theta\alpha_0^2 y}{8R(y)^4}.\end{aligned} \tag{2.18}$$

Furthermore, we obtain from $d_{1,0}(-1)=0$, $d_{1,1}(-1)=1$ and (2.9) that

$$g_{1,\pm}(y)=-\frac{\theta y}{2}\psi_0'(y)\pm\psi_1(y).$$

Substituting (2.15) and (2.17) into the above formula gives

$$g_{1,\pm}(y)=\frac{-\theta\alpha_0 y\pm 2(\alpha_1 y-\beta_1)}{4R(y)}\pm\frac{\theta\alpha_0 yL(y)}{4R(y)^2}, \tag{2.19}$$

and

$$\begin{aligned}\frac{[g_{1,\pm}(y)]^2}{2}&=\frac{(\theta\alpha_0 y)^2+4(\alpha_1 y-\beta_1)^2\mp 4\theta\alpha_0 y(\alpha_1 y-\beta_1)}{32[R(y)]^2}+\frac{(\theta\alpha_0 y)^2[L(y)]^2}{32R(y)^4}\\ &\quad+\frac{2\theta\alpha_0 yL(y)(\alpha_1 y-\beta_1)\mp(\theta\alpha_0 y)^2 L(y)}{16[R(y)]^3}.\end{aligned} \tag{2.20}$$

When $s=2$, we have from (2.14)

$$\begin{aligned}\psi_2(y)&=\frac{\alpha_2 y-\beta_2}{2R(y)}-\frac{1}{6}[d_{2,0}(-1)\psi_0(y)+d_{2,1}(-1)(-\theta y)\psi_0'(y)\\ &\quad+d_{2,2}(-1)(-\theta y)^2\psi_0''(y)]\\ &\quad-\frac{L(y)}{2R(y)}[d_{1,0}(0)\psi_1(y)+(-\theta y)\psi_1'(y)]\\ &\quad-\frac{L(y)+R(y)}{2R(y)}\cdot\frac{[g_{1,+}(y)]^2}{2}-\frac{L(y)-R(y)}{2R(y)}\cdot\frac{[g_{1,-}(y)]^2}{2}.\end{aligned}$$

Note that $d_{2,0}(-1)=0$, $d_{2,1}(-1)=-\theta-1$, $d_{2,2}(-1)=1$, $d_{1,0}(0)=-1$ and $d_{1,1}(0)=1$. We substitute (2.15), (2.17), (2.18) and (2.20) into the above equation to obtain

$$\begin{aligned}\psi_2(y)&=\frac{\alpha_2 y-\beta_2}{2R(y)}-\frac{1}{6}\left\{\frac{\theta(\theta+1)\alpha_0 y}{2R(y)}-\frac{(\theta\alpha_0 y)^2 L(y)}{4[R(y)]^3}\right\}\\ &\quad+\frac{L(y)}{2R(y)}\left\{\frac{\alpha_1 y-\beta_1}{2R(y)}+\frac{\theta\alpha_0 yL(y)}{4R(y)^2}\right.\\ &\quad\left.+\frac{2\theta\alpha_1 y[R(y)]^2-(\alpha_1 y-\beta_1)\theta\alpha_0 yL(y)}{4R(y)^3}-\frac{\theta^2\alpha_0 y\beta_0[R(y)]^2+2\theta^2\alpha_0^2 y^2}{8R(y)^4}\right\}\\ &\quad-\frac{L(y)}{R(y)}\left\{\frac{(\theta\alpha_0 y)^2+4(\alpha_1 y-\beta_1)^2}{32[R(y)]^2}+\frac{(\theta\alpha_0 y)^2[L(y)]^2}{32R(y)^4}+\frac{\theta\alpha_0 yL(y)(\alpha_1 y-\beta_1)}{8[R(y)]^3}\right\}\\ &\quad+\frac{\theta\alpha_0 y(\alpha_1 y-\beta_1)}{8[R(y)]^2}+\frac{(\theta\alpha_0 y)^2 L(y)}{16[R(y)]^3}.\end{aligned}$$

Simplifying the above formula gives (2.16). This completes the proof. □

From the calculation, we observe that $\psi_1(y)$ and $\psi_2(y)$ are simply sums of fractions with polynomial functions divided by a nonnegative integer power of $R(y)$. To better explain this property, we introduce the definition of *quasi-rational functions* and the *index* for these functions.

Definition 2.8. Let $[R(y)]^2$ be a quadratic function such that $R(y)$ cannot be simplified as a linear function. A quasi-rational function (associated with $R(y)$) is a function that can be expressed as the finite sum:

$$\sum_{k=0}^{m} \frac{f_k(y)}{[R(y)]^k},$$

where $f_k(y)$ are polynomial functions that do not contain $[R(y)]^2$ as a factor, and $f_m(y) \neq 0$. The highest power of $R(y)$ in the denominator (i.e., m) is called the index, denoted by $\mathrm{ind}_R f$. If further, the polynomial degree of $f_k(y)$ is no more than k for all $k = 0, \ldots, m$, then we say the quasi-rational function f is bounded, in the sense that, as $y \to \infty$, the limit of $f(y)$ exists.

We have the following properties for the quasi-rational function. The proof is trivial and we omit it here.

Proposition 2.9. *Let $[R(y)]^2$ be a quadratic function such that $R(y)$ cannot be simplified as a linear function. Let $f(y)$ and $g(y)$ be quasi-rational functions associated with $R(y)$.*

1. *$\mathrm{ind}_R f(y) = 0$ if and only if $f(y)$ is a polynomial function.*
2. *$\mathrm{ind}_R f(y) = m \geq 1$ if and only if there exist two polynomial functions $f_1(y)$ and $f_2(y)$ such that $f_2(y)$ does not contain $[R(y)]^2$ as a factor and*
$$f(y) = \frac{f_1(y)}{[R(y)]^{m-1}} + \frac{f_2(y)}{[R(y)]^m}.$$
Moreover, the decomposition is unique.
3. *$\mathrm{ind}_R f^{(j)}(y) = \mathrm{ind}_R f(y) + 2j$ if $\mathrm{ind}_R f(y) > 0$.*
4. *$\mathrm{ind}_R[f(y)g(y)] \leq \mathrm{ind}_R f(y) + \mathrm{ind}_R g(y)$.*
5. *If $f(y)$ is bounded, so is $y^j f^{(j)}(y)$.*

We observe from (2.15) and (2.19) that $\mathrm{ind}_R \psi_1(y) = \mathrm{ind}_R g_{1,\pm}(y) = 2$. Actually, we have the following results.

Corollary 2.10. *For $s \geq 1$, we have $\mathrm{ind}_R \psi_s(y) \leq 3s - 1$ and $\mathrm{ind}_R g_{s,\pm}(y) \leq 3s - 1$. Especially, there exist polynomial functions $\psi_{s,1}(y)$ and $\psi_{s,2}(y)$ such that*

$$\psi_s(y) = \frac{\psi_{s,1}(y)}{[R(y)]^{3s-2}} + \frac{\psi_{s,2}(y)}{[R(y)]^{3s-1}}. \tag{2.21}$$

Proof. We will prove the results by induction. On account of (2.15) and (2.19), we see $\mathrm{ind}_R \psi_1(y) \leq 2$ and $\mathrm{ind}_R g_{1,\pm}(y) \leq 2$. Assuming $\mathrm{ind}_R \psi_l(y) \leq 3l - 1$ and $\mathrm{ind}_R g_{l,\pm}(y) \leq 3l - 1$ for all $l = 1, \ldots, s - 1$, we need to prove that $\mathrm{ind}_R \psi_s(y) \leq 3s - 1$ and $\mathrm{ind}_R g_{s,\pm}(y) \leq 3s - 1$. First, we obtain from (2.17) and Proposition 2.9 that $\mathrm{ind}_R \psi_1^{(j)} \leq 2j - 1$ for $j \geq 1$. Second, we recall from Remark 2.6 that $\psi_0(y)$ does not appear in the expression of $\psi_s(y)$. By induction and Proposition 2.9, we have

$$\mathrm{ind}_R \psi_{s-2k}^{(j)}(y) \leq 3(s - 2k) - 1 + 2j \leq 3s - 2k - 1 \leq 3s - 3$$

for $0 \le j \le 2k$ and $1 \le k \le s/2$, and

$$\mathrm{ind}_R \psi^{(j)}_{s-2k+1}(y) \le 3(s-2k+1) - 1 + 2j \le 3s - 2k \le 3s - 2$$

for $0 \le j \le 2k-1$ and $1 \le k \le (s+1)/2$, and

$$\mathrm{ind}_R \prod_{l=1}^{s-1} \frac{[g_{l,\pm}(y)]^{i_l}}{i_l!} \le \sum_{l=1}^{s-1} [(3l-1)i_l] = 3s - m \le 3s - 2$$

for $i_1 + \cdots + i_{s-1} = m$, $1i_1 + \cdots + (s-1)i_{s-1} = s$ and $2 \le m \le s$. It then follows from (2.14) that

$$\mathrm{ind}_R \psi_s(y) \le \max\{1, 3s-3, 1+3s-2, 1+3s-2\} = 3s - 1.$$

Again, we recall from Remark 2.6 that $\psi_0(y)$ does not appear in the expression of $g_{s,\pm}(y)$. Furthermore,

$$\mathrm{ind}_R \psi^{(j)}_{s-l}(y) \le 3(s-l) - 1 + 2j \le 3s - l - 1 \le 3s - 1$$

for $0 \le j \le l$ and $0 \le l \le s$ such that $s - l$ and j cannot be identically zero. Hence, it is observed from (2.9) that

$$\mathrm{ind}_R g_{s,\pm}(y) \le 3s - 1.$$

This completes the proof. □

Remark 2.11. For $s \ge 1$, we observe from Remark 2.6 that the term $\psi_0(y)$ does not appear in the expression of $g_{s,\pm}(y)$. Thus, it is easily seen from (2.9), (2.14), (2.17), (2.15), Proposition 2.9, and an argument of induction that both $\psi_s(y)$ and $g_{s,\pm}(y)$ are bounded as $y \to \infty$.

Finally, we are ready to solve the differential equation (2.5) and determine the kth order term in the asymptotic expansion for $\Phi_n(n^\theta y)$. It is easily seen that

$$\phi_k(y) = \frac{-1}{\theta y^{k/\theta}} \int_{y_k}^{y} t^{k/\theta - 1} \psi_{k+1}(t) dt, \tag{2.22}$$

where y_k is an integration constant to be determined by asymptotic matching. Especially, we have from (2.15) and (2.17)

$$\phi_{-1}(y) = c_{-1} y^{1/\theta} + \ln[L(y) + R(y)] + \int_y^\infty \frac{\alpha_0 y^{1/\theta}}{2t^{1/\theta} R(t)} dt, \tag{2.23}$$

$$\phi_0(y) = c_0 - \frac{\alpha_1}{\theta\alpha_0} \ln[L(y) + R(y)] + \frac{\beta_1}{2\theta\beta_0^*} \ln \frac{\beta_0 L(y) + \beta_0^* R(y) + 2}{\beta_0 L(y) - \beta_0^* R(y) + 2} - \frac{1}{2} \ln R(y), \tag{2.24}$$

where c_{-1} and c_0 are some integration constants, and

$$\beta_0^* = \begin{cases} \sqrt{\beta_0^2 - 4}, & |\beta_0| \ge 2, \\ i\sqrt{4 - \beta_0^2}, & |\beta_0| < 2. \end{cases} \tag{2.25}$$

The singularity in the expression of $\phi_0(y)$ at $\beta_0^* = 0$ (i.e., $\beta_0 = \pm 2$) can be removed by taking the limit $\beta_0 \to \pm 2$. Substituting the above formulas into (1.3) gives

$$\Phi_n(n^\theta y) \sim \frac{e^{c_{-1}ny^{1/\theta}+c_0}}{\sqrt{R(y)}}[L(y)+R(y)]^{n-\alpha_1/(\theta\alpha_0)}\left[\frac{\beta_0 L(y)+\beta_0^* R(y)+2}{\beta_0 L(y)-\beta_0^* R(y)+2}\right]^{\beta_1/(2\theta\beta_0^*)}$$
$$\times \exp\left\{\int_y^\infty \frac{\alpha_0 n y^{1/\theta}}{2t^{1/\theta}R(t)}dt - \sum_{k=1}^\infty \int_{y_k}^y \frac{t^{k/\theta-1}\psi_{k+1}(t)}{\theta n^k y^{k/\theta}}dt\right\}. \tag{2.26}$$

3. Asymptotic matching

In this section, we will connect the initial conditions to the asymptotic solution by principle of asymptotic matching. For convenience, we first introduce several notations. Note from (2.22) that $\phi_k(y)$ may have singularities at the origin $y = 0$ or at the points when $R(y) = 0$. Solving $R(y) = 0$ gives two solutions $y_\pm = (\beta_0 \pm 2)/\alpha_0$, which are referred to as the *turning points*. The *oscillatory interval*, denoted by I, is the union of two disconnected open intervals whose end points are the origin or the turning points. To be more specific, there are three cases:

(i) if $y_- \le 0 \le y_+$; namely, $-2 \le \beta_0 \le 2$, then $I = (y_-, 0) \cup (0, y_+)$;
(ii) if $0 < y_- < y_+$; namely, $\beta_0 > 2$, then $I = (0, y_-) \cup (y_-, y_+)$;
(iii) if $y_- < y_+ < 0$; namely, $\beta_0 < -2$, then $I = (y_-, y_+) \cup (y_+, 0)$.

For some orthogonal polynomials, the support of the orthogonal measure may be a union of multiple disjoint intervals and more than two turning points may occur [2]. However, under the assumption (1.2), there are only two turning points which are the roots of the quadratic equation $L(y)^2 = 1$; see also [23,24]. The *oscillatory region* is a complex neighborhood of the oscillatory interval which contains all complex numbers z such that $\mathrm{Re}\, z \in I$ and $|\mathrm{Im}\, z| < \delta$, where $\delta > 0$ is a fixed small number to be determined later in Lemma 3.3. We also define the complex complement of the closure of the oscillatory interval (i.e., $\mathbf{C} \setminus \bar{I}$) as the *outer region*. In this region, the quadratic equation $R(y)^2 = L(y)^2 - 1$ has two different solutions $\pm R^o(y)$, where $R^o(y) = \sqrt{[L(y)]^2 - 1}$ can be regarded as an analytic function in $\mathbf{C} \setminus \bar{I}$ such that $R^o(y)/L(y) \to 1$ as $y \to \infty$. When y lies in the outer region $\mathbf{C}\setminus\bar{I}$, we substitute $R(y) = \pm R^o(y)$ into (2.26) to obtain two linearly independent solutions, denoted by $\Phi^o_{n,1}(n^\theta y)$ and $\Phi^o_{n,2}(n^\theta y)$, to the difference Eq. (1.1). In general, neither $\Phi^o_{n,1}(n^\theta y)$ nor $\Phi^o_{n,2}(n^\theta y)$ should be a polynomial solution. But they all satisfy the same linear difference equation (1.1) with different initial conditions. Moreover, if we use these two solutions as a basis, then the orthogonal polynomials have the following representation:

$$P_n(n^\theta y) = K_1^o(x)\Phi^o_{n,1}(n^\theta y) + K_2^o(x)\Phi^o_{n,2}(n^\theta y), \tag{3.1}$$

for some coefficients $K_1^o(x)$ and $K_2^o(x)$ depending on the original variable $x = n^\theta y$. To determine the coefficients in the above linear combination, we need to apply the principle of asymptotic matching. Note from (1.1) with the initial conditions that the leading term of $P_n(n^\theta y)$ is $A_0 \cdots A_{n-1}(n^\theta y)^n$. In view of the asymptotic expansion (1.2), we have

$$A_0 \cdots A_{n-1} = A_0 \prod_{k=1}^{n-1}[k^{-\theta}(\alpha_0 + \alpha_1/k)] \times \prod_{k=1}^{n-1} \frac{A_k}{k^{-\theta}(\alpha_0 + \alpha_1/k)}.$$

As $n \to \infty$, the second product on the right-hand side converges because $\frac{A_n}{n^{-\theta}(\alpha_0+\alpha_1/n)} = 1 + O(1/n^2)$. It then follows Stirling's formula that, as $n \to \infty$,

$$A_0 \cdots A_{n-1} \sim k_0 n^{\alpha_1/\alpha_0+\theta/2} \alpha_0^n (e/n)^{n\theta}$$

for some constant k_0. Now we let both $n \to \infty$ and $y \to \infty$. It follows that

$$P_n(n^\theta y) \sim k_0 n^{\alpha_1/\alpha_0+\theta/2} (\alpha_0 y)^n e^{n\theta}.$$

On the other hand, we set $R(y) = \pm R^o(y) = \pm\sqrt{[L(y)]^2 - 1}$ in (2.26) to obtain

$$\Phi^o_{n,1}(n^\theta y) \sim 2^{1/2} e^{c_{-1} n y^{1/\theta} + c_0} (\alpha_0 y)^{n-\alpha_1/(\theta\alpha_0)-1/2} \left(\frac{\beta_0 + \beta_0^*}{\beta_0 - \beta_0^*}\right)^{\beta_1/(2\theta\beta_0^*)} e^{n\theta},$$

$$\Phi^o_{n,2}(n^\theta y) \sim -i 2^{1/2} e^{\tilde{c}_{-1} n y^{1/\theta} + \tilde{c}_0} (\alpha_0 y)^{-n+\alpha_1/(\theta\alpha_0)-1/2} \left(\frac{\beta_0 - \beta_0^*}{\beta_0 + \beta_0^*}\right)^{\beta_1/(2\theta\beta_0^*)} e^{-n\theta},$$

as $n \to \infty$ and $y \to \infty$. Here c_k and $\tilde{c}_k$ are some integration constants for the two linearly independent solutions. By substituting the above three formulas into (3.1) and matching the asymptotic leading term, we obtain $c_{-1} = 0$, $c_0 = \ln(k_0/\sqrt{2})$, and

$$K_1^o(x) = \left(\frac{\beta_0 - \beta_0^*}{\beta_0 + \beta_0^*}\right)^{\beta_1/(2\theta\beta_0^*)} x^{\alpha_1/(\theta\alpha_0)+1/2}.$$

Here, we cannot determine the coefficient $K_2^o(x)$ and integration constants $\tilde{c}_k$, but the term $K_2^o(x)\Phi^o_{n,2}(n^\theta y)$ is exponentially small with respect to $K_1^o(x)\Phi^o_{n,1}(n^\theta y)$, and thus it can be ignored in the asymptotic expansion:

$$P_n(n^\theta y) \sim K_1^o(x)\Phi^o_{n,1}(n^\theta y), \quad n \to \infty.$$

Summarizing the above arguments, we arrive at the following result.

Theorem 3.1. *Let $L(y) = (\alpha_0 y - \beta_0)/2$ and $R^o(y) = \sqrt{[L(y)]^2 - 1}$. Set*

$$k_0 := \lim_{n\to\infty} \frac{A_0 \cdots A_{n-1}}{n^{\alpha_1/\alpha_0+\theta/2} \alpha_0^n (e/n)^{n\theta}}. \tag{3.2}$$

As $n \to \infty$, we have

$$\begin{aligned} P_n(n^\theta y) \sim{}& \frac{k_0 (n^\theta y)^{\alpha_1/(\theta\alpha_0)+1/2}}{\sqrt{2R^o(y)}} [L(y) + R^o(y)]^{n-\alpha_1/(\theta\alpha_0)} \\ &\times \left[\frac{\beta_0 L(y) + \beta_0^* R^o(y) + 2}{\beta_0 L(y) - \beta_0^* R(y) + 2}\right]^{\beta_1/(2\theta\beta_0^*)} \\ &\times \left(\frac{\beta_0 - \beta_0^*}{\beta_0 + \beta_0^*}\right)^{\beta_1/(2\theta\beta_0^*)} \exp\left\{\int_y^\infty \frac{\alpha_0 n y^{1/\theta}}{2t^{1/\theta} R^o(t)} dt \right.\\ &\left. - \sum_{k=1}^\infty \int_{y_k}^y \frac{t^{k/\theta-1}\psi^o_{k+1}(t)}{\theta n^k y^{k/\theta}} dt \right\}, \end{aligned} \tag{3.3}$$

for $y \in \mathbf{C} \setminus \bar{I}$. Here, β_0^* is defined as in (2.25), and $\psi^o_{k+1}(y)$ is the same as $\psi_{k+1}(y)$ in (2.14) with $R(y) = R^o(y)$.

Remark 3.2. The integration constants y_k in the higher-order terms of the asymptotic expansion (3.3) can be determined by asymptotic matching. We will illustrate this in the next section by finding asymptotic expansion of Hermite polynomials.

We denote the truncated expansion in (3.3) as

$$\begin{aligned}\Psi_m(n,y) &:= \frac{k_0(n^\theta y)^{\alpha_1/(\theta\alpha_0)+1/2}}{\sqrt{2R^o(y)}}[L(y)+R^o(y)]^{n-\alpha_1/(\theta\alpha_0)}\\ &\times\left[\frac{\beta_0 L(y)+\beta_0^* R^o(y)+2}{\beta_0 L(y)-\beta_0^* R(y)+2}\right]^{\beta_1/(2\theta\beta_0^*)}\\ &\times\left(\frac{\beta_0-\beta_0^*}{\beta_0+\beta_0^*}\right)^{\beta_1/(2\theta\beta_0^*)}\exp\left\{\int_y^\infty\frac{\alpha_0 n y^{1/\theta}}{2t^{1/\theta}R^o(t)}dt\right.\\ &\left.-\sum_{k=1}^m\int_{y_k}^y\frac{t^{k/\theta-1}\psi_{k+1}^o(t)}{\theta n^k y^{k/\theta}}dt\right\},\quad m\ge 0. \end{aligned}\tag{3.4}$$

It is obvious that $\Psi_m(n,y)$ is analytic in $\mathbf{C}\setminus\bar{I}$. If $y\in I$, we define

$$\Psi_{m,\pm}(n,y) := \lim_{\varepsilon\to 0^+}\Psi_m(n,y\pm i\varepsilon). \tag{3.5}$$

It is readily seen that $\Psi_{m,\pm}(n,y)$ can be analytically continued in a complex neighborhood of any compact subset of I. Remark that the end points of I should be bounded away from the complex neighborhood for the analytic continuation. Moreover, $\Psi_{m,+}(n,y)=\Psi_m(n,y)$ if Im $y>0$ and $\Psi_{m,-}(n,y)=\Psi_m(n,y)$ if Im $y<0$. We then have the following lemma.

Lemma 3.3. *There exists $\delta>0$ such that as $n\to\infty$, the ratio $\Psi_{m,-}(n,y)/\Psi_{m,+}(n,y)$ is exponentially small for any y such that Re $y\in I$ and $0<$ Im $y<\delta$, and exponentially large for any y such that Re $y\in I$ and $-\delta<$ Im $y<0$.*

Proof. Define

$$\varphi(y) := \ln[L(y)+R^o(y)]+\int_y^\infty\frac{\alpha_0 y^{1/\theta}}{2t^{1/\theta}R^o(t)}dt$$

for $y\in\mathbf{C}\setminus I$, and

$$\varphi_\pm(y) := \lim_{\varepsilon\to 0^+}\varphi(y\pm i\varepsilon),\quad \bar\varphi(y) := \varphi_-(y)-\varphi_+(y)$$

for $y\in I$. Remark that

$$\varphi'(y) = \int_y^\infty\frac{\alpha_0 y^{1/\theta-1}}{2\theta t^{1/\theta}R^o(t)}dt$$

for $y\in\mathbf{C}\setminus I$. By analytic continuation, $\varphi_\pm(y)$ and $\bar\varphi(y)$ are analytic in a complex neighborhood of I. Furthermore,

$$\frac{1}{n}\ln\frac{\Psi_{m,-}(n,y)}{\Psi_{m,+}(n,y)}\to\bar\varphi(y),$$

as $n\to\infty$. Let $y=y_r+i\varepsilon$ with $y_r\in I$ and $\varepsilon\in\mathbf{R}$. It suffices to show that there exists $\delta>0$ such that εRe $\bar\varphi(y_r+i\varepsilon)<0$ for all $0<|\varepsilon|<\delta$. Note that

$$\bar\varphi(y_r+i\varepsilon)\sim\bar\varphi(y_r)+i\varepsilon\bar\varphi'(y_r),$$

and

$$\text{Re }\bar\varphi(y_r+i\varepsilon)\sim\text{Re }\bar\varphi(y_r)-\varepsilon\text{Im }\bar\varphi'(y_r),$$

as $\varepsilon \to 0$. We claim Re $\bar{\varphi}(y_r) = 0$ and Im $\bar{\varphi}'(y_r) > 0$. If this is true, then Re $\bar{\varphi}(y_r + i\varepsilon)$ has a different sign with ε. We have to consider three cases.

Case 1: $-2 \le \beta_0 \le 2$. In this case, $I = (y_-, 0) \cup (0, y_+)$. If $y_r \in (0, y_+)$, then

$$\varphi_\pm(y_r) = \pm i \arccos[L(y_r)] \mp i \int_{y_r}^{y_+} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{1-[L(t)]^2}} dt + \int_{y_+}^{\infty} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{[L(t)]^2-1}} dt,$$

which implies that Re $\bar{\varphi}(y_r) = 0$. On the other hand,

$$\varphi'_\pm(y_r) = \mp i \int_{y_r}^{y_+} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{1-[L(t)]^2}} dt + \int_{y_+}^{\infty} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{[L(t)]^2-1}} dt.$$

Thus,

$$\text{Im } \bar{\varphi}'(y_r) = \int_{y_r}^{y_+} \frac{\alpha_0 y_r^{1/\theta-1}}{\theta t^{1/\theta}\sqrt{1-[L(t)]^2}} dt > 0.$$

If $y_r \in (y_-, 0)$, then

$$\varphi_\pm(y_r) = \pm i \arccos[L(y_r)] \mp i \int_{y_r}^{y_-} \frac{\alpha_0 (-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{1-[L(t)]^2}} dt - \int_{y_-}^{-\infty} \frac{\alpha_0 (-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{|[L(t)]^2-1|}} dt,$$

which again implies that Re $\bar{\varphi}(y_r) = 0$. On the other hand,

$$\varphi'_\pm(y_r) = \mp i \int_{y_-}^{y_r} \frac{\alpha_0 (-y_r)^{1/\theta-1}}{2\theta(-t)^{1/\theta}\sqrt{1-[L(t)]^2}} dt - \int_{-\infty}^{y_-} \frac{\alpha_0 (-y_r)^{1/\theta-1}}{2\theta(-t)^{1/\theta}\sqrt{|[L(t)]^2-1|}} dt.$$

We then have

$$\text{Im } \bar{\varphi}'(y_r) = \int_{y_-}^{y_r} \frac{\alpha_0 (-y_r)^{1/\theta-1}}{\theta(-t)^{1/\theta}\sqrt{1-[L(t)]^2}} dt > 0.$$

Case 2: $\beta_0 > 2$. In this case, $I = (0, y_-) \cup (y_-, y_+)$. If $y_r \in (y_-, y_+)$, then

$$\varphi_\pm(y_r) = \pm i \arccos[L(y_r)] \mp i \int_{y_r}^{y_+} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{1-[L(t)]^2}} dt + \int_{y_+}^{\infty} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{[L(t)]^2-1}} dt,$$

which implies that Re $\bar{\varphi}(y_r) = 0$. On the other hand,

$$\varphi'_\pm(y_r) = \mp i \int_{y_r}^{y_+} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{1-[L(t)]^2}} dt + \int_{y_+}^{\infty} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{[L(t)]^2-1}} dt.$$

Thus,

$$\text{Im } \bar{\varphi}'(y_r) = \int_{y_r}^{y_+} \frac{\alpha_0 y_r^{1/\theta-1}}{\theta t^{1/\theta}\sqrt{1-[L(t)]^2}} dt > 0.$$

If $y_r \in (0, y_-)$, then

$$\varphi_\pm(y_r) = \log\{|L(y)| + \sqrt{|[L(y)]^2-1|}\} \pm i\pi - \int_{y_r}^{y_-} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{|[L(t)]^2-1|}}dt$$
$$\mp i \int_{y_-}^{y_+} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{1-[L(t)]^2}}dt + \int_{y_+}^{\infty} \frac{\alpha_0 y_r^{1/\theta}}{2t^{1/\theta}\sqrt{[L(t)]^2-1}}dt,$$

which again implies that Re $\bar{\varphi}(y_r) = 0$. On the other hand,

$$\varphi'_\pm(y_r) = -\int_{y_r}^{y_-} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{|[L(t)]^2-1|}}dt$$
$$\mp i \int_{y_-}^{y_+} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{1-[L(t)]^2}}dt + \int_{y_+}^{\infty} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{[L(t)]^2-1}}dt.$$

We then have

$$\text{Im}\, \bar{\varphi}'(y_r) = \int_{y_-}^{y_+} \frac{\alpha_0 y_r^{1/\theta-1}}{\theta t^{1/\theta}\sqrt{1-[L(t)]^2}}dt > 0.$$

Case 3: $\beta_0 < -2$. In this case, $I = (y_-, y_+) \cup (y_+, 0)$. If $y_r \in (y_-, y_+)$, then

$$\varphi_\pm(y_r) = \pm i \arccos[L(y_r)] \mp i \int_{y_r}^{y_-} \frac{\alpha_0(-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{1-[L(t)]^2}}dt$$
$$- \int_{y_-}^{\infty} \frac{\alpha_0(-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{|[L(t)]^2-1|}}dt,$$

which implies that Re $\bar{\varphi}(y_r) = 0$. On the other hand,

$$\varphi'_\pm(y_r) = \mp i \int_{y_-}^{y_r} \frac{\alpha_0(-y_r)^{1/\theta-1}}{2\theta(-t)^{1/\theta}\sqrt{1-[L(t)]^2}}dt - \int_{-\infty}^{y_-} \frac{\alpha_0 y_r^{1/\theta-1}}{2\theta t^{1/\theta}\sqrt{[L(t)]^2-1}}dt.$$

Thus,

$$\text{Im}\, \bar{\varphi}'(y_r) = \int_{y_-}^{y_r} \frac{\alpha_0(-y_r)^{1/\theta-1}}{\theta(-t)^{1/\theta}\sqrt{1-[L(t)]^2}}dt > 0.$$

If $y_r \in (y_+, 0)$, then

$$\varphi_\pm(y_r) = \log\{L(y)| + \sqrt{[L(y)]^2-1}\} + \int_{y_r}^{y_+} \frac{\alpha_0(-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{[L(t)]^2-1}}dt$$
$$\mp i \int_{y_+}^{y_-} \frac{\alpha_0(-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{1-[L(t)]^2}}dt - \int_{y_-}^{-\infty} \frac{\alpha_0(-y_r)^{1/\theta}}{2(-t)^{1/\theta}\sqrt{|[L(t)]^2-1|}}dt,$$

which again implies that Re $\bar{\varphi}(y_r) = 0$. On the other hand,

$$\varphi'_\pm(y_r) = -\int_{y_+}^{y_r} \frac{\alpha_0(-y_r)^{1/\theta-1}}{2\theta(-t)^{1/\theta}\sqrt{[L(t)]^2-1}}dt$$
$$\mp i \int_{y_-}^{y_+} \frac{\alpha_0(-y_r)^{1/\theta-1}}{2\theta(-t)^{1/\theta}\sqrt{1-[L(t)]^2}}dt - \int_{-\infty}^{y_-} \frac{\alpha_0(-y_r)^{1/\theta-1}}{2\theta(-t)^{1/\theta}\sqrt{|[L(t)]^2-1|}}dt.$$

We then have

$$\text{Im}\, \bar{\varphi}'(y_r) = \int_{y_-}^{y_+} \frac{\alpha_0(-y_r)^{1/\theta-1}}{\theta(-t)^{1/\theta}\sqrt{1-[L(t)]^2}}dt > 0.$$

This completes the proof. □

From the above lemma, we find a solution to the difference Eq. (1.1) with asymptotic expansion $\Phi^i_{n,1}(n^\theta y) \sim \Psi_{m,-}(n,y)+\Psi_{m,+}(n,y)$ for any $m \geq 0$ and y in a complex neighborhood of oscillatory interval I. Let $\Phi^i_{n,2}(n^\theta y)$ be any other linearly independent solution, then

$$P_n(n^\theta y) = K^i_1(x)\Phi^i_{n,1}(n^\theta y) + K^i_2(x)\Phi^i_{n,2}(n^\theta y),$$

where $K^i_1(x)$ and $K^i_2(x)$ are two coefficients depending only on x. Since $P_n(n^\theta y) \sim \Psi_m(n,y) \sim \Phi^i_{n,1}(n^\theta y)$ for Re $y \in I$ and $0 < |\mathrm{Im}\, y| < \delta$. We obtain $K^i_1(x) = 1$ and $K^i_2(x)\Phi^i_{n,2}(n^\theta y)$ can be ignored in the asymptotic expansion.

According to Szegö [19, pages 395-396], the asymptotic formula of $P_n(n^\theta y)$ for y in the oscillatory interval I should be twice of the real part of $\Psi_{m,+}(n,y)$; see also [30]. Note that $\Psi_{m,+}(n,y)$ and $\Psi_{m,-}(n,y)$ are complex conjugates when $y \in I$. The argument in the previous paragraph suggests that, for any $m \geq 0$ and y in a complex neighborhood of oscillatory interval I,

$$P_n(n^\theta y) \sim \Psi_{m,-}(n,y) + \Psi_{m,+}(n,y), \tag{3.6}$$

as $n \to \infty$, where $\Psi_{m,\pm}(n,y)$ is defined in (3.5).

4. Examples

As an illustrative example, we consider the Hermite polynomials $H_n(x)$ which can be defined from the difference equation:

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \quad n \geq 1,$$

together with the initial conditions: $H_0(x) = 1$ and $H_1(x) = 2x$. Let

$$P_n(x) = \frac{\sqrt{\pi}H_n(x/\sqrt{2})}{2^n\Gamma(n/2+1/2)}.$$

We then have

$$P_{n+1}(x) + P_{n-1}(x) = \frac{x\Gamma(n/2+1/2)}{\sqrt{2}\Gamma(n/2+1)}P_n(x), \quad n \geq 1. \tag{4.1}$$

The initial conditions are $P_0(x) = 1$ and $P_1(x) = x\sqrt{\pi/2}$. Note that $B_n = 0$ and

$$A_n = \frac{\Gamma(n/2+1/2)}{\sqrt{2}\Gamma(n/2+1)} \sim n^{-1/2}\left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \cdots\right), \quad n \to \infty.$$

Thus, $\theta = 1/2$, $\beta_k = 0$, $\alpha_0 = 1$, $\alpha_1 = -1/4, \ldots$, and the scale is $x = n^{1/2}y$. Moreover, the leading coefficient of $P_n(x)$ is

$$A_0 \cdots A_{n-1} = \frac{\Gamma(1/2)}{2^{n/2}\Gamma(n/2+1/2)} \sim \frac{e^{n/2}}{\sqrt{2}n^{n/2}}, \quad n \to \infty.$$

This implies that $k_0 = 1/\sqrt{2}$. We observe $L(y) = y/2$, $R^o(y) = \sqrt{y^2-4}/2$, and

$$\int_y^\infty \frac{\alpha_0 n y^{1/\theta}}{2t^{1/\theta}R^o(t)}dt = \int_y^\infty \frac{ny^2}{t^2\sqrt{t^2-4}}dt = \frac{ny(y-\sqrt{y^2-4})}{4}.$$

Substituting the above formulas into (3.3) gives

$$P_n(n^{1/2}y) = \frac{1}{\sqrt{2}}\left(\frac{y+\sqrt{y^2-4}}{2}\right)^n\left(\frac{y+\sqrt{y^2-4}}{2\sqrt{y^2-4}}\right)^{1/2} \times \exp\left\{\frac{ny(y-\sqrt{y^2-4})}{4} + O\left(\frac{1}{n}\right)\right\} \tag{4.2}$$

for $y \in \mathbf{C} \setminus [-2, 2]$. A further application of (3.6) yields

$$\begin{aligned} P_n(n^{1/2}y) = &\frac{1}{\sqrt{2}}\left(\frac{y+i\sqrt{4-y^2}}{2}\right)^n\left(\frac{y+i\sqrt{4-y^2}}{2i\sqrt{4-y^2}}\right)^{1/2} \\ &\times \exp\left\{\frac{ny(y-i\sqrt{4-y^2})}{4} + O\left(\frac{1}{n}\right)\right\} \\ &+ \frac{1}{\sqrt{2}}\left(\frac{y-i\sqrt{4-y^2}}{2}\right)^n\left(\frac{y-i\sqrt{4-y^2}}{-2i\sqrt{4-y^2}}\right)^{1/2} \\ &\times \exp\left\{\frac{ny(y+i\sqrt{4-y^2})}{4} + O\left(\frac{1}{n}\right)\right\} \end{aligned} \tag{4.3}$$

for y in a complex neighborhood of $(-2, 0) \cup (0, 2)$. By analytic continuation, the above asymptotic expansion remains valid in a complex neighborhood of $y = 0$.

Now, we continue to calculate the higher-order terms. From (2.16) we obtain

$$\psi_2(y) = \frac{y^5 - 10y^3 - 36}{24(y^2-4)^{5/2}} + \frac{8t^2 - t^4}{8(y^2-4)^2}.$$

Substituting this into (2.22) gives

$$\phi_1(y) = \frac{1}{8} + \frac{2}{y^2(y^2-4)} - \frac{y^3+6y}{24(y^2-4)^{3/2}} + \frac{c_1}{y^2}.$$

Therefore,

$$\begin{aligned} P_n(n^{1/2}y) = &\frac{1}{\sqrt{2}}\left(\frac{y+\sqrt{y^2-4}}{2}\right)^n\left(\frac{y+\sqrt{y^2-4}}{2\sqrt{y^2-4}}\right)^{1/2} \exp\left\{\frac{ny(y-\sqrt{y^2-4})}{4}\right\} \\ &\times \exp\left\{\frac{1}{8n} + \frac{2}{ny^2(y^2-4)} - \frac{y^3+6y}{24n(y^2-4)^{3/2}} + \frac{c_1}{ny^2} + O\left(\frac{1}{n^2}\right)\right\} \end{aligned} \tag{4.4}$$

for $y \in \mathbf{C} \setminus [-2, 2]$. To determine the integration constant c_1, we need to match both sides of (4.4) by letting $n \to \infty$ and $y \to \infty$. Note from (4.1) and the initial conditions $P_0(x) = 1$ and $P_1(x) = x\sqrt{\pi/2}$ that

$$P_n(x) = A_0 \cdots A_{n-1}[x^n - \frac{n(n-1)}{2}x^{n-2} + \cdots].$$

Thus,

$$P_n(n^{1/2}y) \sim \frac{\Gamma(1/2)n^{n/2}y^n}{2^{n/2}\Gamma(n/2+1/2)}(1 - \frac{n-1}{2y^2}) \sim \frac{e^{n/2}y^n}{\sqrt{2}}(1 + \frac{1}{12n} - \frac{n-1}{2y^2}).$$

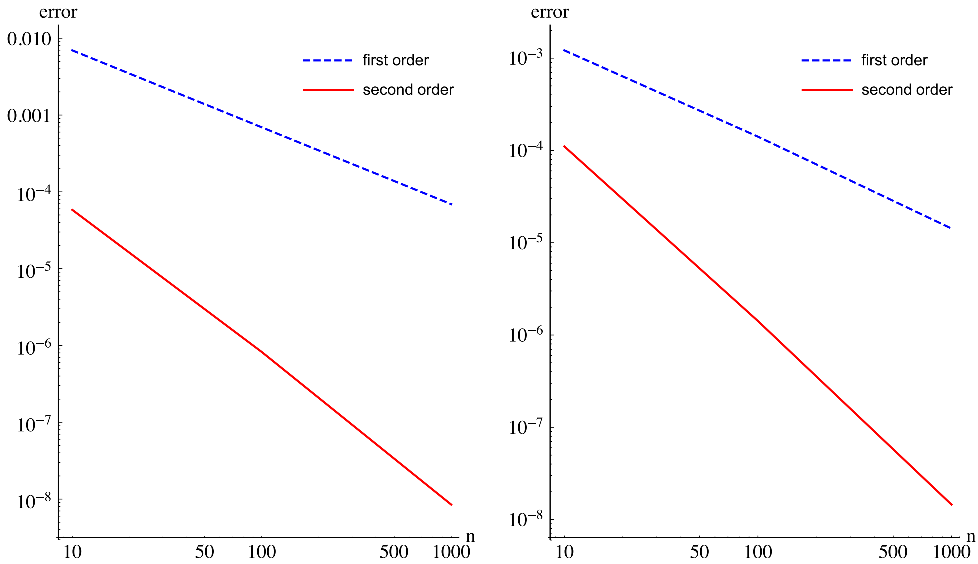

Fig. 1. The log–log plot of relative errors for the first-order and second order approximations of the Hermite polynomials, respectively. In the left panel, we choose $y = 3.3$ and use (4.2) and (4.4); while in the right panel, we set $y = 0.39$ and apply (4.3) and (4.5).

On the other hand, the right-hand side of (4.4) is asymptotically equal to

$$\frac{1}{\sqrt{2}} y^n \Big(1 - \frac{n}{y^2}\Big)\Big(1 + \frac{1}{2y^2}\Big) \exp\Big\{\frac{n}{2} + \frac{n}{2y^2} + \frac{1}{8n} - \frac{1}{24n} - \frac{1}{2ny^2} + \frac{c_1}{ny^2}\Big\}.$$

By asymptotic matching, we obtain $c_1 = 1/2$. It then follows from (3.6) that

$$\begin{aligned} P_n(n^{1/2}y) &= \frac{1}{\sqrt{2}}\left(\frac{y + i\sqrt{4-y^2}}{2}\right)^n \left(\frac{y + i\sqrt{4-y^2}}{2i\sqrt{4-y^2}}\right)^{1/2} \exp\left\{\frac{ny(y - i\sqrt{4-y^2})}{4}\right\} \\ &\times \exp\left\{\frac{1}{8n} + \frac{2}{ny^2(y^2-4)} + i\frac{y^3+6y}{24n(4-y^2)^{3/2}} + \frac{1}{2ny^2} + O\left(\frac{1}{n^2}\right)\right\} \\ &+ \frac{1}{\sqrt{2}}\left(\frac{y - i\sqrt{4-y^2}}{2}\right)^n \left(\frac{y - i\sqrt{4-y^2}}{-2i\sqrt{4-y^2}}\right)^{1/2} \exp\left\{\frac{ny(y + i\sqrt{4-y^2})}{4} + O\left(\frac{1}{n}\right)\right\} \\ &\times \exp\left\{\frac{1}{8n} + \frac{2}{ny^2(y^2-4)} - i\frac{y^3+6y}{24n(4-y^2)^{3/2}} + \frac{1}{2ny^2} + O\left(\frac{1}{n^2}\right)\right\} \end{aligned} \tag{4.5}$$

for y in a complex neighborhood of $(-2, 2)$.

To illustrate the accuracy of our asymptotic expansions, we choose $y = 3.3$ and use (4.2) and (4.4), respectively, to approximate the Hermite polynomial with different values of n. From the log–log plot (see left panel in Fig. 1), we observe that the first-order approximation (4.2) has linear convergence rate, while the second-order approximation (4.4) converges quadratically. Similarly, we choose $y = 0.39$ and use (4.3) and (4.5), respectively, to approximate the Hermite polynomials in the oscillatory region. Again, we observe from the right panel of Fig. 1 that (4.3) is accurate up to an error of order $O(1/n)$, while (4.5) is accurate with error $O(1/n^2)$.

To demonstrate that our systematic method has a wide application, we consider the monic continuous dual Hahn polynomials [16, (9.3.1)]

$$\pi_n(z) = (-1)^n S_n(x^2; a, b, c) = (-1)^n (a+b)_n (a+c)_n {}_3F_2\left(\left. \begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix} \right| 1 \right),$$

where a, b, c are parameters and $z = x^2$. The corresponding difference equation is given by [16, (9.3.5)]

$$\begin{aligned} \pi_{n+1}(z) =& [z - 2n^2 - (2a+2b+2c-1)n - ab - bc - ca]\pi_n(z) \\ & - n(n+a+b-1)(n+b+c-1)(n+c+a-1)\pi_{n-1}(z), \end{aligned}$$

with initial conditions $\pi_0(z) = 1$ and $\pi_1(z) = z - ab - bc - ca$. By setting $P_n(z) = \gamma_n \pi_n(z)$ with

$$\gamma_n = \frac{\Gamma((a+b)/2)\Gamma((b+c)/2)\Gamma((c+a)/2)\Gamma(1/2)}{4^n \Gamma((n+a+b)/2)\Gamma((n+b+c)/2)\Gamma((n+c+a)/2)\Gamma((n+1)/2)},$$

we arrive at the difference Eq. (1.1), where the coefficients A_n and B_n have the asymptotic expansions in (1.2). Especially, one can calculate $\theta = 2$, $\alpha_0 = 1$, $\beta_0 = 2$, $\alpha_1 = 1/2 - a - b - c$, $\beta_1 = 0$, $\alpha_2 = a^2 + b^2 + c^2 + 3(ab + bc + ca)/2 - (a + b + c) + 1/8$, and $\beta_2 = -1/4$. Since $A_n = \gamma_{n+1}/\gamma_n$ and $\gamma_0 = 1$, we have

$$A_0 \cdots A_{n-1} = \gamma_n \sim k_0 n^{3/2-a-b-c} (e/n)^{2n},$$

where

$$k_0 = 2^{a+b+c-7/2} \pi^{-2} \Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{b+c}{2}\right) \Gamma\left(\frac{c+a}{2}\right) \Gamma\left(\frac{1}{2}\right).$$

A further computation on the expression (3.3) gives

$$\begin{aligned} P_n(n^2 y) =& k_0 n^{3/2-a-b-c} y^n \left(\frac{y}{y-4}\right)^{1/4} \left(\frac{y/2 - 1 + \sqrt{y^2/4 - y}}{y}\right)^{n+(a+b+c)/2-1/4} \\ & \times \exp\left\{ -in\sqrt{y} \ln \frac{\sqrt{y-4} + 2i}{\sqrt{y}} + \frac{\phi_1(y)}{n} + O\left(\frac{1}{n^2}\right) \right\} \end{aligned} \quad (4.6)$$

for $y \in \mathbf{C} \setminus [0, 4]$, where

$$\phi_1(y) = -\alpha_2 \sqrt{\frac{y-4}{y}} - \frac{\alpha_1(y-2)}{2(y-4)} + \frac{\alpha_1^2(y-6)}{2\sqrt{y(y-4)}} - \frac{2y^2 - 11y + 32}{12(y-4)\sqrt{y(y-4)}} + \frac{c_1}{y^{1/2}}. \quad (4.7)$$

By a tedious but simple asymptotic matching, we find $c_1 = 0$. Numerical simulation (not shown here) also verifies that the asymptotic formula has an accuracy of second order.

5. Discussion

In this paper, we provide a unified method to derive asymptotic expansion for orthogonal polynomials from their difference equations. The key idea is to find two linearly independent solutions to the difference equation whose logarithms have asymptotic expansions in terms of powers of the polynomial degree. This logarithmic-type asymptotic method has the advantage of preserving the symmetry in the difference equation, and thus we are able to find explicit expressions for the higher-order terms. To be more specific, the coefficient functions in the asymptotic expansion can be explicitly written as a simple integral whose integrand has nice structure and satisfies a recurrence relation; see (2.22), Theorem 2.4 and Corollary 2.10.

Therefore, it is easy to compute the integrands and obtain an asymptotic expansion up to an arbitrary high order.

Another contribution of this paper is to connect the general asymptotic solutions of the difference equation with initial conditions by applying the principle of asymptotic matching. As a result, we derive asymptotic expansions for the orthogonal polynomials in the outer and oscillatory regions, respectively. For the sake of simplicity, we only use Hermite polynomials as an illustrative example, but it is remarked that our formula is general such that it can be directly applied to any difference equation that can be converted to the symmetric form (1.1) with coefficients satisfying a relatively general asymptotic condition (1.2). It is not surprising, but still worthwhile noting that, to obtain an asymptotic expansion of the general solution up to mth order, one would need and only need the $(m+1)$th asymptotic expansion for the coefficients in the difference equation. However, to find asymptotic expansion for the orthogonal polynomials with given initial conditions, one would need more global information on the coefficients such as the product $A_0 \cdots A_{n-1}$, and so on.

It is noted that we have restricted our focus only on asymptotic analysis in the outer and oscillatory regions. For the study of uniform asymptotic expansions near the turning points, we refer the authors to the papers by Wang-Wong [22–24] and by Cao-Li [6]. There is still one interesting problem which has not been resolved in this paper or in the literature; that is, how to find a uniform asymptotic expansion for the orthogonal polynomials near the origin from their difference equations. For Hermite polynomials, this is not a problem because the asymptotic formula can be analytically continued to a complex neighborhood of the origin. However, in many other cases when the asymptotic expansion (3.3) has non-removable singularity near the origin, it becomes a challenging problem to cancel out the singularity. One may need to introduce an auxiliary function (gamma function, for example). We will leave this as a future project.

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