

Asymptotics of delay differential equations via polynomials

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Abstract. In this paper, we introduce an innovative and systematic technique to study delay differential equations via polynomials. First, we review an intrinsic relation between delay differential equations and polynomials. From this relation, we obtain long time behaviors of the solutions to delay differential equations via asymptotic analysis of the corresponding polynomials. Moreover, we derive asymptotic formulas and upper bounds for the intrinsic growth rate of delay differential equations, as well as a Gronwall-type inequality for delay differential inequalities.

Keywords: delay differential equations, asymptotic analysis, Gronwall-type inequality, intrinsic growth rate

1. Introduction

Delay differential equations arise from various mathematical models in the areas of engineering and science (cf. [7,14,20,24]). In the context of dynamical systems with time delay, we are interested in finding the conditions when a certain equilibrium is asymptotically stable, and the conditions when a periodic solution bifurcates from the equilibrium. These stability and bifurcation problems can be transformed to asymptotic analysis of linear delay differential equations. For instance, the dynamical behavior of a monotone system can be determined by the spectral radius of an abstract monodromy operator generated from its linearized system (cf. [27, Theorem 2.3.4]). In general, it is difficult to find an explicit formula of the spectral radius in terms of model parameters (see [11] for example). In order to obtain qualitative and more decent information about the delay differential equations, it is important to derive asymptotic formulas and upper bounds for the intrinsic growth rate which characterizes long time behavior of the solution.

Various techniques have been developed in asymptotic analysis of integrals [23], ordinary differential equations [16], recurrence relations/difference equations [22], and partial differential equations [6]. However, not much is known about the asymptotic formulas of solutions to delay differential equations. The main objective of this paper is to develop an innovative and unified technique in asymptotic analysis of delay differential equations. This technique is quite different from the Krein–Rutman theorem and theory of monotone dynamical systems where only qualitative results can be obtained (see [27, Theorem 2.3.4] for example). Our method is based on an intrinsic relation between delay differential equations and polynomials, as well as some classical techniques in asymptotic analysis. To illustrate our idea, we consider the following nonlinear delay differential equation

$$u'(t) = f(t, u(t), u(t-1)), \quad t \geq 0. \tag{1.1}$$

Here, for convenience, we have normalized the time delay to be 1. Linearizing the above equation at a certain equilibrium gives

$$u'(t) = f_1(t)u(t) + f_2(t)u(t-1),$$

where f_1 and f_2 correspond to the partial derivatives of f at the equilibrium. Upon a transformation

$$u(t) \rightarrow u(t) \exp\left\{\int_0^t f_1(s) ds\right\},$$

we obtain following linear delay differential equation

$$u'(t) = a(t)u(t-1), \quad t \geq 0. \quad (1.2)$$

We will show that this equation can be solved successively and its solution has an intrinsic relation with polynomials. Therefore, the asymptotic formula of solution to the linear differential equation follows from asymptotic formula of the corresponding polynomials. Especially, we are able to derive an explicit formula for the intrinsic growth rate of the delay differential equation. Note that the intrinsic growth rate of linearized equation (1.2) determines the stability of the equilibrium and locations of bifurcation points for the nonlinear equation (1.1). Thus, our systematic method can be applied in stability and bifurcation analysis of nonlinear delay differential equations.

The rest of this paper is organized as follows. In Sections 2, we focus on linear delay differential equations with constant or periodic coefficients. We obtain asymptotic behaviors and intrinsic growth rates of the solutions to linear delay differential equations. In Section 3, we use two examples to illustrate the applications of our innovative method in stability and bifurcation analysis of nonlinear delay differential equations. In Section 4, we derive a Gronwall-type inequality for delay differential inequality. This inequality enables us to find a sharp upper bound for the intrinsic growth rate and thus to obtain a sufficient condition for the stability of equilibrium for nonlinear delay differential equations with periodic coefficients. In Section 5, we conclude our paper with discussions on several open problems.

2. Linear delay differential equations

In this section, we study the linear delay differential equation (1.2) with constant initial value, say, 1, on the interval $[-1, 0]$. The nonlinear equations with general initial values will be investigated in the next section. We will consider the cases when the coefficient $a(t)$ is a constant and a periodic function, respectively.

In the first part of this section, we focus on the autonomous case when the coefficient $a(t)$ is a constant, denoted by α . We have the following equation and will solve it in a successive manner.

$$u'(t) = \alpha u(t-1), \quad t \geq 0, \quad (2.1)$$

$$u(\theta) = 1, \quad \theta \in [-1, 0]. \quad (2.2)$$

It should be mentioned that the solution of the above equation can be easily obtained by Laplace transformation and has an explicit integral representation. However, the asymptotic analysis of the corresponding integral expression is not trivial. Since we are mainly interested in the long time behavior of the solution, we will find an equivalent expression of the solution given in terms of polynomials and the problem can

be solved via asymptotic analysis of corresponding polynomials. First, we define a sequence of functions as follows:

$$u_n(t) := u(n+t), \quad n = -1, 0, 1, 2, \dots$$

It is readily seen that for $n \geq 0$,

$$u_n'(t) = \alpha u_{n-1}(t), \quad t \geq 0,$$

$$u_n(0) = u_{n-1}(1).$$

Recall the initial condition $u_{-1}(t) = 1$ for $t \in [0, 1]$. We obtain by induction that $u_n(t)$ is a polynomial of t with degree $n+1$ on the interval $[0, 1]$. Set

$$u_n(t) = \sum_{k=0}^{n+1} u_{n,k} t^k, \quad n \geq -1, t \in [0, 1]. \quad (2.3)$$

It follows that for $n \geq 0$,

$$u_{n,k} = \alpha u_{n-1,k-1}/k, \quad 1 \leq k \leq n,$$

$$u_{n,0} = \sum_{k=0}^n u_{n-1,k}.$$

Therefore, we have

$$u_{n,k} = \frac{\alpha^k}{k!} u_{n-k,0} \quad (2.4)$$

and

$$u_{n,0} = \sum_{k=0}^n \frac{\alpha^k}{k!} u_{n-1-k,0}.$$

The initial condition for the above recurrence relation is $u_{-1,0} = 1$. By induction, we have for $n \geq 0$,

$$u_{n,0} = P_n(\alpha) := \sum_{k=0}^n (n-k+1)^k \frac{\alpha^k}{k!}. \quad (2.5)$$

To prove the above formula, we first observe that $u_{0,0} = 1$. So, we only need to verify that if the above formula is true for all $u_{0,0}, \dots, u_{n-1,0}$, then it is also true for $u_{n,0}$. By induction, we have

$$\begin{aligned} u_{n,0} - \frac{\alpha^n}{n!} &= \sum_{k=0}^{n-1} \frac{\alpha^k}{k!} u_{n-1-k,0} = \sum_{k=0}^{n-1} \frac{\alpha^k}{k!} \sum_{j=0}^{n-1-k} (n-1-k-j+1)^j \frac{\alpha^j}{j!} \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1-k} (n-k-j)^j \frac{\alpha^{j+k}}{j!k!} = \sum_{k=0}^{n-1} \sum_{l=k}^{n-1} (n-l)^{l-k} \frac{\alpha^l}{(l-k)!k!} \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{n-1} (n-l)^l \frac{\alpha^l}{l!} \sum_{k=0}^l \frac{(n-l)^{-k} l!}{(l-k)! k!} = \sum_{l=0}^{n-1} (n-l)^l \frac{\alpha^l}{l!} [1 - 1/(n-l)]^l \\
&= \sum_{l=0}^{n-1} (n-l+1)^l \frac{\alpha^l}{l!}.
\end{aligned}$$

This proves (2.5). Note that $u_{n,0} = u_n(0) = u(n)$. We also note that the polynomial $P_n(\alpha)$ in (2.5) is called delayed exponential function in the literature; see [12,13]. Next, we investigate asymptotic behavior of $P_n(\alpha)$ as $n \rightarrow \infty$. Recall the definition of Lambert W-function $W_p(x)$ as the principle solution to the equation $We^W = x$; see [8, Section 4.13]. It is natural that Lambert W-function plays an important role in the study of delay differential equations; see [9,26] and references therein. We have the following theorem.

Theorem 2.1. *As $n \rightarrow \infty$, we have*

$$P_n(\alpha) \sim \frac{e^{(n+1)W_p(\alpha)}}{1 + W_p(\alpha)}, \quad \alpha \in \mathcal{C} \setminus (-\infty, -e^{-1}], \quad (2.6)$$

and

$$P_n(\alpha) \sim \frac{e^{(n+1)W_p^+(\alpha)}}{1 + W_p^+(\alpha)} + \frac{e^{(n+1)W_p^-(\alpha)}}{1 + W_p^-(\alpha)}, \quad \alpha \in (-\infty, -e^{-1}), \quad (2.7)$$

where

$$W_p^\pm(\alpha) := \lim_{\varepsilon \rightarrow 0^+} W_p(\alpha \pm i\varepsilon), \quad \alpha \in (-\infty, -e^{-1}).$$

Proof. Let us first consider the case when $\alpha > 0$. For convenience, denote $w = W_p(\alpha)$. Then, we have $\alpha = we^w$ and

$$u_{n,0} = P_n(\alpha) := \sum_{k=0}^n (n-k+1)^k \frac{(we^w)^k}{k!}.$$

If $k/n = s$ and s is bounded away from 0 and 1, we obtain by Stirling's formula

$$\begin{aligned}
(n-k+1)^k \frac{(we^w)^k}{k!} &= (n-ns+1)^{ns} \frac{(we^w)^{ns}}{(ns)!} \\
&\sim n^{ns} (1-s)^{ns} e^{s/(1-s)} \frac{(we^w)^{ns}}{\sqrt{2\pi ns} (ns/e)^{ns}} \\
&= \frac{e^{s/(1-s)}}{\sqrt{2\pi ns}} \exp\{ns[w + \log w + 1 + \log(1-s) - \log s]\}.
\end{aligned}$$

Denote

$$\varphi(s) := s[w + \log w + 1 + \log(1-s) - \log s].$$

We have

$$\varphi'(s) = w + \log w - \log\left(\frac{s}{1-s}\right) - \frac{s}{1-s},$$

$$\varphi''(s) = -\frac{1}{s} - \frac{1}{1-s} - \frac{1}{(1-s)^2} < 0.$$

Hence, $\varphi(s)$ attains its maximum at $s_0 = w/(1+w)$. By trapezoidal rule and Laplace's method [4, Chapter 5], we obtain for any small $\delta > 0$,

$$\begin{aligned} \sum_{k=\lfloor n\delta \rfloor}^{n-\lfloor n\delta \rfloor} (n-k+1)^k \frac{(we^w)^k}{k!} &\sim n \int_{\delta}^{1-\delta} (n-ns+1)^{ns} \frac{(we^w)^{ns}}{(ns)!} ds \\ &\sim n \frac{e^{s_0/(1-s_0)}}{\sqrt{2\pi ns_0}} \exp\{n\varphi(s_0)\} \sqrt{\frac{2\pi}{-n\varphi''(s_0)}}. \end{aligned}$$

Here, $\delta > 0$ is small such that $\delta < s_0 < 1 - \delta$. The symbol $\lfloor n\delta \rfloor$ denotes the largest integer that is less than or equal to $n\delta$. Note that $s_0/(1-s_0) = w$, $\varphi(s_0) = w$ and $s_0\varphi''(s_0) = -(1+w)^2$. It follows that

$$\sum_{k=\lfloor n\delta \rfloor}^{n-\lfloor n\delta \rfloor} (n-k+1)^k \frac{(we^w)^k}{k!} \sim \frac{e^{(n+1)w}}{1+w}, \quad \alpha > 0.$$

Next, we intend to show that the contribution of

$$\sum_{k=0}^{\lfloor n\delta \rfloor - 1} (n-k+1)^k \frac{(we^w)^k}{k!} + \sum_{k=n-\lfloor n\delta \rfloor + 1}^n (n-k+1)^k \frac{(we^w)^k}{k!}$$

in the sum expression of $P_n(\alpha)$ is negligible as $n \rightarrow \infty$. For any $0 \leq k \leq \lfloor n\delta \rfloor - 1$, we estimate the ratio

$$\begin{aligned} &\frac{(n-k+1)^k (we^w)^k / k!}{(n - \lfloor ns_0 \rfloor - k + 1)^{\lfloor ns_0 \rfloor + k} (we^w)^{\lfloor ns_0 \rfloor + k} / (\lfloor ns_0 \rfloor + k)!} \\ &= \left(\frac{n-k+1}{n - \lfloor ns_0 \rfloor - k + 1} \right)^k [we^w (n - \lfloor ns_0 \rfloor - k + 1)]^{-\lfloor ns_0 \rfloor} \frac{(\lfloor ns_0 \rfloor + k)!}{k!} \\ &\leq \left(\frac{n+1}{n - \lfloor ns_0 \rfloor + 1} \right)^{\lfloor n\delta \rfloor} [we^w (n - \lfloor ns_0 \rfloor - \lfloor n\delta \rfloor + 1)]^{-\lfloor ns_0 \rfloor} \frac{(\lfloor ns_0 \rfloor + \lfloor n\delta \rfloor)!}{(\lfloor n\delta \rfloor)!} \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\frac{1}{1-s_0} \right)^{n\delta} [we^w n(1-s_0-\delta)]^{-ns_0+1} \frac{\sqrt{2\pi n(s_0+\delta)} [n(s_0+\delta)/e]^{n(s_0+\delta)}}{\sqrt{2\pi/(n\delta)} (n\delta/e)^{n\delta}} \\
&\leq C n^2 (1-s_0)^{-n\delta} [we^w n(1-s_0-\delta)]^{-ns_0} e^{-ns_0} (s_0+\delta)^{n(s_0+\delta)} \delta^{-n\delta} \\
&\leq C n^2 e^{nf(\delta)},
\end{aligned}$$

where $C > 0$ denotes a generic constant independent of n , and

$$f(\delta) := -\delta \ln(1-s_0) - s_0 \ln[we^w(1-s_0-\delta)] - s_0 + (s_0+\delta) \ln(s_0+\delta) - \delta \ln \delta.$$

Since $f(\delta) \rightarrow -w < 0$ as $\delta \rightarrow 0^+$, we obtain $f(\delta) < 0$ for sufficiently small $\delta > 0$. Thus,

$$\sum_{k=0}^{\lfloor n\delta \rfloor - 1} (n-k+1)^k \frac{(we^w)^k}{k!}$$

is exponentially small compared with

$$\sum_{k=0}^{\lfloor n\delta \rfloor - 1} (n - \lfloor ns_0 \rfloor - k + 1)^{\lfloor ns_0 \rfloor + k} \frac{(we^w)^{\lfloor ns_0 \rfloor + k}}{(\lfloor ns_0 \rfloor + k)!} = \sum_{k=\lfloor ns_0 \rfloor}^{\lfloor ns_0 \rfloor + \lfloor n\delta \rfloor - 1} (n-k+1)^k \frac{(we^w)^k}{k!}.$$

For any $0 \leq k \leq \lfloor n\delta \rfloor - 1$, we estimate the ratio

$$\begin{aligned}
&\frac{(k+1)^{n-k} (we^w)^{n-k} / (n-k)!}{(n - \lfloor ns_0 \rfloor + k + 1)^{\lfloor ns_0 \rfloor - k} (we^w)^{\lfloor ns_0 \rfloor - k} / (\lfloor ns_0 \rfloor - k)!} \\
&= \left(\frac{k+1}{n - \lfloor ns_0 \rfloor + k + 1} \right)^{n-k} [we^w (n - \lfloor ns_0 \rfloor + k + 1)]^{n - \lfloor ns_0 \rfloor} \frac{(\lfloor ns_0 \rfloor - k)!}{(n-k)!} \\
&\leq \left(\frac{\lfloor n\delta \rfloor}{n - \lfloor ns_0 \rfloor + \lfloor n\delta \rfloor} \right)^{n - \lfloor n\delta \rfloor + 1} [we^w (n - \lfloor ns_0 \rfloor + \lfloor n\delta \rfloor)]^{n - \lfloor ns_0 \rfloor} \frac{(\lfloor ns_0 \rfloor - \lfloor n\delta \rfloor)!}{(n - \lfloor n\delta \rfloor)!} \\
&\leq C \left(\frac{\delta}{1-s_0+\delta} \right)^{n(1-\delta)} [we^w n(1-s_0+\delta)]^{n - ns_0 + 1} \frac{\sqrt{2\pi n(s_0-\delta)} [n(s_0-\delta)/e]^{n(s_0-\delta)}}{\sqrt{2\pi/[n(1-\delta)]} [n(1-\delta)/e]^{n(1-\delta)}} \\
&\leq C n^2 \left(\frac{\delta}{1-s_0+\delta} \right)^{n(1-\delta)} [we^w n(1-s_0+\delta)]^{n(1-s_0)} \frac{e^{n(1-s_0)} (s_0-\delta)^{n(s_0-\delta)}}{(1-\delta)^{n(1-\delta)}} \\
&\leq C n^2 e^{ng(\delta)},
\end{aligned}$$

where $C > 0$ denotes a generic constant independent of n , and

$$\begin{aligned}
g(\delta) &:= (1-\delta) [\ln \delta - \ln(1-s_0+\delta)] + (1-s_0) [w + \ln w + \ln(1-s_0+\delta) + 1] \\
&\quad + (s_0-\delta) \ln(s_0-\delta) - (1-\delta) \ln(1-\delta).
\end{aligned}$$

Since $g(\delta) \rightarrow -\infty$ as $\delta \rightarrow 0^+$, we obtain $g(\delta) < 0$ for sufficiently small $\delta > 0$. Thus,

$$\sum_{k=0}^{\lfloor n\delta \rfloor - 1} (k+1)^{n-k} \frac{(we^w)^{n-k}}{(n-k)!} = \sum_{k=n-\lfloor n\delta \rfloor + 1}^n (n-k+1)^k \frac{(we^w)^k}{k!}$$

is exponentially small compared with

$$\sum_{k=0}^{\lfloor n\delta \rfloor - 1} (n - \lfloor ns_0 \rfloor + k + 1)^{\lfloor ns_0 \rfloor - k} \frac{(we^w)^{\lfloor ns_0 \rfloor - k}}{(\lfloor ns_0 \rfloor - k)!} = \sum_{k=\lfloor ns_0 \rfloor - \lfloor n\delta \rfloor + 1}^{\lfloor ns_0 \rfloor} (n - k + 1)^k \frac{(we^w)^k}{k!}.$$

Combining the above arguments, we have

$$P_n(\alpha) \sim \frac{e^{(n+1)w}}{1+w}, \quad \alpha > 0.$$

By analytic continuation, the above formula is still valid for $\alpha \in \mathcal{C} \setminus (-\infty, -e^{-1}]$. Thus, (2.6) is proved. Next, we investigate the asymptotic behavior of $P_n(\alpha)$ for α in the Stokes line $(-\infty, -e^{-1})$. Note that $W_p(\alpha)$ has a branch cut on this line, and $W_p^\pm(\alpha) = u \pm iv$, where $u = -v \cot v$ and v is the positive root of the equation

$$\frac{-ve^{-v \cot v}}{\sin v} = \alpha.$$

The functions $W_p^\pm(\alpha)$ can be analytically extended to a complex neighborhood of $(-\infty, -e^{-1})$. Moreover, given $\alpha \in (-\infty, -e^{-1})$, we have $\Re[W_p^+(\alpha + i\delta) - W_p^-(\alpha + i\delta)] > 0$ and $\Re[W_p^+(\alpha - i\delta) - W_p^-(\alpha - i\delta)] < 0$ for small $\delta > 0$. This, together with (2.6) implies that

$$P_n(\alpha) \sim \frac{e^{(n+1)W_p^+(\alpha)}}{1 + W_p^+(\alpha)} + \frac{e^{(n+1)W_p^-(\alpha)}}{1 + W_p^-(\alpha)}$$

for any α near but not lying on the Stokes line $(-\infty, -e^{-1})$; noting that $\exp\{(n+1)[W_p^+(\alpha) - W_p^-(\alpha)]\}$ is exponentially small for α below the Stokes line and exponentially large for α above the Stokes line. By analytical continuation, the above formula is also valid on the Stokes line $(-\infty, -e^{-1})$. Thus, (2.7) follows. \square

As a numerical evidence of our formulas (2.6) and (2.7), we use the software Mathematica to compute the exact and approximate values of $P_n(\alpha)$ with $n = 10$ and α varies in \mathbb{R} ; see Table 1.

Remark 2.2. The formula (2.7) can be also written as

$$P_n(\alpha) \sim \frac{2e^{(n+1)u}\{(1+u)\cos[(n+1)v] + v\sin[(n+1)v]\}}{(1+u)^2 + v^2}, \quad (2.8)$$

Table 1

Numerical evidence of asymptotic formulas of $P_n(\alpha)$ in (2.6) and (2.7), where $n = 10$

α	Exact value	Approximate value	Relative error
-2	1.122328042	1.122328010	2.9×10^{-8}
-1.58	-0.927559	-0.927559	4.54311×10^{-9}
$-\pi/2$	-0.9060366969	-0.9060367009	4.4×10^{-9}
-1.57	-0.90414	-0.90414	4.38739×10^{-9}
-1.5	-0.723344	-0.723344	3.53141×10^{-9}
-1	0.02024112654	0.02024112651	1.8×10^{-9}
-0.37	0.000304486	0.000304486	5.18091×10^{-13}
-0.36	0.000720525	0.000726992	0.00897488
-0.2	0.0780117	0.0780117	5.92226×10^{-12}
0.2	5.4849	5.4849	1.61932×10^{-16}
1	326.7913170	326.7913170	5.0×10^{-11}

where $u = -v \cot v$ and v is the positive root of the equation

$$\frac{-ve^{-v \cot v}}{\sin v} = \alpha$$

with $\alpha < -e^{-1}$. Especially, if $\alpha = -\pi/2$, then we have $v = \pi/2$, $u = 0$ and

$$P_n(-\pi/2) \sim \begin{cases} \frac{\pi(-1)^k}{1 + (\pi/2)^2}, & n = 2k, \\ \frac{2(-1)^k}{1 + (\pi/2)^2}, & n = 2k - 1. \end{cases}$$

Remark 2.3. The relation between (2.6) and (2.7) is universal in asymptotics of polynomials in the following sense. Assume that

$$P_n(z) \sim \Phi(n, z), \quad z \in \mathcal{C} \setminus \bar{S}_0,$$

where S_0 is the union of one-dimensional (open) branch cuts/Stokes lines of $\Phi(n, z)$ with \bar{S}_0 being its closure. We can obtain the asymptotic formula of $P_n(z)$ on S_0 as follows. Denote by $\Phi^\pm(n, x)$ the one-sided limits of $\Phi(n, z)$ on S_0 . If $\Phi^\pm(n, x)$ can be analytically extended in a neighborhood of S_0 and $\Phi^+(n, z)$ dominates on the + side of S_0 , whereas $\Phi^-(n, z)$ dominates on the - side of S_0 , we then have

$$P_n(x) \sim \Phi^+(n, x) + \Phi^-(n, x), \quad x \in S_0.$$

Examples of polynomials having this property include classical ones such as Legendre polynomials [21, Section 2], Hermite polynomials [21, Section 3]; multiple orthogonal polynomials from higher-order three-term recurrences [1, Section 7]; and the polynomials arising from birth and death processes such as Chen–Ismail polynomials [5, Section 2], Berg–Letessier–Valent polynomials [5, Section 3], Conrad–Flajolet polynomials [5, Sections 4, 5].

Now, we are ready to study the asymptotic behavior of $u(t)$ as $t \rightarrow \infty$. First, it follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned} u_n(t) &= \sum_{k=0}^n \sum_{i=0}^{n-k} (n-k-i+1)^i \frac{\alpha^i}{i!} \frac{\alpha^k}{k!} t^k + \frac{(\alpha t)^{n+1}}{(n+1)!} \\ &= \sum_{k=0}^n \sum_{j=k}^n (n-j+1)^{j-k} \frac{\alpha^j}{(j-k)!k!} t^k + \frac{(\alpha t)^{n+1}}{(n+1)!} \\ &= \sum_{j=0}^n \sum_{k=0}^j (n-j+1)^{j-k} \frac{\alpha^j}{(j-k)!k!} t^k + \frac{(\alpha t)^{n+1}}{(n+1)!} = \sum_{j=0}^n (n-j+1+t)^j \frac{\alpha^j}{j!} + \frac{(\alpha t)^{n+1}}{(n+1)!}. \end{aligned}$$

By a similar argument as that in the proof of Theorem 2.1, we obtain the asymptotic formulas of $u(t)$ in the following theorem.

Theorem 2.4. *As $t \rightarrow \infty$, we have*

$$u(t) \sim \frac{e^{(1+t)W_p(\alpha)}}{1+W_p(\alpha)}, \quad \alpha \in \mathcal{C} \setminus (-\infty, -e^{-1}], \quad (2.9)$$

and

$$u(t) \sim \frac{e^{(1+t)W_p^+(\alpha)}}{1+W_p^+(\alpha)} + \frac{e^{(1+t)W_p^-(\alpha)}}{1+W_p^-(\alpha)}, \quad \alpha \in (-\infty, -e^{-1}), \quad (2.10)$$

where

$$W_p^\pm(\alpha) := \lim_{\varepsilon \rightarrow 0^+} W_p(\alpha \pm i\varepsilon), \quad \alpha \in (-\infty, -e^{-1}).$$

Especially, the intrinsic growth rate of the solution to (2.1) with real α is given by

$$\limsup_{t \rightarrow \infty} \frac{\log |u(t)|}{t} = W(\alpha) := \begin{cases} W_p(\alpha), & \alpha > -e^{-1}, \\ \Re\{W_p^+(\alpha)\}, & \alpha < -e^{-1}. \end{cases} \quad (2.11)$$

Proof. The proof is similar to that of Theorem 2.1. Here, we note that for $k/n = s$ bounded away from 0 and 1, we have as $n \rightarrow \infty$,

$$\begin{aligned} (n-k+1+t)^k \frac{(w e^w)^k}{k!} &= (n-ns+1+t)^{ns} \frac{(w e^w)^{ns}}{(ns)!} \\ &\sim n^{ns} (1-s)^{ns} e^{s(1+t)/(1-s)} \frac{(w e^w)^{ns}}{\sqrt{2\pi ns} (ns/e)^{ns}} \\ &= \frac{e^{s(1+t)/(1-s)}}{\sqrt{2\pi ns}} \exp\{ns[w + \log w + 1 + \log(1-s) - \log s]\}. \end{aligned}$$

Furthermore, the last term in the expression of $u_n(t)$ is always small:

$$\frac{(\alpha t)^{n+1}}{(n+1)!} \sim \frac{1}{\sqrt{2\pi(n+1)}} \left(\frac{\alpha t e}{n+1} \right)^{n+1}.$$

Therefore, we have as $n \rightarrow \infty$,

$$u_n(t) \sim \frac{e^{(n+1+t)W_p(\alpha)}}{1 + W_p(\alpha)}, \quad \alpha \in \mathcal{C} \setminus (-\infty, -e^{-1}],$$

and

$$u_n(t) \sim \frac{e^{(n+1+t)W_p^+(\alpha)}}{1 + W_p^+(\alpha)} + \frac{e^{(n+1+t)W_p^-(\alpha)}}{1 + W_p^-(\alpha)}, \quad \alpha \in (-\infty, -e^{-1}),$$

where

$$W_p^\pm(\alpha) := \lim_{\varepsilon \rightarrow 0^+} W_p(\alpha \pm i\varepsilon), \quad \alpha \in (-\infty, -e^{-1}).$$

Since $u(t+n) = u_n(t)$, we obtain (2.9) and (2.10). Finally, noting that $W_p^+(\alpha)$ and $W_p^-(\alpha)$ are complex conjugates if $\alpha < -e^{-1}$, (2.11) follows from (2.9) and (2.10). \square

Remark 2.5. The formula (2.10) can be also written as

$$u(t) \sim \frac{2e^{(1+t)u} \{(1+u) \cos[(1+t)v] + v \sin[(1+t)v]\}}{(1+u)^2 + v^2}, \quad (2.12)$$

where $u = -v \cot v$ and v is the positive root of the equation

$$\frac{-ve^{-v \cot v}}{\sin v} = \alpha$$

with $\alpha < -e^{-1}$. Especially, if $\alpha = -\pi/2$, then we have $v = \pi/2$, $u = 0$ and the solution eventually approaches a periodic solution with period 4:

$$u(t) \sim \frac{2 \sin[(1+t)\pi/2 + \arctan(2/\pi)]}{\sqrt{1 + (\pi/2)^2}}, \quad t \rightarrow \infty.$$

In the second part of this section, we consider the case when the coefficient $a(t)$ is a periodic function with period $T = 1/m$, where m is a positive integer. Since the time delay has been normalized to be 1, this assumption means that the time delay is an integer multiplication of the period of the coefficient. We study the following delay differential equation

$$u'(t) = a(t)u(t-1), \quad t \geq 0, \quad (2.13)$$

$$u(\theta) = 1, \quad \theta \in [-1, 0]. \quad (2.14)$$

Note that any integer multiplication of the period is still a period of $a(t)$. Especially, we have $a(t+1) = a(t)$. For convenience, we denote by \bar{a} the average of $a(t)$ over a period:

$$\bar{a} := \int_0^1 a(t) dt.$$

Similar to the autonomous case, we will solve Eq. (2.13) successively. First, we define a sequence of functions as follows:

$$u_n(t) := u(n+t), \quad n = -1, 0, 1, 2, \dots$$

It is readily seen that for $n \geq 0$,

$$\begin{aligned} u_n'(t) &= a(t)u_{n-1}(t), \quad t \geq 0, \\ u_n(0) &= u_{n-1}(1). \end{aligned}$$

For $t \in [0, 1]$, we introduce the functions $M_n(t)$ as follows:

$$\begin{aligned} M_0(t) &= 1, \\ M_n(t) &= \int_0^t a(s)M_{n-1}(s) ds, \quad n = 1, 2, \dots \end{aligned}$$

Recall the initial condition $u_{-1}(t) = 1$ for $t \in [0, 1]$. It follows by induction that

$$u_n(t) = \sum_{k=0}^{n+1} u_{n-k}(0)M_k(t).$$

Furthermore, since $M_k(1) = \bar{a}^k/k!$, we obtain

$$u_{n+1}(0) = u_n(1) = \sum_{k=0}^{n+1} u_{n-k}(0) \frac{\bar{a}^k}{k!}.$$

The initial condition for the above recurrence relation is $u_{-1}(0) = 1$. By induction, we have for $n \geq 0$,

$$u_n(0) = P_n(\bar{a}) := \sum_{k=0}^n (n-k+1)^k \frac{\bar{a}^k}{k!}. \quad (2.15)$$

The following theorem will be useful in determining the asymptotic stability of nonlinear delay differential equations with periodic coefficients.

Theorem 2.6. *If $a(t+1) = a(t)$ and $a(t) > 0$ for all $t \geq 0$, then the solution of (2.13) with initial condition (2.14) is exponentially growing and*

$$u(n) = u_n(0) \sim \frac{e^{(n+1)W_p(\bar{a})}}{1 + W_p(\bar{a})}, \quad n \rightarrow \infty, \quad (2.16)$$

where \bar{a} is the average of $a(t)$ over a period. Especially, the exponential growth rate of the solution is given by

$$\lim_{t \rightarrow \infty} \frac{\log u(t)}{t} = W_p(\bar{a}). \quad (2.17)$$

Proof. Applying (2.6) to (2.15) gives (2.16). Since $u(t)$ is an increasing function, we have $u(n) \leq u(t) \leq u(n+1)$ for $n \leq t \leq n+1$. The formula (2.17) now follows from (2.16) and the squeeze theorem. \square

3. Nonlinear delay differential equations

In this section, we will use two examples to illustrate how our asymptotic results of linear delay differential equations can be applied in stability and bifurcation analysis of nonlinear delay differential equations.

Example 1 (Stability and bifurcation analysis of autonomous delay differential equations). Let us consider the following general delay differential equation generated from population dynamics:

$$N'(t) = -\mu N(t) + e^{-\delta\tau} b(N(t-\tau)), \quad (3.1)$$

where $N(t)$ denotes the matured population, μ is the death rate for the matured population, b is a general birth function with $b(0) = 0$ and $b'(0) > 0$, τ stands for the time delay for maturation, and δ counts for the death rate during maturation. The above model has been studied extensively in previous literature; see [2, 3, 10] and references therein. Here, we intend to reproduce the well-known stability results of the above equation using our Theorem 2.4. Assume that the initial population is nonnegative and nontrivial on the interval $[-\tau, 0]$. It can be shown that the solution $N(t) > 0$ for all $t > \tau$; see [18, Proposition 2.3]. We have the following result about the intrinsic growth rate of the matured population and stability results.

Proposition 3.1. *For any nonnegative and nontrivial initial conditions, the intrinsic growth rate of the delay differential equation (3.1) at the trivial equilibrium is given by*

$$\lim_{t \rightarrow \infty} \frac{\log N(t)}{t} = \frac{W_p[\tau e^{\mu\tau - \delta\tau} b'(0)]}{\tau} - \mu.$$

Especially, the trivial equilibrium of (3.1) is locally asymptotically unstable (stable) if

$$e^{-\delta\tau} b'(0) > (<) \mu.$$

Moreover, if $b(N) \leq b'(0)N$ for any $N > 0$ and $e^{-\delta\tau} b'(0) < \mu$, then the trivial equilibrium is globally asymptotically stable.

Proof. We linearize Eq. (3.1) about the trivial equilibrium to obtain

$$N'(t) = -\mu N(t) + e^{-\delta\tau} b'(0)N(t-\tau).$$

Denote $u(t) := e^{\mu t} N(t)$ and introduce a new time scale $s := t/\tau$. It follows that

$$\frac{du}{ds}(\tau s) = \alpha u[\tau(s-1)],$$

where

$$\alpha := \tau e^{\mu\tau - \delta\tau} b'(0) > 0.$$

Upon a shift of s by 1, we may assume the initial values are positive. Thus, there exist two positive constants m and M such that $m < u(\tau\theta) < M$ for all $\theta \in [-1, 0]$. From (2.11), comparison principle and squeeze theorem, we obtain

$$\lim_{s \rightarrow \infty} \frac{\log u(\tau s)}{s} = W_p[\tau e^{\mu\tau - \delta\tau} b'(0)].$$

Since $s = t/\tau$ and $u(t) = e^{\mu t} N(t)$, it is readily seen that

$$\lim_{t \rightarrow \infty} \frac{\log N(t)}{t} = \frac{W_p[\tau e^{\mu\tau - \delta\tau} b'(0)]}{\tau} - \mu.$$

Recall that Lambert W-function $W_p(\alpha)$ is the solution of the equation $W e^W = \alpha$ and it is an increasing function on the positive real line. If $e^{-\delta\tau} b'(0) > \mu$, then

$$W_p[\tau e^{\mu\tau - \delta\tau} b'(0)] > W_p(\mu\tau e^{\mu\tau}) = \mu\tau$$

and the intrinsic growth rate is positive, which implies that the trivial equilibrium is locally asymptotically unstable. On the other hand, if $e^{-\delta\tau} b'(0) < \mu$, then

$$W_p[\tau e^{\mu\tau - \delta\tau} b'(0)] < W_p(\mu\tau e^{\mu\tau}) = \mu\tau$$

and the intrinsic growth rate is negative, which implies that the trivial equilibrium is locally asymptotically stable. Moreover, if $b(N) \leq b'(0)N$ for any $N > 0$ and $e^{-\delta\tau} b'(0) < \mu$, then the nonlinear delay differential equation (3.1) is dominated by its linearized equation about the trivial equilibrium, and thus a comparison principle [18, Proposition 1.1] yields global asymptotic stability of the trivial equilibrium. \square

Remark 3.2. Note that if $b'(0) > 0$, then the local asymptotic stability of trivial equilibrium for the delay differential equation (3.1) can be also obtained via the theory of monotone dynamical systems; see [19, Chapter 5]. We also note that the formula of the intrinsic growth rate coincides with the principal eigenvalue of the characteristic equation $\lambda = -\mu + e^{-\delta\tau} b'(0)e^{-\lambda\tau}$.

Let us now further assume Eq. (3.1) possesses a positive equilibrium, denoted by N^* . We linearize (3.1) about N^* and obtain

$$N'(t) = -\mu N(t) + e^{-\delta\tau} b'(N^*) N(t - \tau). \quad (3.2)$$

Note that $b'(N^*)$ may be negative and the comparison principle for monotone dynamical system does not apply. We are interested in the solution of (3.2) with constant initial condition

$$N(\theta) = 1, \quad \theta \in [-\tau, 0]. \quad (3.3)$$

Proposition 3.3. *Let $N(t)$ be the solution of (3.2) with initial condition (3.3). If $\alpha := \tau e^{\mu\tau - \delta\tau} b'(N^*) > -e^{-1}$, then as $t \rightarrow \infty$,*

$$N(t) \sim \frac{e^{(t+1)W_p(\alpha)/\tau - \mu t}}{1 + W_p(\alpha)}.$$

If $\alpha := \tau e^{\mu\tau - \delta\tau} b'(N^) < -e^{-1}$, then as $t \rightarrow \infty$,*

$$N(t) \sim \frac{e^{(t+1)W_p^+(\alpha)/\tau - \mu t}}{1 + W_p^+(\alpha)} + \frac{e^{(t+1)W_p^-(\alpha)/\tau - \mu t}}{1 + W_p^-(\alpha)}.$$

The intrinsic growth rate is given by

$$\limsup_{t \rightarrow \infty} \frac{\log |N(t)|}{t} = \frac{W(\alpha)}{\tau} - \mu = \begin{cases} W_p(\alpha)/\tau - \mu, & \alpha > -e^{-1}, \\ \Re\{W_p^+(\alpha)\}/\tau - \mu, & \alpha < -e^{-1}. \end{cases}$$

Moreover, if $\alpha := \tau e^{\mu\tau - \delta\tau} b'(N^) < -e^{-1}$ and*

$$\Re[W_p^+(\alpha)] = \mu\tau,$$

then $N(t)$ will approach a periodic solution with period $2\pi\tau/\Im[W_p^+(\alpha)]$.

Proof. Set $u(t) := e^{\mu t} N(t)$ and introduce a new time scale $s = t/\tau$. The results now follow from Theorem 2.3 and a similar argument as in the proof of Proposition 3.1. \square

Remark 3.4. Recall that for $\alpha := \tau e^{\mu\tau - \delta\tau} b'(N^*) < -e^{-1}$, we have $W_p^+(\alpha) = u + iv$ with $u = -v \cot v$ and v is the positive root of the equation

$$\frac{-ve^{-v \cot v}}{\sin v} = \tau e^{\mu\tau - \delta\tau} b'(N^*).$$

The bifurcation condition is the same as $-v \cot v = \mu\tau$. A simple calculation yields

$$v = \tau \sqrt{[e^{-\delta\tau} b'(N^*)]^2 - \mu^2}, \quad \cos v = \frac{\mu}{e^{-\delta\tau} b'(N^*)}.$$

This coincides with the formula [17, (3.8)] when the birth function $b(N)$ takes the special form as in the Nicholson's blowflies equation studied in [17].

Example 2 (Stability analysis of non-autonomous delay differential equations). We study the following delay differential equation with periodic coefficients

$$N'(t) = -\mu(t)N(t) + e^{-\delta\tau}b(t, N(t - \tau)), \tag{3.4}$$

where $\mu(t + T) = \mu(t) \geq 0$ and $b(t + T, N) = b(t, N) \geq 0$ for all $t \geq 0$ and $N \geq 0$. Furthermore, we assume $b(t, 0) = 0$, $\partial_N b(t, 0) > 0$ and the period $T = \tau/m$ for some positive integer m . If the initial values are nonnegative and nontrivial on $[-\tau, 0]$, then the solution will be positive for all $t > \tau$; see [18, Proposition 2.3]. Similar to the autonomous case, we obtain the following results about the intrinsic growth rate of the equation and stability criterion of trivial equilibrium.

Proposition 3.5. *Let $\bar{\mu}$ and \bar{b} be the average of $\mu(t)$ and $\partial_N b(t, 0)$ over a period, respectively. Given any nonnegative and nontrivial initial conditions, the intrinsic growth rate of the delay differential equation (3.4) at the trivial equilibrium is given by*

$$\frac{W_p[\tau e^{\bar{\mu}\tau - \delta\tau\bar{b}}]}{\tau} - \bar{\mu}.$$

Epecially, the trivial equilibrium of (3.4) is locally asymptotically unstable (stable) if

$$e^{-\delta\tau\bar{b}} > (<) \bar{\mu}.$$

Moreover, if $b(t, N) \leq \partial_N b(t, 0)N$ for any $N > 0$ and $e^{-\delta\tau\bar{b}} < \bar{\mu}$, then the trivial equilibrium is globally asymptotically stable.

Proof. We linearize Eq. (3.4) about the trivial equilibrium to obtain

$$N'(t) = -\mu(t)N(t) + e^{-\delta\tau}\partial_N b(t, 0)N(t - \tau).$$

Next, we denote $u(t) := N(t) \exp\{\int_0^t \mu(r) dr\}$ and introduce a new time scale $s := t/\tau$. It follows that

$$\frac{du}{ds}(\tau s) = a(s)u[\tau(s - 1)],$$

where

$$a(s) := \tau e^{\bar{\mu}\tau - \delta\tau} \partial_N b(\tau s, 0).$$

Epecially, $a(s + 1) = a(s)$ and the average of $a(s)$ over a period is given by

$$\bar{a} = \int_0^1 a(r) dr = \tau e^{\bar{\mu}\tau - \delta\tau\bar{b}}.$$

Without loss of generality, we may assume that the initial values are positive. Especially, there exist two positive constants m and M such that $m < u(\tau\theta) < M$ for all $\theta \in [-1, 0]$. On account of (2.17), comparison principle and squeeze theorem, the intrinsic growth rate of $u(\tau s)$ is

$$\lim_{s \rightarrow \infty} \frac{\log u(\tau s)}{s} = W_p[\tau e^{\bar{\mu}\tau - \delta\tau\bar{b}}].$$

Note that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(r) \, dr = \bar{\mu}.$$

It follows from $u(t) := N(t) \exp\{\int_0^t \mu(r) \, dr\}$ and $s := t/\tau$ that

$$\lim_{t \rightarrow \infty} \frac{\log N(t)}{t} = \frac{W_p[\tau e^{\bar{\mu}\tau - \delta\tau\bar{b}}]}{\tau} - \bar{\mu}.$$

The local asymptotic stability results of trivial equilibrium follows from the formula of intrinsic growth rate and monotonicity of Lambert W-function on the positive real line. Furthermore, if $b(t, N) \leq \partial_N b(t, 0)N$ for any $N > 0$ and $e^{-\delta\tau} b'(0) < \bar{\mu}$, then the global asymptotic stability of trivial equilibrium follows from a comparison principle for monotone dynamical systems; see [18, Proposition 1.1]. \square

Remark 3.6. The local stability of trivial equilibrium was also obtained in [25, Proposition 2.1] using Krein–Rutman theorem and spectral properties of bounded linear operators. However, our formula of intrinsic growth rate seems to be new.

4. Gronwall-type inequality

In this section, we will derive a Gronwall-type inequality for the delay differential inequality with periodic coefficient:

$$u'(t) \leq a(t)u(t-1), \quad t \geq 0, \quad (4.1)$$

where $a(t) \geq 0$ for $t \geq 0$, and $u(t) \geq 0$ for $t \in [-1, \infty)$. We assume that $a(t+T) = a(t)$ for $t \geq 0$, where $T > 0$ is the period of $a(t)$. Remark that T may not be divided by the delay 1, so Theorem 2.6 does not apply. Denote the average of $a(t)$ by

$$\bar{a} := \frac{1}{T} \int_0^T a(t) \, dt.$$

First, we note that

$$\begin{aligned} u(t) &\leq u(0) + \int_0^t a(s)u(s-1) \, ds \\ &\leq u(0) + \int_{-1}^0 a(s+1)u(s) \, ds + \int_0^{t-1} a(s+1)u(s) \, ds \\ &\leq u(0) + \int_{-1}^0 a(s+1)u(s) \, ds + \int_0^t a(s+1)u(s) \, ds. \end{aligned}$$

It follows from Gronwall's inequality that

$$u(t) \leq Me^{\bar{a}t},$$

where $M > 0$ is a constant independent of t . The above inequality gives an upper bound for the intrinsic growth rate of $u(t)$:

$$\lim_{t \rightarrow 0} \frac{\log u(t)}{t} \leq \bar{a}.$$

In the following theorem, we will give a sharper upper bound of the intrinsic growth rate.

Theorem 4.1. *Assume $u(t) \geq 0$ satisfies the delay differential inequality (4.1) with periodic coefficient $a(t) \geq 0$. Let $a_- \geq 0$ and $a_+ > 0$ be the minimum and maximum of $a(t)$, respectively. Then we have*

$$u(t) \leq M \exp\{W_p(a_+)t\},$$

and

$$u(t) \leq M \exp\left\{\frac{\bar{a}t}{\exp[W_p(a_-)]}\right\},$$

where $M > 0$ is a constant independent of t . Especially,

$$\lim_{t \rightarrow 0} \frac{\log u(t)}{t} \leq \min\left\{W_p(a_+), \frac{\bar{a}}{\exp[W_p(a_-)]}\right\}.$$

Proof. First, we note that

$$u(t) \leq a_+ u(t-1).$$

By comparison principle, it follows from (2.9) that

$$u(t) \leq M \exp\{W_p(a_+)t\},$$

where $M > 0$ denotes a large constant independent of t . Next, we define

$$v(t) := u(t)e^{-w_-t},$$

where $w_- := W_p(a_-)$. It follows that

$$v'(t) \leq a(t)e^{-w_-}v(t-1) - w_-v(t).$$

Integrating from 0 to t yields

$$\begin{aligned} v(t) &\leq v(0) + \int_0^t a(s)e^{-w_-}v(s-1) ds - \int_0^t w_-v(s) ds \\ &\leq v(0) + \int_{-1}^0 a(s+1)e^{-w_-}v(s) ds + \int_0^t [a(s+1)e^{-w_-} - w_-]v(s) ds. \end{aligned}$$

An application of Gronwall's inequality gives

$$v(t) \leq M \exp\{[\bar{a}e^{-w_-} - w_-]t\},$$

where $M > 0$ denotes a generic constant. Thus,

$$u(t) \leq M \exp\left\{\frac{\bar{a}t}{e^{w_-}}\right\}.$$

Finally, the upper bound of intrinsic growth rate comes from the two inequalities for $u(t)$. \square

Remark 4.2. If $a(t) = \alpha$ is a constant, then the upper bound is $W_p(\alpha)$, which is sharper than the one obtained directly from the Gronwall's inequality; noting that $W_p(\alpha) < \alpha$ for all $\alpha > 0$. If $a(t)$ vanishes at some time, then $a_- = 0$ and the upper bound becomes the minimum of $W_p(a_+)$ and \bar{a} .

As a corollary of our Gronwall-type inequality, we obtain a sufficient condition for the stability of trivial equilibrium for the following nonlinear delay differential equation:

$$N'(t) = -\mu(t)N(t) + e^{-\delta\tau}b(t, N(t-\tau)), \quad (4.2)$$

where $\mu(t+T) = \mu(t) \geq 0$ and $b(t+T, N) = b(t, N) \geq 0$ for all $t \geq 0$ and $N \geq 0$. Furthermore, we assume $b(t, 0) = 0$, $\partial_N b(t, 0) > 0$. Here, we do not assume the period is an integer multiplication of the delay. For simplicity, we denote

$$\beta_+ := \max_{t \in [0, T]} \partial_N b(t, 0),$$

$$\beta_- := \min_{t \in [0, T]} \partial_N b(t, 0),$$

$$\bar{\beta} := \frac{1}{T} \int_0^T \partial_N b(t, 0) dt$$

and

$$\mu_+ := \max_{t \in [0, T]} \frac{1}{\tau} \int_t^{t+\tau} \mu(s) ds,$$

$$\bar{\mu} := \frac{1}{T} \int_0^T \mu(t) dt.$$

Corollary 4.3. *The trivial equilibrium of (4.2) is locally asymptotically stable if*

$$\min\left\{W_p[\beta_+ \tau e^{(\mu_+ - \delta)\tau}], \frac{\bar{\beta} \tau e^{(\mu_+ - \delta)\tau}}{\exp[W_p(\beta_- \tau e^{(\mu_+ - \delta)\tau})]}\right\} < \bar{\mu} \tau.$$

Proof. Define

$$u(t) := N(t) \exp \left\{ \int_0^t \mu(s) \, ds \right\}.$$

It follows that

$$\begin{aligned} u'(t) &= \exp \left\{ -\delta\tau + \int_{t-\tau}^t \mu(s) \, ds \right\} \partial_N b(t, 0) u(t - \tau) \\ &\leq \exp \{ (\mu_+ - \delta)\tau \} \partial_N b(t, 0) u(t - \tau). \end{aligned}$$

Next, we introduce a new time scale $s = t/\tau$ to obtain

$$\frac{du}{ds}(\tau s) \leq a(s) u(\tau(s - 1)),$$

where

$$a(s) = \tau \exp \{ (\mu_+ - \delta)\tau \} \partial_N b(\tau s, 0).$$

From Theorem 4.1, we obtain the upper bound of the intrinsic growth rate for $u(\tau s)$ as follows.

$$\lim_{s \rightarrow \infty} \frac{\log u(\tau s)}{s} \leq \min \left\{ W_p[\beta_+ \tau e^{(\mu_+ - \delta)\tau}], \frac{\bar{\beta} \tau e^{(\mu_+ - \delta)\tau}}{\exp[W_p(\beta_- \tau e^{(\mu_+ - \delta)\tau})]} \right\}.$$

Since

$$\lim_{t \rightarrow \infty} \frac{\log N(t)}{t} = \lim_{s \rightarrow \infty} \frac{\log u(\tau s)}{\tau s} - \bar{\mu},$$

the desired result follows. \square

5. Conclusions and open problems

We make use of an intrinsic relation between delay differential equations and polynomials to find asymptotic behaviors of solutions to the delay differential equations. Moreover, we obtain asymptotic formulas and upper bounds for the intrinsic growth rate, as well as a Gronwall-type inequality. Our innovative method provides a systematic framework in stability and bifurcation analysis of nonlinear delay differential equations.

Note that we use a single delay differential equation with one discrete delay to illustrate our new idea. It would be interesting to explore our technique to more general cases where multiple discrete delays or multiple species/groups are taken into consideration. There are plenty of work on delay differential system using the method of characteristic equations; see [2,10,15] for example. Here, we propose a new technique in studying the linearized delay differential systems about an equilibrium, namely, we transform the linearized delay differential systems into linear recurrence relations/difference equations.

The main challenge would be the asymptotic analysis of the solutions to corresponding recurrence relations/difference equations. We will investigate this problem in our future work.

The polynomial $P_n(\alpha)$ defined in (2.5) warrants further investigations. For instance, one may be interested in finding an integral representation of $P_n(\alpha)$ and studying its asymptotic behavior using steepest-descent method. It is also possible to derive asymptotic formulas of $P_n(\alpha)$ via the differential-difference equation: $P'_n(\alpha) = nP_{n-1}(\alpha) - \alpha P'_{n-1}(\alpha)$. Another interesting open problem is the uniform asymptotic analysis of the polynomial $P_n(\alpha)$ for α in a neighborhood of the turning point $-e^{-1}$. We conjecture that the Airy function (or other special functions) should be used in the uniform asymptotic formula.

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References

- [1] A.I. Aptekarev, V.A. Kalyagin and E.B. Saff, Higher-order three-term recurrences and asymptotics of multiple orthogonal polynomials, *Constr. Approx.* **30**(2) (2009), 175–223.
- [2] E. Beretta and Y. Kuang, Geometric stability switch criteria in delay differential systems with delay dependent parameters, *SIAM J. Math. Anal.* **33**(5) (2002), 1144–1165.
- [3] K. Cooke, P. van den Driessche and X. Zou, Interaction of maturation delay and nonlinear birth in population and epidemic models, *J. Math. Biol.* **39**(4) (1999), 1999; Erratum, *J. Math. Biol.* **45**(5) (2002), 470.
- [4] E.T. Copson, *Asymptotic Expansions*, Cambridge Tracts in Math. and Math. Phys., No. 55, Cambridge University Press, London, 1965.
- [5] D. Dai, M.E.H. Ismail and X.-S. Wang, Plancherel–Rotach asymptotic expansion for some polynomials from indeterminate moment problems, *Constructive Approximation* **40** (2014), 61–104.
- [6] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotic for the MKdV equation, *Ann. of Math.* **137**(2) (1993), 295–368.
- [7] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel and H.-O. Walther, *Delay Equations. Functional, Complex, and Non-linear Analysis*, Applied Mathematical Sciences, Vol. 110, Springer-Verlag, New York, 1995.
- [8] Digital Library of Mathematical Functions, 2011-08-29, National Institute of Standards and Technology, available at: <http://dlmf.nist.gov/>.
- [9] J. Forde and P.W. Nelson, Applications of Sturm sequences to bifurcation analysis of delay differential equation models, *J. Math. Anal. Appl.* **300**(2) (2004), 273–284.
- [10] S.A. Gourley and Y. Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, *J. Math. Biol.* **49**(2) (2004), 188–200.
- [11] S.A. Gourley, R. Liu and J. Wu, Spatiotemporal distributions of migratory birds: patchy models with delay, *SIAM J. Appl. Dyn. Syst.* **9** (2010), 589–610.
- [12] D.Ya. Khusainov, A.F. Ivanov and I.V. Kovarzh, Solution of one heat equation with delay, *Nonlinear Oscil.* **12**(2) (2009), 260–282.
- [13] D.Ya. Khusainov and G.V. Shuklin, On relative controllability in systems with pure delay, *Internat. Appl. Mech.* **41**(2) (2005), 210–221.
- [14] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, Vol. 191, Academic Press, Boston, MA, 1993.
- [15] Y. Kuang and J.M. Cushing, Global stability in a nonlinear difference-delay equation model of flour beetle population growth, *J. Differ. Equations Appl.* **2**(1) (1996), 31–37.
- [16] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974. (Reprinted by A.K. Peters, Wellesley, MA, 1997.)
- [17] H. Shu, L. Wang and J. Wu, Global dynamics of Nicholson’s blowflies equation revisited: Onset and termination of nonlinear oscillations, *J. Differential Equations* **255**(9) (2013), 2565–2586.
- [18] H. Smith, Monotone semiflows generated by functional-differential equations, *J. Differential Equations* **66**(3) (1987), 420–442.

- [19] H. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, Vol. 41, American Mathematical Society, Providence, RI, 1995.
- [20] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Texts in Applied Mathematics, Vol. 57, Springer-Verlag, New York, 2011.
- [21] X.-S. Wang and R. Wong, Asymptotics of orthogonal polynomials via recurrence relations, *Anal. Appl.* **10** (2012), 215–235.
- [22] Z. Wang, Asymptotic equations for second-order linear difference equations with a turning point, PhD thesis, City University of Hong Kong, 2001.
- [23] R. Wong, *Asymptotic Approximations of Integrals*, Academic Press, Boston, 1989. (Reprinted by SIAM, Philadelphia, PA, 2001.)
- [24] J. Wu, *Theory and Applications of Partial Functional-Differential Equations*, Applied Mathematical Sciences, Vol. 119, Springer-Verlag, New York, 1996.
- [25] D. Xu and X.-Q. Zhao, Dynamics in a periodic competitive model with stage structure, *J. Math. Anal. Appl.* **311**(2) (2005), 417–438.
- [26] S. Yi, P.W. Nelson and A.G. Ulsoy, Delay differential equations via the matrix Lambert W function and bifurcation analysis: Application to machine tool chatter, *Math. Biosci. Eng.* **4**(2) (2007), 355–368.
- [27] X.-Q. Zhao, *Dynamical Systems in Population Biology*, CMS Books Math./Ouvrages Math. SMC, Vol. 16, Springer-Verlag, New York, 2003.

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