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Plancherel–Rotach asymptotics of second-order difference equations with linear coefficients

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Abstract

In this paper, we provide a complete Plancherel–Rotach asymptotic analysis of polynomials that satisfy a second-order difference equation with linear coefficients. According to the signs of the parameters, we classify the difference equations into six cases and derive explicit asymptotic formulas of the polynomials in the outer and oscillatory regions, respectively. It is remarkable that the zero distributions of the polynomials may locate on the imaginary line or even on a sideways Y-shape curve in some cases. Finally, we apply our results to find asymptotic formulas for associated Hermite and associated Charlier polynomials.

MSC: 39A06; 41A60

Keywords: Difference equations; Polynomials; Plancherel–Rotach asymptotics; Zero distributions

1. Introduction

All of the classical hypergeometric (monic) orthogonal polynomials $\pi_n(x)$ within Askey scheme [10] satisfy the following second-order linear difference equation:

$$\pi_{n+1}(x) = (x - A_n)\pi_n(x) - B_n\pi_{n-1}(x), \quad \pi_0(x) = 1,\ \pi_1(x) = x - A_0, \tag{1.1}$$

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where the coefficients A_n and B_n are polynomials or rational functions of n. For instance, the Charlier polynomials correspond to $A_n = n + a$ and $B_n = na$; the Hermite polynomials correspond to $A_n = 0$ and $B_n = n/2$; and the Chebyshev polynomials correspond to $A_n = 0$ and $B_n = 1/4$. In this paper, we will provide a complete Plancherel–Rotach asymptotic analysis of second-order difference equations with linear coefficients, namely, A_n and B_n are linear functions of n. Upon a shift on x, we may assume $A_n = dn$ and $B_n = an + b$.

There are plenty of methods developed for asymptotic analysis of orthogonal polynomials: if the polynomials can be expressed in terms of an integral, one may adopt the classical Laplace's method and steepest-descent method [19]; if the polynomials satisfy a second-order linear differential equation, the well-known WKB method [12] can be applied; if the polynomials have an explicit orthogonal weight with certain nice properties, we may use the Riemann–Hilbert approach and Deift–Zhou nonlinear steepest-descent method [2,5,6]. However, few studies in the previous literature were considering asymptotic analysis of polynomials via difference equations due to the loss of continuity. Van Assche and Geronimo [13] did some pioneer works in this field and obtained asymptotic formulas in the outer region, where trapezoidal rule was used to build a bridge from discreteness to continuity. Wong and Li [21,22] derived two linearly independent solutions in the oscillatory region, while determining the coefficients of the linear combination of the two solutions with given initial values was left as an open problem. In a series of work [15–18], Wang and Wong established some beautiful lemmas on Airy functions and Bessel functions to derive uniform asymptotic formulas near the turning points. Wang and Wong classified the turning points into three cases and considered two of them in [16,17]. The third case was recently resolved by Cao and Li [3]. It is noted that the turning point theory developed in [15–18] and [3] was based on the assumption that the asymptotic formulas in the oscillatory region were given. In [14], Wang and Wong completed this framework by introducing a matching method to determine the coefficients of linear combination of Wong–Li solutions in the oscillatory region from Van Assche–Geronimo solutions in the outer region. Therefore, a systematic method of asymptotic analysis on difference equations was formulated. This method was successively applied in the study of several indeterminate moment problems [4] where only difference equations were known and thus the classical Laplace's method, steepest descent method, WKB method, Riemann–Hilbert approach and Deift–Zhou nonlinear steepest-descent method seem to be inapplicable. For a review of asymptotic analysis for difference equations, we refer to the survey paper by Wong [20].

To further develop the difference equation technique, we study a general second-order linear difference equation with linear coefficients. We are interested in the Plancherel–Rotach asymptotic formulas of solutions in the outer region and oscillatory region. According to the signs of the parameters d and a, we classify the equations into six cases: (I.A) $d > 0$ and $a > 0$; (I.B) $d > 0$ and $a < 0$; (I.C) $d > 0$ and $a = 0$; (II.A) $d = 0$ and $a > 0$; (II.B) $d = 0$ and $a < 0$; (II.C) $d = 0$ and $a = 0$. The cases with $d < 0$ can be transformed to the cases with $d > 0$ by a simple reflection. Note that the classical orthogonal polynomials (Charlier, Hermite and Chebyshev, for instance) always have nonnegative $a \geq 0$ and their zeros are always real. However, if we choose $a < 0$, as we shall see later, the zero distributions of the polynomials $\pi_n(x)$ may lie on the imaginary line (subcase II.B) or even on a sideways Y-shape curve (subcase I.B).

The rest of this paper is organized as follows. In Section 2, we focus on the case $d \neq 0$ and divide this case into three subcases according to the sign of a. In Section 3, we investigate the special case $d = 0$ and again consider three subcases $a > 0$, $a < 0$ and $a = 0$ in three subsections, respectively. In Section 4, we derive asymptotic formulas for associated Hermite and associated Charlier polynomials using our results.

2. Case I: $d \neq 0$

Upon a transformation $x \to -x$ and $\pi_n \to (-1)^n \pi_n$, we may assume without loss of generality that $d > 0$. In the following three subsections, we shall consider three subcases $a > 0$, $a < 0$ and $a = 0$, respectively.

2.1. Subcase I.A: $a > 0$

We first state our theorem.

Theorem 2.1. *Assume $d > 0$ and $a > 0$. Let $x = ny$ and $y = d + z/\sqrt{n}$. The polynomials $\pi_n(x)$ defined in (1.1) have the following large-n asymptotics. As $n \to \infty$, for $z \in \mathcal{C} \setminus [-\sqrt{n}d, 2\sqrt{a}]$, we have*

$$\pi_n(nd+\sqrt{n}z) \sim (n/e)^n \left(\frac{z+\sqrt{z^2-4a}}{2\sqrt{n}}\right)^n \left(\frac{z+\sqrt{z^2-4a}}{2(\sqrt{n}d+z)}\right)^{-a/d^2-\sqrt{n}(\sqrt{n}d+z)/d} \times \left(\frac{\sqrt{n}d+z}{\sqrt{z^2-4a}}\right)^{1/2} \times \exp\left[\frac{2a-z^2-4\sqrt{n}dz+(z+4\sqrt{n}d)\sqrt{z^2-4a}}{4d^2}\right]; \tag{2.1}$$

and for z in a complex neighborhood of any compact subset in $(-2\sqrt{a}, 2\sqrt{a})$, we have

$$\pi_n(nd+\sqrt{n}z) \sim (n/e)^n \left(\frac{\sqrt{a}}{\sqrt{n}}\right)^{-a/d^2-\sqrt{n}z/d} \left(d+\frac{z}{\sqrt{n}}\right)^{a/d^2+\sqrt{n}(\sqrt{n}d+z)/d} \times \left(\frac{\sqrt{n}d+z}{\sqrt{4a-z^2}}\right)^{1/2} \exp\left(\frac{2a-z^2-4\sqrt{n}dz}{4d^2}\right) \times 2\cos\left[\left(-\frac{a}{d^2}-\frac{\sqrt{n}z}{d}\right)\arccos\frac{z}{2\sqrt{a}}-\frac{\pi}{4}+\frac{(z+4\sqrt{n}d)\sqrt{4a-z^2}}{4d^2}\right]; \tag{2.2}$$

and for z in a complex neighborhood of any compact subset in $(-\sqrt{n}d, -2\sqrt{a})$, we have

$$\pi_n(nd+\sqrt{n}z) \sim (n/e)^n \left(\frac{-z+\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}{2\sqrt{n}}\right)^{-a/d^2-\sqrt{n}z/d} \times (d+z/\sqrt{n})^{a/d^2+\sqrt{n}(\sqrt{n}d+z)/d} \left(\frac{\sqrt{n}d+z}{\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}\right)^{1/2} \times \exp\left[\frac{2a-z^2-4\sqrt{n}dz-(z+4\sqrt{n}d)\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}{4d^2}\right] \times 2\cos[\pi(-a/d^2-\sqrt{n}z/d-1/2)]. \tag{2.3}$$

Proof. For x in the outer region such that $\pi_{k-1}(x) \neq 0$, we define

$$w_k(x) := \frac{\pi_k(x)}{\pi_{k-1}(x)}$$

for any $k \geq 1$. It follows that

$$w_{k+1}(x) = x - dk - \frac{ak+b}{w_k(x)}, \quad k \geq 1.$$

Let $x = ny$ with $y \in \mathcal{C} \setminus [0, d + 2\sqrt{a}/\sqrt{n}]$. We have as $n \to \infty$,

$$w_k(x) \sim \frac{x - dk + \sqrt{(x-dk)^2 - 4ak}}{2} \times \left\{ 1 + \frac{d}{2\sqrt{(x-dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x-dk)^2 - 4ak]} \right\}.$$

The above asymptotic formula can be obtained by successive approximation and proved rigorously by induction on k. Since $(x-dk)^2 - 4ak$ is of order $O(n^2)$ for any $k = 1, \ldots, n$, we have

$$\ln \pi_n \sim \sum_{k=1}^{n} \left\{ \ln \frac{x - dk + \sqrt{(x-dk)^2 - 4ak}}{2} + \frac{d}{2\sqrt{(x-dk)^2 - 4ak}} + \frac{dx - d^2k}{2[(x-dk)^2 - 4ak]} \right\}.$$

We will use trapezoidal rule to approximate the three summations on the right-hand side of the above formula. Firstly, we obtain

$$\begin{aligned} &\sum_{k=1}^{n} \ln \frac{x - dk + \sqrt{(x-dk)^2 - 4ak}}{2} \\ &\quad = \sum_{k=1}^{n} \ln \frac{ny - dk + \sqrt{(ny-dk)^2 - 4ak}}{2} \\ &\quad \sim n \ln \frac{n}{2} + n \int_0^1 \ln[y - dt + \sqrt{(y-dt)^2 - 4at/n}]dt \\ &\qquad + \frac{1}{2} \ln \frac{y - d + \sqrt{(y-d)^2 - 4a/n}}{2y}. \end{aligned}$$

A simple integration gives

$$\begin{aligned} &n \int_0^1 \ln[y - dt + \sqrt{(y-dt)^2 - 4at/n}]dt \\ &\quad \sim n \left\{ t \ln[y - dt + \sqrt{(y-dt)^2 - 4at/n}] + \frac{\sqrt{(y-dt)^2 - 4at/n}}{2d} - \frac{t}{2} \right. \\ &\qquad \left. - \left(\frac{a}{nd^2} + \frac{y}{d} \right) \ln[dy + 2a/n - d^2t + d\sqrt{(y-dt)^2 - 4at/n}] \right\} \Big|_0^1 \\ &\quad \sim n \ln[y - d + \sqrt{(y-d)^2 - 4a/n}] + \frac{n}{2d}(\sqrt{(y-d)^2 - 4a/n} - y) - \frac{n}{2} \\ &\qquad - \left(\frac{a}{d^2} + \frac{ny}{d} \right) \ln \frac{y - d + \sqrt{(y-d)^2 - 4a/n} + 2a/(nd)}{2y + 2a/(nd)}. \end{aligned}$$

For the sake of convenience, we introduce a new scale: $y = d + z/\sqrt{n}$ with $z \in \mathcal{C} \setminus [-\sqrt{n}d, 2\sqrt{a}]$. It follows from the above two formulas that

$$\sum_{k=1}^{n} \ln \frac{x - dk + \sqrt{(x-dk)^2 - 4ak}}{2}$$

$$\sim n\ln\frac{n}{2}+n\ln\frac{z+\sqrt{z^2-4a}}{\sqrt{n}}+\frac{\sqrt{n}}{2d}(\sqrt{z^2-4a}-z)-n$$
$$-\left(\frac{a}{d^2}+\frac{\sqrt{n}(\sqrt{n}d+z)}{d}\right)\ln\frac{z+\sqrt{z^2-4a}+2a/(\sqrt{n}d)}{2(\sqrt{n}d+z)+2a/(\sqrt{n}d)}+\frac{1}{2}\ln\frac{z+\sqrt{z^2-4a}}{2(\sqrt{n}d+z)}.$$

A further application of trapezoidal rule yields

$$\sum_{k=1}^{n}\left\{\frac{d}{2\sqrt{(x-dk)^2-4ak}}+\frac{dx-d^2k}{2[(x-dk)^2-4ak]}\right\}$$
$$\sim\int_0^1\frac{d}{2\sqrt{(y-dt)^2-4at/n}}+\frac{dy-d^2t}{2[(y-dt)^2-4at/n]}dt$$
$$\sim\frac{1}{2}\ln\frac{2(\sqrt{n}d+z)}{z+\sqrt{z^2-4a}}+\frac{1}{2}\ln\frac{\sqrt{n}d+z}{\sqrt{z^2-4a}}.$$

Adding the above two formulas gives

$$\ln\pi_n\sim n\ln\frac{n}{2}+n\ln\frac{z+\sqrt{z^2-4a}}{\sqrt{n}}+\frac{\sqrt{n}}{2d}(\sqrt{z^2-4a}-z)-n$$
$$-\left(\frac{a}{d^2}+\frac{\sqrt{n}(\sqrt{n}d+z)}{d}\right)\ln\frac{z+\sqrt{z^2-4a}+2a/(\sqrt{n}d)}{2(\sqrt{n}d+z)+2a/(\sqrt{n}d)}+\frac{1}{2}\ln\frac{\sqrt{n}d+z}{\sqrt{z^2-4a}}.$$

Since

$$\ln\frac{z+\sqrt{z^2-4a}+2a/(\sqrt{n}d)}{2(\sqrt{n}d+z)+2a/(\sqrt{n}d)}\sim\ln\frac{z+\sqrt{z^2-4a}}{2(\sqrt{n}d+z)}+\frac{2a}{(\sqrt{n}d)(z+\sqrt{z^2-4a})}$$
$$-\frac{2a^2}{d^2n(z+\sqrt{z^2-4a})^2}-\frac{a}{(\sqrt{n}d)(\sqrt{n}d+z)},$$

we have

$$\ln\pi_n\sim n\ln n+n\ln\frac{z+\sqrt{z^2-4a}}{2\sqrt{n}}-n-\left(\frac{a}{d^2}+\frac{\sqrt{n}(\sqrt{n}d+z)}{d}\right)\ln\frac{z+\sqrt{z^2-4a}}{2(\sqrt{n}d+z)}$$
$$+\frac{\sqrt{n}(\sqrt{z^2-4a}-z)}{2d}-\left[\frac{2a(\sqrt{n}d+z)}{(d^2)(z+\sqrt{z^2-4a})}\right.$$
$$\left.-\frac{2a^2}{d^2(z+\sqrt{z^2-4a})^2}-\frac{a}{(d^2)}\right]+\frac{1}{2}\ln\frac{\sqrt{n}d+z}{\sqrt{z^2-4a}}.$$

A simple calculation yields

$$\frac{\sqrt{n}(\sqrt{z^2-4a}-z)}{2d}-\left[\frac{2a(\sqrt{n}d+z)}{(d^2)(z+\sqrt{z^2-4a})}-\frac{2a^2}{d^2(z+\sqrt{z^2-4a})^2}-\frac{a}{(d^2)}\right]$$
$$=\frac{\sqrt{n}d(\sqrt{z^2-4a}-z)}{2d^2}-\left[\frac{(\sqrt{n}d+z)(z-\sqrt{z^2-4a})}{(2d^2)}-\frac{(z-\sqrt{z^2-4a})^2}{8d^2}-\frac{a}{(d^2)}\right]$$
$$=\frac{2\sqrt{n}d(\sqrt{z^2-4a}-z)}{4d^2}-\left[\frac{2(\sqrt{n}d+z)(z-\sqrt{z^2-4a})}{(4d^2)}\right.$$

$$- \frac{(z^2 - 2a - z\sqrt{z^2-4a})}{4d^2} - \frac{4a}{4d^2}\Bigg]$$
$$= \frac{-z^2 + 2a - 4\sqrt{n}dz + (z + 4\sqrt{n}d)\sqrt{z^2-4a}}{4d^2}.$$

Consequently,

$$\ln \pi_n \sim n \ln n - n + n \ln \frac{z + \sqrt{z^2-4a}}{2\sqrt{n}} - \left(\frac{a}{d^2} + \frac{\sqrt{n}(\sqrt{n}d + z)}{d}\right) \ln \frac{z + \sqrt{z^2-4a}}{2(\sqrt{n}d + z)}$$
$$+ \frac{-z^2 + 2a - 4\sqrt{n}dz + (z + 4\sqrt{n}d)\sqrt{z^2-4a}}{4d^2} + \frac{1}{2} \ln \frac{\sqrt{n}d + z}{\sqrt{z^2-4a}}.$$

Recall that $x = ny$ and $y = d + z/\sqrt{n}$. For any $z \in \mathcal{C} \setminus [-\sqrt{n}d, 2\sqrt{a}]$, we have $\pi_n(nd + \sqrt{n}z) \sim \Phi_n(z)$ as $n \to \infty$, where

$$\Phi_n(z) := (n/e)^n \left(\frac{z + \sqrt{z^2-4a}}{2\sqrt{n}}\right)^n \left(\frac{z + \sqrt{z^2-4a}}{2(\sqrt{n}d + z)}\right)^{-a/d^2 - \sqrt{n}(\sqrt{n}d+z)/d}$$
$$\times \left(\frac{\sqrt{n}d + z}{\sqrt{z^2-4a}}\right)^{1/2} \times \exp\left[\frac{2a - z^2 - 4\sqrt{n}dz + (z + 4\sqrt{n}d)\sqrt{z^2-4a}}{4d^2}\right].$$

This proves (2.1). Note that $\Phi_n(z)$ has a branch cut on $[-\sqrt{n}d, 2\sqrt{a}]$. We take the one-sided limits and define

$$\Phi_n^{\pm}(z) := \lim_{\varepsilon \to 0^+} \Phi_n(z \pm i\varepsilon), \quad z \in (-\sqrt{n}d, 2\sqrt{a}).$$

It is readily seen that $\Phi_n^{\pm}(z)$ can be analytically extended to a neighborhood of $(-\sqrt{n}d, 2\sqrt{a})$. Moreover, if $z = z_1 + iz_2$ with $z_1 \in (-\sqrt{n}d, 2\sqrt{a})$ and $z_2 > 0$, then $\Phi_n(z) = \Phi_n^+(z)$ and $\Phi_n^-(z)/\Phi_n^+(z)$ is exponentially small as $n \to \infty$. On the other hand, if $z = z_1 + iz_2$ with $z_1 \in (-\sqrt{n}d, 2\sqrt{a})$ and $z_2 < 0$, then $\Phi_n(z) = \Phi_n^-(z)$ and $\Phi_n^+(z)/\Phi_n^-(z)$ is exponentially small as $n \to \infty$. It follows that as $n \to \infty$, $\pi_n(nd + \sqrt{n}z) \sim \Phi_n(z) \sim \Phi_n^+(z) + \Phi_n^-(z)$ for all $z = z_1 + iz_2$ with $z_1 \in (-\sqrt{n}d, 2\sqrt{a})$ and $z_2 \neq 0$. By analytically continuity, we obtain $\pi_n(nd + \sqrt{n}z) \sim \Phi_n^+(z) + \Phi_n^-(z)$ for z in a complex neighborhood of any compact subset in $(-\sqrt{n}d, 2\sqrt{a})$. A simple calculation gives

$$\Phi_n^+(z) + \Phi_n^-(z) = (n/e)^n \left(\frac{\sqrt{a}}{\sqrt{n}}\right)^{-a/d^2 - \sqrt{n}z/d}$$
$$\times (d + z/\sqrt{n})^{a/d^2 + \sqrt{n}(\sqrt{n}d+z)/d} \left(\frac{\sqrt{n}d + z}{\sqrt{4a - z^2}}\right)^{1/2} \exp\left(\frac{2a - z^2 - 4\sqrt{n}dz}{4d^2}\right)$$
$$\times 2\cos\left[\left(-\frac{a}{d^2} - \frac{\sqrt{n}z}{d}\right) \arccos\frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z + 4\sqrt{n}d)\sqrt{4a - z^2}}{4d^2}\right]$$

for z in a complex neighborhood of any compact subset in $(-2\sqrt{a}, 2\sqrt{a})$, and

$$\Phi_n^+(z) + \Phi_n^-(z) = (n/e)^n \left(\frac{-z + \sqrt{-z - 2\sqrt{a}}\sqrt{-z + 2\sqrt{a}}}{2\sqrt{n}}\right)^{-a/d^2 - \sqrt{n}z/d}$$

$$\times (d+z/\sqrt{n})^{a/d^2+\sqrt{n}(\sqrt{n}d+z)/d}\left(\frac{\sqrt{n}d+z}{\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}\right)^{1/2}$$
$$\times \exp\left[\frac{2a-z^2-4\sqrt{n}dz-(z+4\sqrt{n}d)\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}{4d^2}\right]$$
$$\times 2\cos[\pi(-a/d^2-\sqrt{n}z/d-1/2)]$$

for z in a complex neighborhood of any compact subset in $(-\sqrt{n}d, -2\sqrt{a})$. This completes the proof of (2.2) and (2.3). □

2.2. Case I.B: $a < 0$

For the case $a < 0$, we observe from numerical simulation that the zeros of π_n are not solely lying on the real line, instead, they will locate on a sideways Y-shape curve (cf. Fig. 1). To describe the Y-shape curve, we introduce the following definition.

Definition 2.2. Given $A > 0$, we define Γ_A to be the curve in the left-half complex plane satisfying the following equation

$$\mathrm{Re}\left\{2\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}-z\ln\frac{z+\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{z-\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}\right\}=0. \tag{2.4}$$

It is noted that the above equation formulates a sideways V-shape curve that is symmetric about the x-axis with two end points $\pm 2i\sqrt{A}$; see Fig. 1. Let $z_A < 0$ be the intersection of Γ_A with the negative real line. To be specific, z_A is the negative real root of the following equation

$$2\sqrt{z_A^2+4A}-z_A\ln\frac{z_A+\sqrt{z_A^2+4A}}{-z_A+\sqrt{z_A^2+4A}}=0. \tag{2.5}$$

Now, we describe the sideways Y-shape curve as the union of the curve Γ_A defined in (2.4) and the interval $[-\sqrt{n}d, z_A]$. We have the following theorem.

Theorem 2.3. *Assume $d > 0$ and $a < 0$. Let $x = ny$ and $y = d + z/\sqrt{n}$. Denote $A = -a > 0$ and let Γ_A and z_A be defined as in Definition 2.2. The polynomials $\pi_n(x)$ defined in (1.1) have the following large-n asymptotics. As $n \to \infty$, we have for $z \in \mathcal{C} \setminus ([-\sqrt{n}d, z_A] \cup \Gamma_A)$,*

$$\pi_n(nd+\sqrt{n}z) \sim (n/e)^n\left(\frac{z+\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{2\sqrt{n}}\right)^{A/d^2-\sqrt{n}z/d}$$
$$\times (d+z/\sqrt{n})^{-A/d^2+\sqrt{n}(\sqrt{n}d+z)/d}\left(\frac{\sqrt{n}d+z}{\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}\right)^{1/2}$$
$$\times \exp\left[\frac{-2A-z^2-4\sqrt{n}dz+(z+4\sqrt{n}d)\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{4d^2}\right]; \tag{2.6}$$

and for z in a complex neighborhood of any compact subset in $(-\sqrt{n}d, z_A)$, we have

$$\pi_n(nd+\sqrt{n}z) \sim (n/e)^n\left(\frac{-z+\sqrt{-z-2i\sqrt{A}}\sqrt{-z+2i\sqrt{A}}}{2\sqrt{n}}\right)^{A/d^2-\sqrt{n}z/d}$$

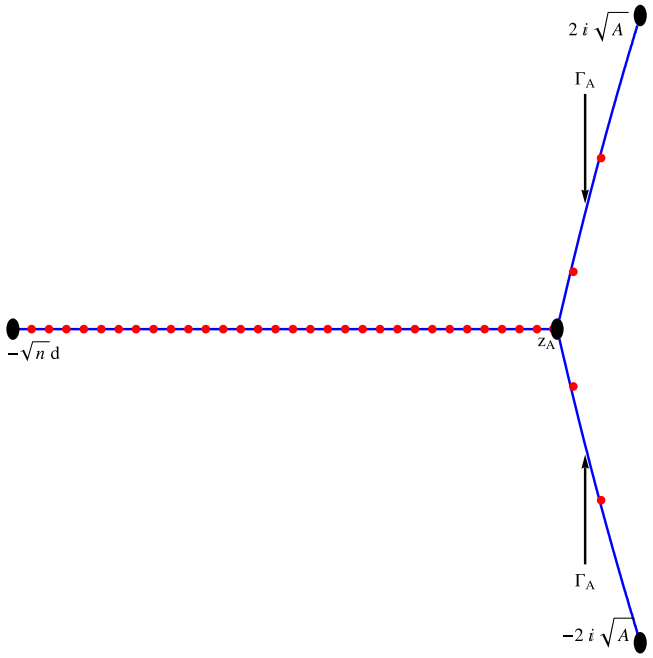

Fig. 1. The sideways Y-shape branch cut (curve) and zero distribution (dots).

$$\times (d + z/\sqrt{n})^{-A/d^2+\sqrt{n}(\sqrt{n}d+z)/d} \left(\frac{\sqrt{n}d + z}{\sqrt{-z - 2i\sqrt{A}}\sqrt{-z + 2i\sqrt{A}}} \right)^{1/2}$$
$$\times \exp\left[\frac{-2A - z^2 - 4\sqrt{n}dz - (z + 4\sqrt{n}d)\sqrt{-z - 2i\sqrt{A}}\sqrt{-z + 2i\sqrt{A}}}{4d^2} \right]$$
$$\times 2\cos[\pi(A/d^2 - \sqrt{n}z/d - 1/2)]; \tag{2.7}$$

and for z in a complex neighborhood of any compact subset in $\mathring{\Gamma}_A := \Gamma_A \setminus \{z_A, \pm 2i\sqrt{A}\}$, we have

$$\pi_n(nd + \sqrt{n}z) \sim (\sqrt{n}/e)^n (\sqrt{n}d + z)^{-A/d^2+\sqrt{n}(\sqrt{n}d+z)/d+1/2}$$
$$\times \exp\left(\frac{-2A - z^2 - 4\sqrt{n}dz}{4d^2} \right)$$
$$\times \left\{ \frac{[(z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2 - \sqrt{n}z/d}}{(\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}} \right.$$
$$\times \exp\left[\frac{(z + 4\sqrt{n}d)\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right]$$
$$+ \frac{[(z - \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2 - \sqrt{n}z/d}}{(-\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}}$$
$$\left. \times \exp\left[\frac{-(z + 4\sqrt{n}d)\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right] \right\}. \tag{2.8}$$

Proof. Similar to the proof of Theorem 2.1, we obtain for $z \in \mathcal{C} \setminus ([-\sqrt{n}d, z_A] \cup \Gamma_A)$,

$$\begin{aligned}
\pi_n(nd+\sqrt{n}z) &\sim (n/e)^n \left(\frac{z+\sqrt{z^2-4a}}{2\sqrt{n}}\right)^{-a/d^2-\sqrt{n}z/d} \\
&\quad \times (d+z/\sqrt{n})^{a/d^2+\sqrt{n}(\sqrt{n}d+z)/d} \left(\frac{\sqrt{n}d+z}{\sqrt{z^2-4a}}\right)^{1/2} \\
&\quad \times \exp\left[\frac{2a-z^2-4\sqrt{n}dz+(z+4\sqrt{n}d)\sqrt{z^2-4a}}{4d^2}\right] \\
&\sim (n/e)^n \left(\frac{z+\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{2\sqrt{n}}\right)^{A/d^2-\sqrt{n}z/d} \\
&\quad \times (d+z/\sqrt{n})^{-A/d^2+\sqrt{n}(\sqrt{n}d+z)/d} \left(\frac{\sqrt{n}d+z}{\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}\right)^{1/2} \\
&\quad \times \exp\left[\frac{-2A-z^2-4\sqrt{n}dz+(z+4\sqrt{n}d)\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{4d^2}\right].
\end{aligned}$$

This gives (2.6). Denote the right-hand side of (2.6) by $\Phi_n(z)$. Note that $\Phi_n(z)$ is analytic on the complex plane except for a Y-shape branch cut $[-\sqrt{n}d, z_A] \cup \Gamma_A$ that connects $-\sqrt{n}d$ and $\pm 2i\sqrt{A}$. Moreover, $\Phi_n(z)$ is one-side continuous on the branch cut. Therefore, the functions

$$\Phi_n^{\pm}(z) := \lim_{\varepsilon \to 0^+} \Phi_n(z + i\varepsilon)$$

are analytic in a complex neighborhood of any compact subset in the branch cut. Note that for z in a complex neighborhood of any compact subset in $(-\infty, z_A)$,

$$\frac{\Phi_n^+(z)}{\Phi_n^-(z)} = \exp[2\pi i(A/d^2 - \sqrt{n}z/d - 1/2)];$$

and for z in a complex neighborhood of any compact subset in $\mathring{\Gamma}_A$,

$$\begin{aligned}
\frac{\Phi_n^+(z)}{\Phi_n^-(z)} &= -i\left(\frac{z+\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{z-\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}\right)^{A/d^2-\sqrt{n}z/d} \\
&\quad \times \exp\left(\frac{z+4\sqrt{n}d}{2d^2}\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}\right).
\end{aligned}$$

It follows from the definition of Γ_A in (2.4) that the ratio Φ_n^+/Φ_n^- is exponentially large on one side and exponentially small on the other side of the branch cut $[-\sqrt{n}d, z_A] \cup \Gamma_A$. Using a similar argument in the proof of Theorem 2.1, we obtain $\pi_n(nd+\sqrt{n}z) \sim [\Phi_n^+(z) + \Phi_n^-(z)]$ for z in a complex neighborhood of any compact subset in $(-\sqrt{n}d, z_A) \cup \mathring{\Gamma}_A$. A simple calculation yields

$$\Phi_n^+(z) + \Phi_n^-(z) = (n/e)^n \left(\frac{-z+\sqrt{z-2i\sqrt{A}}\sqrt{z+2i\sqrt{A}}}{2\sqrt{n}}\right)^{A/d^2-\sqrt{n}z/d}$$

$$\times (d + z/\sqrt{n})^{-A/d^2+\sqrt{n}(\sqrt{n}d+z)/d} \left(\frac{\sqrt{n}d + z}{\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}} \right)^{1/2}$$
$$\times \exp\left[\frac{-2A - z^2 - 4\sqrt{n}dz - (z + 4\sqrt{n}d)\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right]$$
$$\times 2\cos[\pi(A/d^2 - \sqrt{n}z/d - 1/2)]$$

for z in a complex neighborhood of any compact subset in $(-\sqrt{n}d, z_A)$, and

$$\Phi_n^+(z) + \Phi_n^-(z)$$
$$= (\sqrt{n}/e)^n(\sqrt{n}d + z)^{-A/d^2+\sqrt{n}(\sqrt{n}d+z)/d+1/2} \times \exp\left(\frac{-2A - z^2 - 4\sqrt{n}dz}{4d^2} \right)$$
$$\times \left\{ \frac{[(z + \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2-\sqrt{n}z/d}}{(\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}} \right.$$
$$\times \exp\left[\frac{(z + 4\sqrt{n}d)\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right]$$
$$+ \frac{[(z - \sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})/2]^{A/d^2-\sqrt{n}z/d}}{(-\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}})^{1/2}}$$
$$\left. \times \exp\left[\frac{-(z + 4\sqrt{n}d)\sqrt{z - 2i\sqrt{A}}\sqrt{z + 2i\sqrt{A}}}{4d^2} \right] \right\}$$

for z in a complex neighborhood of any compact subset in $\mathring{I}_A$. This proves (2.7) and (2.8). □

2.3. Case I.C: $a = 0$

Theorem 2.4. *Assume $d > 0$ and $a = 0$. Let $x = ny$, the polynomials $\pi_n(x)$ defined in (1.1) have the following large-n asymptotics. As $n \to \infty$, we have*

$$\pi_n(ny) \sim (n/e)^n \left(\frac{y}{y - d} \right)^{ny/d+1/2} (y - d)^n \tag{2.9}$$

for $y \in \mathcal{C} \setminus [0, d]$; and

$$\pi_n(ny) \sim (n/e)^n (d - y)^n \left(\frac{y}{d - y} \right)^{ny/d+1/2} \times 2\cos[\pi(n - ny/d - 1/2)] \tag{2.10}$$

for y in a complex neighborhood of any compact subset in $(0, d)$.

Proof. Setting $z = \sqrt{n}(y-d)$ in (2.1) and taking limit $a \to 0^+$ yields (2.9). A standard argument of analytical continuity as in the proof of Theorem 2.1 gives (2.10). □

3. Case II: $d = 0$

In this section, we consider the critical case $d = 0$. Again, we investigate three subcases according to the sign of a.

3.1. Case II.A: $a > 0$

Theorem 3.1. *Assume $d = 0$ and $a > 0$. Let $x = \sqrt{n}y$. The polynomials $\pi_n(x)$ defined in (1.1) have the following large-n asymptotics. As $n \to \infty$, we have for $y \in \mathcal{C} \setminus [-2\sqrt{a}, 2\sqrt{a}]$,*

$$\pi_n(\sqrt{n}y) \sim \left(\frac{n}{4e}\right)^{n/2} (y + \sqrt{y^2 - 4a})^n \left(\frac{y + \sqrt{y^2 - 4a}}{2\sqrt{y^2 - 4a}}\right)^{1/2} \left(\frac{y + \sqrt{y^2 - 4a}}{2y}\right)^{b/a} \times \exp\left[\frac{ny}{4a}(y - \sqrt{y^2 - 4a})\right]; \tag{3.1}$$

and for y in a complex neighborhood of any compact subset in $(0, 2\sqrt{a})$, we have

$$\pi_n(\sqrt{n}y) \sim \left(\frac{na}{e}\right)^{n/2} \left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a} - y}\sqrt{2\sqrt{a} + y}}\right)^{1/2} \left(\frac{\sqrt{a}}{y}\right)^{b/a} \times \exp\left(\frac{ny^2}{4a}\right) \times 2\cos\left[(n + 1/2 + b/a)\arccos\frac{y}{2\sqrt{a}} - \pi/4 - \frac{ny}{4a}\sqrt{2\sqrt{a} - y}\sqrt{2\sqrt{a} + y}\right]; \tag{3.2}$$

and for y in a complex neighborhood of any compact subset in $(-2\sqrt{a}, 0)$, we have

$$\pi_n(\sqrt{n}y) \sim \left(\frac{na}{e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a} - y}\sqrt{2\sqrt{a} + y}}\right)^{1/2} \left(\frac{\sqrt{a}}{-y}\right)^{b/a} \times \exp\left(\frac{ny^2}{4a}\right) \times 2\cos\left[(n + 1/2 + b/a)\arccos\frac{-y}{2\sqrt{a}} - \pi/4 + \frac{ny}{4a}\sqrt{2\sqrt{a} - y}\sqrt{2\sqrt{a} + y}\right]. \tag{3.3}$$

Proof. For x in the outer region such that $\pi_{k-1}(x) \neq 0$, we define

$$w_k(x) := \frac{\pi_k(x)}{\pi_{k-1}(x)}$$

for any $k \geq 1$. It follows that

$$w_{k+1}(x) = x - \frac{ak + b}{w_k(x)}, \quad k \geq 1.$$

Let $x = \sqrt{n}y$ with $y \in \mathcal{C} \setminus [-2\sqrt{a}, 2\sqrt{a}]$. We have as $n \to \infty$,

$$w_k(x) \sim \frac{x + \sqrt{x^2 - 4ak}}{2} \times \left\{1 + \frac{a}{x^2 - 4ak} - \frac{2b}{(x + \sqrt{x^2 - 4ak})\sqrt{x^2 - 4ak}}\right\}.$$

By trapezoidal rule, we obtain

$$\begin{aligned}\ln \pi_n &\sim n \ln(\sqrt{n}/2) + n \int_0^1 \ln(y + \sqrt{y^2 - 4at})dt + \frac{1}{2}\ln\frac{y + \sqrt{y^2 - 4a}}{2y} \\ &\quad + \int_0^1 \frac{a}{y^2 - 4at}dt - \frac{2b}{(y + \sqrt{y^2 - 4at})\sqrt{y^2 - 4at}}dt \\ &\sim n \ln(\sqrt{n}/2) + n\left[t \ln(y + \sqrt{y^2 - 4at}) - \frac{y}{4a}\sqrt{y^2 - 4at} - \frac{t}{2}\right]\Bigg|_0^1\end{aligned}$$

$$+\frac{1}{2}\ln\frac{y+\sqrt{y^2-4a}}{2y}+\frac{1}{4}\ln\frac{y^2}{y^2-4a}+\frac{b}{a}\ln(y+\sqrt{y^2-4at})\Big|_0^1$$
$$\sim n\ln(\sqrt{n}/2)+n\ln(y+\sqrt{y^2-4a})-\frac{ny}{4a}(\sqrt{y^2-4a}-y)-\frac{n}{2}$$
$$+\frac{1}{2}\ln\frac{y+\sqrt{y^2-4a}}{2y}+\frac{1}{4}\ln\frac{y^2}{y^2-4a}+\frac{b}{a}\ln\frac{y+\sqrt{y^2-4a}}{2y}.$$

Recall that $x = ny$. We then obtain $\pi_n(ny) \sim \Phi_n(y)$, where

$$\Phi_n(y) := \left(\frac{n}{4e}\right)^{n/2}(y+\sqrt{y^2-4a})^n\left(\frac{y+\sqrt{y^2-4a}}{2\sqrt{y^2-4a}}\right)^{1/2}$$
$$\times\left(\frac{y+\sqrt{y^2-4a}}{2y}\right)^{b/a}\times\exp\left[\frac{ny}{4a}(y-\sqrt{y^2-4a})\right].$$

By a standard argument of analytical continuity, we obtain $\pi_n(ny) \sim \Phi_n^+(y)+\phi_n^-(y)$ for y in a complex neighborhood of any compact subset in $(-2\sqrt{a}, 0)\cup(0, 2\sqrt{a})$, where

$$\Phi_n^{\pm}(y) := \lim_{\varepsilon\to 0^+}\Phi_n(y+i\varepsilon).$$

For y in a complex neighborhood of any compact subset in $(0, 2\sqrt{a})$, a simple calculation gives

$$\Phi_n^+(y)+\phi_n^-(y) = \left(\frac{n}{4e}\right)^{n/2}(2\sqrt{a})^n\left(\frac{2\sqrt{a}}{2\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2}$$
$$\times\left(\frac{2\sqrt{a}}{2y}\right)^{b/a}\times\exp\left[\frac{ny}{4a}(y)\right]$$
$$\times 2\cos\left[(n+1/2+b/a)\arccos\frac{y}{2\sqrt{a}}\right.$$
$$\left.-\pi/4-\frac{ny}{4a}\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}\right]$$
$$\sim\left(\frac{na}{e}\right)^{n/2}\left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2}\left(\frac{\sqrt{a}}{y}\right)^{b/a}\times\exp\left[\frac{ny^2}{4a}\right]$$
$$\times 2\cos\left[(n+1/2+b/a)\arccos\frac{y}{2\sqrt{a}}\right.$$
$$\left.-\pi/4-\frac{ny}{4a}\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}\right].$$

Thus, (3.2) follows. Note that for $\mathrm{Re}\, y < 0$, we can write

$$\Phi_n(y) = \left(\frac{n}{4e}\right)^{n/2}(-1)^n(-y+\sqrt{-y-2\sqrt{a}}\sqrt{-y+2\sqrt{a}})^n$$
$$\times\left(\frac{-y+\sqrt{-y-2\sqrt{a}}\sqrt{-y+2\sqrt{a}}}{2\sqrt{-y-2\sqrt{a}}\sqrt{-y+2\sqrt{a}}}\right)^{1/2}$$

$$\times \left(\frac{-y+\sqrt{-y-2\sqrt{a}}\sqrt{-y+2\sqrt{a}}}{-2y}\right)^{b/a}$$
$$\times \exp\left[\frac{ny}{4a}(y+\sqrt{-y-2\sqrt{a}}\sqrt{-y+2\sqrt{a}})\right].$$

It follows that for y in a complex neighborhood of any compact subset in $(-2\sqrt{a},0)$,

$$\begin{aligned}\Phi_n^+(y)+\phi_n^-(y) &= \left(\frac{n}{4e}\right)^{n/2}(-1)^n(2\sqrt{a})^n\left(\frac{2\sqrt{a}}{2\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2}\\ &\quad\times\left(\frac{2\sqrt{a}}{-2y}\right)^{b/a}\times\exp\left[\frac{ny}{4a}(y)\right]\\ &\quad\times 2\cos\left[(n+1/2+b/a)\arccos\frac{-y}{2\sqrt{a}}\right.\\ &\quad\left.-\ \pi/4+\frac{ny}{4a}\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}\right]\\ &\sim\left(\frac{na}{e}\right)^{n/2}(-1)^n\left(\frac{\sqrt{a}}{\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}}\right)^{1/2}\left(\frac{\sqrt{a}}{-y}\right)^{b/a}\\ &\quad\times\exp\left[\frac{ny^2}{4a}\right]2\cos\left[(n+1/2+b/a)\arccos\frac{-y}{2\sqrt{a}}\right.\\ &\quad\left.-\ \pi/4+\frac{ny}{4a}\sqrt{2\sqrt{a}-y}\sqrt{2\sqrt{a}+y}\right].\end{aligned}$$

This proves (3.3). □

3.2. Case II.B: $a<0$

Theorem 3.2. *Assume $d=0$ and $a<0$. Let $x=i\sqrt{n}y$, $A=-a>0$ and $B=-b$. The polynomials $\pi_n(x)$ defined in (1.1) have the following large-n asymptotics. As $n\to\infty$, we have for $y\in\mathcal{C}\setminus[-2\sqrt{A},2\sqrt{A}]$,*

$$\begin{aligned}\pi_n(i\sqrt{n}y) &\sim i^n\left(\frac{n}{4e}\right)^{n/2}(y+\sqrt{y^2-4A})^n\\ &\quad\times\left(\frac{y+\sqrt{y^2-4A}}{2\sqrt{y^2-4A}}\right)^{1/2}\left(\frac{y+\sqrt{y^2-4A}}{2y}\right)^{B/A}\\ &\quad\times\exp\left[\frac{ny}{4A}(y-\sqrt{y^2-4A})\right];\end{aligned}\tag{3.4}$$

and for y in a complex neighborhood of any compact subset in $(0,2\sqrt{A})$, we have

$$\begin{aligned}\pi_n(i\sqrt{n}y) &\sim i^n\left(\frac{nA}{e}\right)^{n/2}\left(\frac{\sqrt{A}}{\sqrt{2\sqrt{A}-y}\sqrt{2\sqrt{A}+y}}\right)^{1/2}\left(\frac{\sqrt{A}}{y}\right)^{B/A}\times\exp\left(\frac{ny^2}{4A}\right)\\ &\quad\times 2\cos\left[(n+1/2+B/A)\arccos\frac{y}{2\sqrt{A}}\right.\end{aligned}$$

$$- \pi/4 - \frac{ny}{4A}\sqrt{2\sqrt{A}-y}\sqrt{2\sqrt{A}+y}\Bigg]; \tag{3.5}$$

and for y in a complex neighborhood of any compact subset in $(-2\sqrt{A}, 0)$, we have

$$\pi_n(i\sqrt{n}y) \sim i^n \left(\frac{nA}{e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{A}}{\sqrt{2\sqrt{A}-y}\sqrt{2\sqrt{A}+y}}\right)^{1/2} \left(\frac{\sqrt{A}}{-y}\right)^{B/A}$$
$$\times \exp\left(\frac{ny^2}{4A}\right) 2\cos\Bigg[(n+1/2+B/A)\arccos\frac{-y}{2\sqrt{A}}$$
$$- \pi/4 + \frac{ny}{4A}\sqrt{2\sqrt{A}-y}\sqrt{2\sqrt{A}+y}\Bigg]. \tag{3.6}$$

Proof. The monic polynomials $p_n(z) := i^{-n}\pi_n(iz)$ satisfy the same difference equation and initial conditions of π_n with a and b replaced by $A = -a$ and $B = -b$ respectively. Theorem 3.2 follows from Theorem 3.1. □

3.3. Case II.C: $a = 0$

Theorem 3.3. *Assume $d = 0$ and $a = 0$. If $b < 0$, we denote $\sqrt{b} = i\sqrt{|b|}$. The polynomials $\pi_n(x)$ defined in (1.1) have the following large-n asymptotics. As $n \to \infty$, we have for $x \in \mathcal{C} \setminus [-2\sqrt{b}, 2\sqrt{b}]$,*

$$\pi_n(x) \sim \left(\frac{x+\sqrt{x^2-4b}}{2}\right)^{n+1} \frac{1}{\sqrt{x^2-4b}}; \tag{3.7}$$

and for x in a complex neighborhood of any compact subset in $(-2\sqrt{b}, 2\sqrt{b})$, we have

$$\pi_n(x) \sim \frac{\sin\left[(n+1)\arccos\frac{x}{2\sqrt{b}}\right]}{2^n\sqrt{4b-x^2}}. \tag{3.8}$$

The above asymptotic formula is actually an equality.

Proof. Note that

$$\pi_n(x) = \frac{(x+\sqrt{x^2-4b})^{n+1} - (x-\sqrt{x^2-4b})^{n+1}}{2^{n+1}\sqrt{x^2-4b}}.$$

For $x \in \mathcal{C} \setminus [-2\sqrt{b}, 2\sqrt{b}]$, we have $\pi_n(x) \sim \Phi_n(x)$ with

$$\Phi_n(x) := \left(\frac{x+\sqrt{x^2-1}}{2}\right)^{n+1} \frac{1}{\sqrt{x^2-1}}.$$

This proves (3.7). To be consistent, we use the argument of analytical continuity and obtain

$$\pi_n(x) \sim \lim_{\varepsilon\to 0^+} [\Phi_n(x+i\varepsilon) + \Phi_n(x-i\varepsilon)] = \frac{\sin[(n+1)\arccos\frac{x}{2\sqrt{b}}]}{2^n\sqrt{4b-x^2}}$$

for x in a complex neighborhood of any compact subset in $(-2\sqrt{b}, 2\sqrt{b})$. This gives (3.8). We remark that the formula (3.8) is actually an equality. □

4. Associated polynomials

In this section, we will make application of our results to associated polynomials [7] which are generalized from classical polynomials. To be specific, we will consider associated Hermite and associated Charlier polynomials in the following two subsections, respectively. These two polynomials satisfy difference equations with linear coefficients and their asymptotic behaviors can be directly obtained from our theorems in the previous two sections.

4.1. Associated Hermite polynomials

The (monic) associated Hermite polynomials [1] satisfy the following difference equation.

$$\pi_{n+1}(x) = x\pi_n(x) - \frac{n+c}{2}\pi_{n-1}(x), \quad \pi_0(x) = 1, \ \ \pi_1(x) = x. \tag{4.1}$$

According to our classification, the associated Hermite polynomials belong to the case II.A with $d = 0$, $a = 1/2$ and $b = c/2$. A direct application of Theorem 3.1 gives the following results.

Corollary 4.1. *Let $x = \sqrt{n}y$. The polynomials $\pi_n(x)$ defined in (4.1) have the following large-n asymptotics. As $n \to \infty$, we have for $y \in \mathcal{C} \setminus [-\sqrt{2}, \sqrt{2}]$,*

$$\begin{aligned}\pi_n(\sqrt{n}y) \sim \left(\frac{n}{4e}\right)^{n/2} (y+\sqrt{y^2-2})^n \left(\frac{y+\sqrt{y^2-2}}{2\sqrt{y^2-2}}\right)^{1/2} \\ \times \left(\frac{y+\sqrt{y^2-2}}{2y}\right)^c \times \exp\left[\frac{ny}{2}(y-\sqrt{y^2-2})\right];\end{aligned} \tag{4.2}$$

and for y in a complex neighborhood of any compact subset in $(0, \sqrt{2})$, we have

$$\begin{aligned}\pi_n(\sqrt{n}y) \sim \left(\frac{n}{2e}\right)^{n/2} \left(\frac{\sqrt{1/2}}{\sqrt{\sqrt{2}-y}\sqrt{\sqrt{2}+y}}\right)^{1/2} \left(\frac{\sqrt{1/2}}{y}\right)^c \times \exp\left(\frac{ny^2}{2}\right) \\ \times 2\cos\left[(n+1/2+c)\arccos\frac{y}{\sqrt{2}} - \pi/4 - \frac{ny}{2}\sqrt{\sqrt{2}-y}\sqrt{\sqrt{2}+y}\right];\end{aligned} \tag{4.3}$$

and for y in a complex neighborhood of any compact subset in $(-\sqrt{2}, 0)$, we have

$$\begin{aligned}\pi_n(\sqrt{n}y) \sim \left(\frac{n}{2e}\right)^{n/2} (-1)^n \left(\frac{\sqrt{1/2}}{\sqrt{\sqrt{2}-y}\sqrt{\sqrt{2}+y}}\right)^{1/2} \left(\frac{\sqrt{1/2}}{-y}\right)^c \times \exp\left(\frac{ny^2}{2}\right) \\ \times 2\cos\left[(n+1/2+c)\arccos\frac{-y}{\sqrt{2}} - \pi/4 + \frac{ny}{2}\sqrt{\sqrt{2}-y}\sqrt{\sqrt{2}+y}\right].\end{aligned} \tag{4.4}$$

4.2. Associated Charlier polynomials

The (monic) associated Charlier polynomials (c.f. [11]) satisfy the following difference equation (with $a > 0$):

$$\begin{aligned}\pi_{n+1}(x) &= (x-n-a-c)\pi_n(x) - a(n+c)\pi_{n-1}(x), \\ \pi_0(x) &= 1, \ \ \pi_1(x) = x-a-c.\end{aligned} \tag{4.5}$$

These polynomials can be viewed as special cases of the associated Wilson polynomials studied in [9]. According to our classification, the associated Charlier polynomials belong to the case I.A with $d = 1$ and $b = ac$. A direct application of Theorem 2.1 gives the following results.

Corollary 4.2. *Let $x = a + c + ny$ and $y = 1 + z/\sqrt{n}$. The polynomials $\pi_n(x)$ defined in (4.5) have the following large-n asymptotics. As $n \to \infty$, for $z \in \mathcal{C} \setminus [-\sqrt{n}, 2\sqrt{a}]$, we have*

$$\pi_n(n+\sqrt{n}z+a+c) \sim (n/e)^n \left(\frac{z+\sqrt{z^2-4a}}{2\sqrt{n}}\right)^n \left(\frac{z+\sqrt{z^2-4a}}{2(\sqrt{n}+z)}\right)^{-a-\sqrt{n}(\sqrt{n}+z)} \times \left(\frac{\sqrt{n}+z}{\sqrt{z^2-4a}}\right)^{1/2} \times \exp\left[\frac{2a-z^2-4\sqrt{n}z+(z+4\sqrt{n})\sqrt{z^2-4a}}{4}\right]; \tag{4.6}$$

and for z in a complex neighborhood of any compact subset in $(-2\sqrt{a}, 2\sqrt{a})$, we have

$$\pi_n(n+\sqrt{n}z+a+c) \sim (n/e)^n \left(\frac{\sqrt{a}}{\sqrt{n}}\right)^{-a-\sqrt{n}z} (1+z/\sqrt{n})^{a+\sqrt{n}(\sqrt{n}+z)} \times \left(\frac{\sqrt{n}+z}{\sqrt{4a-z^2}}\right)^{1/2} \exp\left(\frac{2a-z^2-4\sqrt{n}z}{4}\right) \times 2\cos\left[(-a-\sqrt{n}z)\arccos\frac{z}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z+4\sqrt{n})\sqrt{4a-z^2}}{4}\right]; \tag{4.7}$$

and for z in a complex neighborhood of any compact subset in $(-\sqrt{n}, -2\sqrt{a})$, we have

$$\pi_n(n+\sqrt{n}z+a+c) \sim (n/e)^n \left(\frac{-z+\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}{2\sqrt{n}}\right)^{-a-\sqrt{n}z} \times (1+z/\sqrt{n})^{a+\sqrt{n}(\sqrt{n}+z)} \left(\frac{\sqrt{n}+z}{\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}\right)^{1/2} \times \exp\left[\frac{2a-z^2-4\sqrt{n}z-(z+4\sqrt{n})\sqrt{-z-2\sqrt{a}}\sqrt{-z+2\sqrt{a}}}{4}\right] \times 2\cos[\pi(-a-\sqrt{n}z-1/2)]. \tag{4.8}$$

Analogous to the treatment on associated Laguerre and Meixner polynomials in [8], we introduce the second family of associated Charlier polynomials as solutions to the following difference equation:

$$\begin{aligned} &P_{n+1}(x) = (x-n-a-c)P_n(x) - a(n+c)P_{n-1}(x), \\ &P_0(x) = 1, \quad P_1(x) = x-c. \end{aligned} \tag{4.9}$$

Note that the difference equations for $\pi_n(x)$ in (4.5) and $P_n(x)$ in (4.9) are the same, but the initial conditions are different; namely, $\pi_1(x) = x - a - c$ while $P_1(x) = x - c$. Define

$$Q_n(x) := \frac{1}{c}[P_{n+1}(x) - \pi_{n+1}(x)]. \tag{4.10}$$

It follows that

$$\begin{aligned} Q_{n+1}(x) &= (x-n-1-a-c)Q_n(x) - a(n+1+c)P_{n-1}(x), \\ Q_0(x) &= 1, \quad Q_1(x) = x-1-a-c. \end{aligned} \tag{4.11}$$

Observe from (4.10) that

$$P_n(n+\sqrt{n}z+a+c) = \pi_n(n+\sqrt{n}z+a+c) + cQ_{n-1}(n+\sqrt{n}z+a+c).$$

Therefore, to find asymptotic behaviors of the second family of associated Charlier polynomials $P_n(x)$, we only need to investigate $Q_{n-1}(x)$. By applying Theorem 2.1 to (4.11), we have the following results.

Corollary 4.3. *Let $z_n := z\sqrt{n/(n-1)}$. We have $z\sqrt{n} = z_n\sqrt{n-1}$ and*

$$Q_{n-1}(n+\sqrt{n}z+a+c) = Q_{n-1}(n-1+\sqrt{n-1}z_n+a+c+1).$$

Moreover, we have the following large-n asymptotics. As $n \to \infty$, for $z \in \mathcal{C} \setminus [-\sqrt{n}, 2\sqrt{a}]$, we have

$$\begin{aligned} Q_{n-1}(n+\sqrt{n}z+a+c) &\sim \left(\frac{n-1}{e}\right)^{n-1} \left(\frac{z_n+\sqrt{z_n^2-4a}}{2\sqrt{n-1}}\right)^{n-1} \\ &\times \left(\frac{z_n+\sqrt{z_n^2-4a}}{2(\sqrt{n-1}+z_n)}\right)^{-a-\sqrt{n-1}(\sqrt{n-1}+z_n)} \left(\frac{\sqrt{n-1}+z_n}{\sqrt{z_n^2-4a}}\right)^{1/2} \\ &\times \exp\left[\frac{2a-z_n^2-4\sqrt{n-1}z_n+(z_n+4\sqrt{n-1})\sqrt{z_n^2-4a}}{4}\right]; \end{aligned} \tag{4.12}$$

and for z in a complex neighborhood of any compact subset in $(-2\sqrt{a}, 2\sqrt{a})$, we have

$$\begin{aligned} &Q_{n-1}(n+\sqrt{n}z+a+c) \\ &\sim \left(\frac{n-1}{e}\right)^{n-1} \left(\frac{\sqrt{a}}{\sqrt{n-1}}\right)^{-a-\sqrt{n-1}z_n} (1+z_n/\sqrt{n-1})^{a+\sqrt{n-1}(\sqrt{n-1}+z_n)} \\ &\quad\times \left(\frac{\sqrt{n-1}+z_n}{\sqrt{4a-z_n^2}}\right)^{1/2} \exp\left(\frac{2a-z_n^2-4\sqrt{n-1}z_n}{4}\right) \\ &\quad\times 2\cos\left[(-a-\sqrt{n-1}z_n)\arccos\frac{z_n}{2\sqrt{a}} - \frac{\pi}{4} + \frac{(z_n+4\sqrt{n-1})\sqrt{4a-z_n^2}}{4}\right]; \end{aligned} \tag{4.13}$$

and for z in a complex neighborhood of any compact subset in $(-\sqrt{n}, -2\sqrt{a})$, we have

$$\begin{aligned} &Q_{n-1}(n+\sqrt{n}z+a+c) \\ &\sim \left(\frac{n-1}{e}\right)^{n-1} \left(\frac{-z_n+\sqrt{-z_n-2\sqrt{a}}\sqrt{-z_n+2\sqrt{a}}}{2\sqrt{n-1}}\right)^{-a-\sqrt{n-1}z_n} \\ &\quad\times (1+z_n/\sqrt{n-1})^{a+\sqrt{n-1}(\sqrt{n-1}+z_n)} \left(\frac{\sqrt{n-1}+z_n}{\sqrt{-z_n-2\sqrt{a}}\sqrt{-z_n+2\sqrt{a}}}\right)^{1/2} \end{aligned}$$

$$\times \exp\left[\frac{2a - z_n^2 - 4\sqrt{n-1}z_n - (z_n + 4\sqrt{n-1})\sqrt{-z_n - 2\sqrt{a}}\sqrt{-z_n + 2\sqrt{a}}}{4}\right]$$
$$\times 2\cos[\pi(-a - \sqrt{n-1}z_n - 1/2)]. \tag{4.14}$$

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