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Discrete analogues of Laplace's approximation

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Abstract. An asymptotic formula is derived for the sum

$$I_n(1|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)}$$

as $n \to \infty$, where $f_n(k)$ and $g_n(k)$ are functions defined on nonnegative integers and $0 < q < 1$. This formula is a discrete analogue of Laplace's approximation for integrals. Corresponding results are also provided for the more general sum

$$I_n(z|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)} z^k$$

which is typically an nth order polynomial. The results obtained are then used to give asymptotic formulas for the q^{-1}-Hermite polynomial $h_n(x|q)$, the Stieltjes–Wigert polynomial $S_n(x;q)$ and the q-Laguerre polynomial $L_n^{\alpha}(x;q)$.

Keywords: Laplace's approximation, q-Airy function, q^{-1}-Hermite polynomial, Stieltjes–Wigert polynomial, q-Laguerre polynomial

1. Introduction

Let $\phi(x)$ and $h(x)$ be two real-valued continuous functions defined in the finite interval $\alpha \leqslant x \leqslant \beta$. Assume that $h(x)$ has a single minimum in the interval, namely at $x = \alpha$, and that the infimum of $h(x)$ in any closed sub-interval not containing α is greater than $h(\alpha)$. Furthermore, assume that $h''(x)$ is continuous, $h'(\alpha) = 0$ and $h''(\alpha) > 0$. Then, Laplace's approximation states that the integral

$$I(\lambda) = \int_{\alpha}^{\beta} \phi(x)\, \mathrm{e}^{-\lambda h(x)}\, \mathrm{d}x \tag{1.1}$$

has the asymptotic formula

$$I(\lambda) \sim \phi(\alpha)\, \mathrm{e}^{-\lambda h(\alpha)} \left[\frac{\pi}{2\lambda h''(\alpha)}\right]^{\frac{1}{2}} \tag{1.2}$$

as $\lambda \to +\infty$; see [1, p. 39] or [6, p. 57].

Now, put $\lambda = n^2$ and make the change of variable $x = \alpha + (\beta - \alpha)t$ so that the integral in (1.1) becomes

$$I(n^2) = (\beta - \alpha)\phi(\alpha)\,\mathrm{e}^{-n^2 h(\alpha)} \int_0^1 f(t)\,\mathrm{e}^{-n^2 g(t)}\,\mathrm{d}t, \tag{1.3}$$

where $f(t) := \phi(x)/\phi(\alpha)$ and $g(t) := h(x) - h(\alpha)$. If we set $q := \mathrm{e}^{-1}, k := nt, f_n(k) := \frac{1}{n} f(\frac{k}{n})$ and $g_n(k) := n^2 g(\frac{k}{n})$, then the integral in (1.3) can be written as

$$\int_0^1 f(t)\,\mathrm{e}^{-n^2 g(t)}\,\mathrm{d}t = \int_0^n f_n(k) q^{g_n(k)}\,\mathrm{d}k. \tag{1.4}$$

A discrete form of the last integral is the finite sum

$$I_n(1|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)}, \tag{1.5}$$

and the purpose of this paper is to investigate the behavior of the sum $I_n(1|q)$ and its more general form

$$I_n(z|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)} z^k \tag{1.6}$$

as $n \to \infty$. The results obtained will be used to give asymptotic formulas for the q^{-1}-Hermite polynomial $h_n(x|q)$, the Stieltjes–Wigert polynomial $S_n(x;q)$ and the q-Laguerre polynomial $L_n^{\alpha}(x;q)$. These formulas will then be compared with those provided recently by Ismail and Zhang [4]. As will be shown, our formulas are simpler and our error estimates are sharper.

2. Behavior of $I_n(1|q)$

We first consider the sum $I_n(1|q)$ given in (1.5). As we shall see, its asymptotic behavior is given in terms of the q-Theta function defined by

$$\Theta_q(z) := \sum_{k=-\infty}^{\infty} q^{k^2} z^k, \quad 0 < q < 1; \tag{2.1}$$

see [5, p. 463]. Note that $\Theta_q(1)$ is a continuous function of $q \in (0, 1)$, since the infinite sum $\sum_{k=-\infty}^{\infty} q^{k^2}$ converges uniformly for q in any compact subset of $(0, 1)$.

Theorem 1. *Assume that the following conditions hold*:

(i) $f_n(0) = 1, g_n(0) = 0$;
(ii) *there exists a constant* $M > 0$ *such that* $|f_n(k)| \leqslant M$ *for* $0 \leqslant k \leqslant n$;
(iii) *for any* $\delta \in (0, 1)$ *there exist a constant* $A_\delta > 0$ *and a positive integer* $N(\delta)$ *such that* $g_n(k) \geqslant A_\delta n^2$ *for all* $n\delta \leqslant k \leqslant n$ *and* $n > N(\delta)$;

(iv) *for some fixed* $c_0 > 0$ *and for any small* $\varepsilon > 0$*, there exist* $\delta(\varepsilon) > 0$ *and* $N(\varepsilon) \in \mathbb{N}$ *such that* $|f_n(k) - 1| < \varepsilon$ *and* $|g_n(k) - c_0 k^2| \leqslant \varepsilon k^2$*, whenever* $0 \leqslant k \leqslant n\delta(\varepsilon)$ *and* $n > N(\varepsilon)$*.*

Then, we have

$$I_n(1|q) := \sum_{k=0}^{n} f_n(k) q^{g_n(k)} \sim \frac{1}{2}\big[\Theta_{\widetilde{q}}(1) + 1\big] \quad \text{as } n \to \infty, \tag{2.2}$$

where $\widetilde{q} := q^{c_0}$*.*

Proof. For any small $\varepsilon > 0$, we choose $\delta := \delta(\varepsilon)$ and $N(\varepsilon)$ as in (iv). Split the sum $I_n(1|q)$ into two so that $I_n(1|q) = I_1^* + I_2^*$, where

$$I_1^* := \sum_{k=0}^{\lfloor n\delta \rfloor} f_n(k) q^{g_n(k)} \quad \text{and} \quad I_2^* := \sum_{k=\lfloor n\delta \rfloor + 1}^{n} f_n(k) q^{g_n(k)}.$$

Simple estimation gives

$$I_1^* < \sum_{k=0}^{\lfloor n\delta \rfloor} (1+\varepsilon) q^{k^2(c_0 - \varepsilon)}$$

and

$$I_1^* > \sum_{k=0}^{\lfloor n\delta \rfloor} (1-\varepsilon) q^{k^2(c_0 + \varepsilon)},$$

from which we obtain

$$\frac{1-\varepsilon}{2}\big[\Theta_{q^{c_0+\varepsilon}}(1) + 1\big] \leqslant \varliminf_{n\to\infty} I_1^* \leqslant \varlimsup_{n\to\infty} I_1^* \leqslant \frac{1+\varepsilon}{2}\big[\Theta_{q^{c_0-\varepsilon}}(1) + 1\big].$$

By conditions (ii) and (iii), we also have

$$|I_2^*| \leqslant \sum_{k=\lfloor n\delta \rfloor + 1}^{n} M q^{n^2 A_\delta} \leqslant n M q^{n^2 A_\delta}.$$

Thus, $\lim_{n\to\infty} I_2^* = 0$ and

$$\frac{1-\varepsilon}{2}\big[\Theta_{q^{c_0+\varepsilon}}(1) + 1\big] \leqslant \varliminf_{n\to\infty} I_n(1|q) \leqslant \varlimsup_{n\to\infty} I_n(1|q) \leqslant \frac{1+\varepsilon}{2}\big[\Theta_{q^{c_0-\varepsilon}}(1) + 1\big].$$

Since ε is arbitrary, the desired result (2.2) follows. □

3. Behavior of $I_n(z|q)$

In order to give applications to q-orthogonal polynomials, we need consider the sum (1.6)

$$I_n(z|q) = \sum_{k=0}^{n} f_n(k) q^{g_n(k)} z^k,$$

where, as before, $q \in (0,1)$, f_n and g_n are real-valued functions defined on $\mathbb{N}$, and z is a complex variable.

Theorem 2. *Assume that the following conditions hold*:

(i) *there is a number* $l \in (0,1)$ *such that* $\lim_{n\to\infty} f_n(\lfloor nl \rfloor) = 1$ *and* $\lim_{n\to\infty} g_n(\lfloor nl \rfloor) = 0$;

(ii) *there exists a constant* $M > 0$ *such that* $|f_n(k)| \leqslant M$ *for* $0 \leqslant k \leqslant n$;

(iii) *for any* $0 < \delta < l$, *there exist* $A_\delta > 0$ *and* $N(\delta) \in \mathbb{N}$ *such that* $g_n(k) \geqslant n^2 A_\delta$ *for all* $k \in [0, n(l-\delta)] \cup [n(l+\delta), n]$ *and* $n > N(\delta)$;

(iv) *for any small* $\varepsilon > 0$, *there exist* $\delta(\varepsilon) > 0$ *and* $N(\varepsilon) \in \mathbb{N}$ *such that* $|f_n(k) - 1| < \varepsilon$ *and* $|g_n(k) - b_n(k - \lfloor nl \rfloor) - c_0(k - \lfloor nl \rfloor)^2| < \varepsilon (k - \lfloor nl \rfloor)^2$ *for* $n(l - \delta(\varepsilon)) \leqslant k \leqslant n(l + \delta(\varepsilon))$ *and* $n > N(\varepsilon)$, *where* $\sup_n |b_n| \leqslant L$.

Then, we have

$$I_n(z|q) = z^{\lfloor nl \rfloor} \left[\Theta_{\widetilde{q}}(w_n) + \mathrm{o}(1) \right] \quad \text{as } n \to \infty, \tag{3.1}$$

for all $z \in T_R := \{z \in \mathbb{C}\colon\ R^{-1} \leqslant |z| \leqslant R\}$, *where* $\widetilde{q} = q^{c_0}$ *and* $w_n = q^{b_n} z$.

Remark 1. Condition (i) in Theorem 2 can always be satisfied, if we consider, instead of $I_n(z|q)$, the sum

$$\overline{I}_n(z|q) = \frac{1}{f_n(\lfloor nl \rfloor)} q^{-g_n(\lfloor nl \rfloor)} I_n(z|q) = \sum_{k=0}^{n} \frac{f_n(k)}{f_n(\lfloor nl \rfloor)} q^{g_n(k) - g_n(\lfloor nl \rfloor)}.$$

Condition (iv) in the theorem is the discrete analogue of the conditions that f_n is continuous and g_n is twice continuously differentiable at $k = \lfloor nl \rfloor$ with $g_n'(\lfloor nl \rfloor) = b_n$ and $g_n''(\lfloor nl \rfloor) = 2c_0$.

Before proving Theorem 2, let us first establish the following stronger result.

Theorem 3. *Assume that the conditions* (i), (ii) *and* (iii) *in Theorem* 2 *hold. If condition* (iv) *in that theorem is strengthened to*

(iv′) *for any small* $\delta > 0$, *there exist a function* $\eta_n(\delta)$ *with* $\lim_{n\to\infty} \eta_n(\delta) = 0$ *and a positive integer* $N(\delta)$ *such that* $|f_n(k) - 1| \leqslant \eta_n(\delta)$ *and* $|g_n(k) - b_n(k - \lfloor nl \rfloor) - c_0(k - \lfloor nl \rfloor)^2| \leqslant \eta_n(\delta)(k - \lfloor nl \rfloor)^2$ *for all* k *in* $n(l - \delta) \leqslant k \leqslant n(l + \delta)$ *and all* $n > N(\delta)$,

then the error $r_n := z^{-\lfloor nl \rfloor} I_n(z|q) - \Theta_{\widetilde{q}}(w_n)$ *in the approximation* (3.1) *satisfies*

$$|r_n| \leqslant C\big(\eta_n(\delta) + q^{n^2 A_\delta (1-\delta)} + q^{c_0 n^2 \delta^2 (1-\delta)}\big) \tag{3.2}$$

for sufficiently large n, where C is a constant depending on q, M, R, L, and c_0. Furthermore, the estimate is uniform for z in the annulus T_R given in Theorem 2.

Proof. Clearly,

$$r_n = \sum_{k=-\lfloor nl\rfloor}^{n-\lfloor nl\rfloor} f_n(k+\lfloor nl\rfloor) q^{g_n(k+\lfloor nl\rfloor)} z^k - \sum_{k=-\infty}^{\infty} q^{k^2 c_0 + k b_n} z^k.$$

We write the first sum as

$$\sum_{k=-\lfloor nl\rfloor}^{-\lfloor n\delta\rfloor-1} + \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} + \sum_{k=\lfloor n\delta\rfloor+1}^{n-\lfloor nl\rfloor},$$

and the second sum as

$$\sum_{k=-\infty}^{-\lfloor n\delta\rfloor-1} + \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} + \sum_{k=\lfloor n\delta\rfloor+1}^{\infty}.$$

Thus,

$$r_n = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$I_1 = \sum_{k=\lfloor n\delta\rfloor+1}^{n-\lfloor nl\rfloor} f_n(k+\lfloor nl\rfloor) q^{g_n(k+\lfloor nl\rfloor)} z^k,$$

$$I_2 = -\sum_{k=\lfloor n\delta\rfloor+1}^{\infty} q^{k^2 c_0 + k b_n} z^k,$$

$$I_3 = \sum_{k=-\lfloor nl\rfloor}^{-\lfloor n\delta\rfloor-1} f_n(k+\lfloor nl\rfloor) q^{g_n(k+\lfloor nl\rfloor)} z^k,$$

$$I_4 = -\sum_{k=-\infty}^{-\lfloor n\delta\rfloor-1} q^{k^2 c_0 + k b_n} z^k,$$

$$I_5 = \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} f_n(k+\lfloor nl\rfloor) \left[q^{g_n(k+\lfloor nl\rfloor)} - q^{k^2 c_0 + k b_n}\right] z^k$$

and

$$I_6 = \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} \left[f_n(k+\lfloor nl\rfloor) - 1\right] q^{k^2 c_0 + k b_n} z^k.$$

For sufficiently large n, we have

$$|I_1| \leqslant \sum_{k=\lfloor n\delta\rfloor+1}^{n-\lfloor nl\rfloor} Mq^{n^2A_\delta}R^k \leqslant nMq^{n^2A_\delta}R^n \leqslant q^{n^2A_\delta(1-\delta)},$$

$$\begin{aligned}|I_2| &\leqslant \sum_{m=0}^{\infty} q^{m^2c_0+(\lfloor n\delta\rfloor+1)^2c_0-(m+\lfloor n\delta\rfloor+1)L}R^{m+\lfloor n\delta\rfloor+1} \\ &\leqslant q^{(\lfloor n\delta\rfloor+1)^2c_0-(\lfloor n\delta\rfloor+1)L}R^{\lfloor n\delta\rfloor+1}\Theta_{q^{c_0}}\left(q^{-L}R\right) \\ &\leqslant q^{c_0n^2\delta^2(1-\delta)},\end{aligned}$$

since $|b_n| \leqslant L$. Similarly, we get

$$|I_3| \leqslant \sum_{k=-\lfloor nl\rfloor}^{-\lfloor n\delta\rfloor-1} Mq^{n^2A_\delta}R^{-k} \leqslant nMq^{n^2A_\delta}R^n \leqslant q^{n^2A_\delta(1-\delta)}$$

and

$$\begin{aligned}|I_4| &\leqslant \sum_{m=-\infty}^{0} q^{m^2c_0+(\lfloor n\delta\rfloor+1)^2c_0+(m-\lfloor n\delta\rfloor-1)L}R^{-m+\lfloor n\delta\rfloor+1} \\ &\leqslant q^{(\lfloor n\delta\rfloor+1)^2c_0-(\lfloor n\delta\rfloor+1)L}R^{\lfloor n\delta\rfloor+1}\Theta_{q^{c_0}}\left(q^{-L}R\right) \\ &\leqslant q^{c_0n^2\delta^2(1-\delta)}\end{aligned}$$

for large enough n.

We next estimate I_5 and I_6. It is evident that

$$I_5 = \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} f_n\left(k+\lfloor nl\rfloor\right)\left[q^{g_n(k+\lfloor nl\rfloor)-k^2c_0-kb_n}-1\right]q^{k^2c_0+kb_n}z^k.$$

By the mean-value theorem, we have

$$|I_5| \leqslant M|\ln q|\eta_n(\delta)\sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} k^2q^{-\eta_n(\delta)k^2+k^2c_0+kb_n}|z|^k,$$

where we have made use of condition (iv′). Since $\mathrm{e}^{|k|} \geqslant \frac{1}{2}k^2$ and $\eta_n(\delta) \to 0$ as $n \to \infty$, the last inequality gives

$$|I_5| \leqslant 4M|\ln q|\eta_n(\delta)\Theta_{q^{c_0/2}}\left(\mathrm{e}q^{-L}R\right)$$

for sufficiently large n. In the same manner, it follows that

$$
\begin{aligned}
|I_6| &\leqslant \sup_{|k|\leqslant\lfloor n\delta\rfloor} \left|f_n(k+\lfloor nl\rfloor)-1\right| \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} q^{k^2c_0+kb_n}|z|^k \\
&\leqslant 2\eta_n(\delta)\Theta_{q^{c_0}}\left(q^{-L}R\right).
\end{aligned}
$$

The desired result (3.2) is obtained by a combination of the estimates for $I_1,\ldots,I_6$. □

Proof of Theorem 2. Here we need to show that $r_n \to 0$ as $n \to \infty$. This can be done as follows:

Let $0 < \varepsilon < c_0/2$, and choose $\delta = \delta(\varepsilon)$ as in condition (iv). We estimate I_1, I_2, I_3 and I_4 as before, and they all tend to zero as $n \to \infty$. As for I_5 and I_6, we also proceed as in Theorem 4, and obtain

$$
\begin{aligned}
|I_5| &\leqslant \varepsilon M|\ln q| \sum_{k=-\lfloor n\delta\rfloor}^{\lfloor n\delta\rfloor} k^2\,\mathrm{e}^{-\varepsilon k^2} q^{k^2c_0+kb_n}|z|^k \\
&\leqslant 4\varepsilon M|\ln q|\Theta_{q^{c_0/2}}\left(\mathrm{e}q^{-L}R\right)
\end{aligned}
$$

and

$$
|I_6| \leqslant 2\varepsilon\Theta_{q^{c_0}}\left(q^{-L}R\right).
$$

Thus, $\overline{\lim}_{n\to\infty}|r_n| \leqslant C\varepsilon$, where C is independent of ε. Since ε is arbitrary, the desired result (3.1) follows. □

4. A generalization

In the previous section, we have always assumed that the function (sequence) $f_n(k)$ behaves like a constant as $n \to \infty$. An example of such is given by $(q^{n-k-1};q)_k := \prod_{i=n-k}^{n-1}(1-q^{i+1})$, which tends to 1 uniformly for $k \in [n\delta, n-n\delta]$ as $n \to \infty$, where δ is any number in $(0, \frac{1}{2})$. However, there are functions $f_n(k)$ whose limits, as $n \to \infty$, are bounded functions of k. For example, the *q-binomial coefficient*

$$
\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \tag{4.1}
$$

where

$$
(a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad a \in \mathbb{C}, \tag{4.2}
$$

is asymptotically equal to $\prod_{i=0}^{k-1}(1-q^{i+1})^{-1}$, as $n \to \infty$, for $k \in [0, n-n\delta]$. In such cases, we will use, instead of the q-Theta function given in (2.1), the more general function defined by

$$
\Phi_q(z) := \sum_{k=0}^{\infty} a_k q^{k^2} z^k. \tag{4.3}
$$

The infinite sum on the right converges uniformly for z in any compact subsets of $\mathbb{C}$, as long as the coefficients a_k are bounded. When $a_k = (-1)^k/(q;q)_k$, the function (4.3) becomes the q-Airy function

$$A_q(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k} q^{k^2} z^k; \tag{4.4}$$

cf. [3].

Theorem 4. *For some a_k with $|a_k| \leqslant K$ and $c_0 > 0$, we assume*

(i) $g_n(0) = 0$,
(ii) *there is a constant $M > 0$ such that $|f_n(k)| \leqslant M$ for $0 \leqslant k \leqslant n$,*
(iii) *for any $\delta \in (0,1)$, there exist $A_\delta > 0$ and $N(\delta) \in \mathbb{N}$ such that $|g_n(k)| \geqslant n^2 A_\delta$ for $n\delta \leqslant k \leqslant n$ and $n > N(\delta)$,*
(iv) *for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $N(\varepsilon) \in \mathbb{N}$ such that $|f_n(k) - a_k| < \varepsilon$ and $|g_n(k) - c_0 k^2| < \varepsilon k^2$ for $0 \leqslant k \leqslant n\delta$ and $n > N(\varepsilon)$.*

Then we have

$$I_n(z|q) = \Phi_{\tilde{q}}(z) + \mathrm{o}(1) \tag{4.5}$$

uniformly for $z \in D_R := \{z \in \mathbb{C}\colon |z| \leqslant R\}$, where $\tilde{q} = q^{c_0}$.
Moreover, if condition (iv) *is replaced by*

(iv′) *for any $\delta > 0$, there exist functions $\eta_n(\delta)$ with $\lim_{n\to\infty} \eta_n(\delta) = 0$ and positive integer $N(\delta)$ such that $|f_n(k) - a_k| \leqslant \eta_n(\delta)$ and $|g_n(k) - c_0 k^2| \leqslant \eta_n(\delta) k^2$ for $0 \leqslant k \leqslant n\delta$ and $n > N(\delta)$,*

then the error $r_n := I_n(z|q) - \Phi_{\tilde{q}}(z)$ in the approximation (4.5) *satisfies*

$$|r_n| \leqslant C\big(\eta_n(\delta) + q^{n^2 A_\delta (1-\delta)} + q^{c_0 n^2 \delta^2 (1-\delta)}\big) \tag{4.6}$$

for sufficiently large n and uniformly for $z \in D_R$, where C is a constant depending on q, M, R, K and c_0.

Proof. Without loss of generality, we may assume that $c_0 = 1$, for otherwise we can replace $g_n(k)$ by $\tilde{g}_n(k) := g_n(k)/c_0$ and q by $\tilde{q} := q^{c_0}$. The assumptions are then satisfied by $\tilde{g}_n(k)$ with $c_0 = 1$.

We now proceed as in the proof of Theorem 3. Write

$$\begin{aligned} r_n &= \sum_{k=0}^{n} f_n(k) q^{g_n(k)} z^k - \sum_{k=0}^{\infty} a_k q^{k^2} z^k \\ &:= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$I_1 := \sum_{k=\lfloor n\delta \rfloor + 1}^{n} f_n(k) q^{g_n(k)} z^k,$$

$$I_2 := \sum_{k=\lfloor n\delta\rfloor+1}^{\infty} a_k q^{k^2} z^k,$$

$$I_3 := \sum_{k=0}^{\lfloor n\delta\rfloor} f_n(k)\left[q^{g_n(k)} - q^{k^2}\right] z^k$$

and

$$I_4 = \sum_{k=0}^{\lfloor n\delta\rfloor} \left[f_n(k) - a_k\right] q^{k^2} z^k.$$

It is clear that for sufficiently large n, we have

$$|I_1| \leqslant M \sum_{k=\lfloor n\delta\rfloor+1}^{n} q^{n^2 A_\delta} R^k \leqslant nMq^{n^2 A_\delta} R^n \leqslant q^{n^2 A_\delta(1-\delta)}$$

and

$$|I_2| \leqslant K \sum_{k=\lfloor n\delta\rfloor+1}^{\infty} q^{k^2} R^k \leqslant K \sum_{l=0}^{\infty} q^{l^2+(\lfloor n\delta\rfloor+1)^2} R^{l+\lfloor n\delta\rfloor+1}$$
$$\leqslant Kq^{(\lfloor n\delta\rfloor+1)^2} R^{\lfloor n\delta\rfloor+1} \Theta_q(R) \leqslant q^{n^2\delta^2(1-\delta)}.$$

Since

$$\left|q^{g_n(k)} - q^{k^2}\right| \leqslant |\ln q|\left|g_n(k) - k^2\right| q^{-|g_n(k)-k^2|+k^2}$$
$$\leqslant |\ln q| \sup_{0<k\leqslant\lfloor n\delta\rfloor} \left|\frac{g_n(k)}{k^2} - 1\right| k^2 q^{k^2/2}$$

for $0 \leqslant k \leqslant \lfloor n\delta\rfloor$, we also have

$$|I_3| \leqslant 2M|\ln q| \sup_{0<k\leqslant\lfloor n\delta\rfloor} \left|\frac{g_n(k)}{k^2} - 1\right| \Theta_{q^{1/2}}(\mathrm{e}R).$$

Similarly, one gets

$$|I_4| \leqslant \sup_{0\leqslant k\leqslant\lfloor n\delta\rfloor} \left|f_n(k) - a_k\right| \Theta_q(R).$$

The required results (4.5) and (4.6) now follow from conditions (iv) and (iv′), respectively. □

5. Applications

In this section, we shall apply our theorems to the q^{-1}-Hermite polynomials $h_n(x|q)$, the Stieltjes–Wigert polynomials $S_n(x;q)$ and the q-Laguerre polynomials $L_n^\alpha(x;q)$. These polynomials are defined by

$$h_n(\sinh\xi|q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2-nk}(-1)^k\, \mathrm{e}^{(n-2k)\xi}, \tag{5.1}$$

$$S_n(x;q) := \frac{1}{(q;q)_n}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}(-x)^k \tag{5.2}$$

and

$$L_n^\alpha(x;q) := \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2+\alpha k}\frac{(-x)^k}{(q^{\alpha+1};q)_k}; \tag{5.3}$$

see [2] and [3].

To derive Plancherel–Rotach asymptotic formulas for these polynomials, we shall rescale the variables as was done in [4]. Thus, for the q^{-1}-Hermite polynomials, we set

$$\sinh\xi_n := \frac{1}{2}\left(q^{-nt}u - q^{nt}u^{-1}\right) \tag{5.4}$$

with $u \neq 0$ and $t \geqslant 0$. For the Stieltjes–Wigert polynomials and the q-Laguerre polynomials, we set $x_n(t,u) := q^{-nt}u$ with $u \neq 0$ and $t \geqslant 1$. After rescaling, we get

$$h_n(\sinh\xi_n|q) = u^n q^{-n^2 t}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}\left(-u^2 q^{n(2t-1)}\right)^k, \tag{5.5}$$

$$S_n\left(x_n(t,u);q\right) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}\left(-u^{-1} q^{n(t-2)}\right)^k \tag{5.6}$$

and

$$L_n^\alpha(x_n(t,u);q) = \frac{(-uq^\alpha)^n q^{n^2(1-t)}}{(q;q)_n}\sum_{k=0}^{n}\left(q^{\alpha+1+n-k};q\right)_k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2}\left(-u^{-1} q^{n(t-2)-\alpha}\right)^k. \tag{5.7}$$

Corollary 1. *Let ξ_n be defined as in* (5.4). *For $t \geqslant 1/2$, we have*

$$h_n(\sinh\xi_n|q) = u^n q^{-n^2 t}\left\{A_q\left(u^{-2}q^{n(2t-1)}\right) + \mathrm{O}\left(q^{n(1-\delta)}\right)\right\}, \tag{5.8}$$

where $0 < \delta < 1$ *is any small number. Furthermore, this asymptotic formula holds uniformly for* $u^{-1} \in D_R := \{z \in \mathbb{C}\colon |z| \leqslant R\}$ *with* $R > 0$ *being any large real number. On the other hand, for* $0 \leqslant t < 1/2$, *we have*

$$h_n(\sinh \xi_n|q) = \frac{(-1)^m u^{n-2m}}{(q;q)_\infty q^{n^2 t + m[n(1-2t)-m]}} \{\Theta_q\big(-u^{-2} q^{2m-n(1-2t)}\big) + \mathrm{O}\big(q^{n(l-\delta)}\big)\}, \tag{5.9}$$

where $l := \frac{1}{2}(1-2t), m := \lfloor nl \rfloor$ *and* $\delta > 0$ *is any small number. The* O*-term in* (5.9) *is uniform with respect to* $u \in T_R := \{z \in \mathbb{C}\colon R^{-1} \leqslant |z| \leqslant R\}$ *with* $R > 0$ *being any large real number.*

Proof. When $t \geqslant 1/2$, we apply Theorem 4 to (5.5) with

$$a_k = \frac{(-1)^k}{(q;q)_k}, \qquad f_n(k) = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad g_n(k) = k^2, \qquad z = u^{-2} q^{n(2t-1)},$$

$$c_0 = 1, \qquad M = \frac{1}{(q;q)_\infty}, \qquad A_\delta = \delta^2, \qquad \eta_n(\delta) = \frac{q^{n(1-\delta)}}{(q;q)_\infty (1-q)}.$$

We only need to verify that $|f_n(k) - a_k| \leqslant \eta_n(\delta)$ for $0 \leqslant k \leqslant n\delta$ and sufficiently large n. The other conditions in Theorem 4 are easily seen to hold. For any positive integers m and k, we have $q^m (q^{m+1}; q)_k < q^m$, and hence

$$\begin{aligned} 1 - (q^m;q)_k &= q^m + \big(1-q^m\big)\big[1 - \big(q^{m+1};q\big)_{k-1}\big] < q^m + 1 - \big(q^{m+1};q\big)_{k-1} \\ &< q^m + q^{m+1} + \cdots + q^{m+k-1} = \frac{q^m - q^{m+k}}{1-q} < \frac{q^m}{1-q}. \end{aligned} \tag{5.10}$$

Letting $k \to \infty$ yields $1 - (q^m;q)_\infty \leqslant \frac{q^m}{1-q}$. Thus,

$$\big|f_n(k) - a_k\big| = \frac{1 - (q^{n-k+1};q)_k}{(q;q)_k} \leqslant \frac{q^{n-k+1}}{(1-q)(q;q)_k} \leqslant \frac{q^{n(1-\delta)}}{(1-q)(q;q)_\infty} = \eta_n(\delta) \tag{5.11}$$

for $0 \leqslant k \leqslant n\delta$. The asymptotic formula (5.8) now follows from (4.5) with $\Phi_{\widetilde{q}}(z)$ replaced by $A_q(z)$; see the statement following (4.3).

When $0 \leqslant t < 1/2$, we apply Theorem 3 with

$$l = \frac{1-2t}{2}, \qquad m = \lfloor nl \rfloor, \qquad f_n(k) = (q;q)_\infty \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad g_n(k) = k^2 - 2nlk + m(2nl - m),$$

$$z = -u^2, \qquad c_0 = 1, \qquad M = 1, \qquad b_n = 2(m - nl), \qquad L = 2,$$

$$A_\delta = \delta^2(1-\delta), \qquad \eta_n(\delta) = \frac{2q^{n(l-\delta)}}{1-q}.$$

To verify condition (ii) in Theorem 2 (which is also assumed in Theorem 3), we choose $N(\delta) = \lfloor 2/\delta^2 \rfloor$. Then it is readily seen that for $k \in [0, n(l-\delta)] \cup [n(l+\delta) - 1, n]$ and $n > N(\delta)$, we have

$$g_n(k) = (k - nl)^2 - (m - nl)^2 > (n\delta - 1)^2 - 1 > n^2 \delta^2 (1-\delta) = n^2 A_\delta. \tag{5.12}$$

To show that condition (iv′) in Theorem 3 also holds, we first note that $(1-a)(1-b) > 0$ for $0 < a, b < 1$, and hence $1 - a + 1 - b > 1 - ab$ and

$$|f_n(k) - 1| = 1 - (q^{k+1}; q)_\infty (q^{n-k+1}; q)_\infty < 1 - (q^{k+1}; q)_\infty + 1 - (q^{n-k+1}; q)_k.$$

Using the inequality following (5.10), we obtain

$$|f_n(k) - 1| \leqslant \frac{q^{k+1}}{1-q} + \frac{q^{n-k+1}}{1-q} \leqslant \frac{2q^{n(l-\delta)}}{1-q} = \eta_n(\delta) \tag{5.13}$$

for $k \in [n(l-\delta), n(l+\delta)]$, since $l \leqslant 1/2$. Next, we observe that

$$\begin{aligned} g_n(k) &= k^2 - 2nlk + m(2nl - m) = k^2 - m^2 - 2nl(k-m) \\ &= (k-m)(k+m-2nl) = 2(k-m)(m-nl) + (k-m)^2. \end{aligned}$$

With $c_0 = 1, m = \lfloor nl \rfloor$ and $b_n = 2(m - nl)$, the last equation becomes

$$g_n(k) = (k - \lfloor nl \rfloor) b_n + c_0 (k - \lfloor nl \rfloor)^2, \tag{5.14}$$

thus establishing condition (iv′). Formula (5.9) now follows from (3.1) and (3.2). □

Remark 2. Comparing Corollary 1 above with Theorem 2.1 in [4], our results have two advantages. First, when $t \geqslant 1/2$, our estimate for the error is $\mathrm{O}(q^{n(1-\delta)})$ for any $\delta > 0$, while the error estimate in [4] is $\mathrm{O}(q^{n/2})$. Second, when $0 \leqslant t < 1/2$, we give a single formula (5.9), whereas it takes two formulas in [4] to cover this case, one when t is rational and the other when t is irrational. Here, probably it should also be pointed out that the reason why the error estimate in [4, Eq. (25)] is only $\mathrm{O}(\log n/n)$ when t is irrational is because of the fact that $\Theta_q(q^{1/n}) - \Theta_q(1) = \mathrm{O}(\log n/n)$.

In a similar manner, we will now apply Theorem 3 and Theorem 4 to (5.6) and (5.7) to obtain asymptotic formulas for the Stieltjes–Wigert polynomials and the q-Laguerre polynomials.

Corollary 2. *Let $x_n(t,u) := q^{-nt}u$ with $u \neq 0$ and $t \geqslant 1$. When $t \geqslant 2$, we have*

$$S_n(x_n(t,u); q) = \frac{(-u)^n q^{n^2(1-t)}}{(q;q)_n} \{ A_q(u^{-1} q^{n(t-2)}) + \mathrm{O}(q^{n(1-\delta)}) \} \tag{5.15}$$

uniformly for $u^{-1} \in D_R := \{z \in \mathbb{C}\colon |z| \leqslant R\}$, where $\delta > 0$ is any small number and $R > 0$ is any large real number. When $1 \leqslant t < 2$, we have

$$S_n(x_n(t,u); q) = \frac{(-u)^{n-m} q^{n^2(1-t) - m[n(2-t)-m]}}{(q;q)_n (q;q)_\infty} \{ \Theta_q(-u^{-1} q^{2m - n(2-t)}) + \mathrm{O}(q^{n(l-\delta)}) \}, \tag{5.16}$$

where $l := \frac{1}{2}(2-t), m := \lfloor nl \rfloor$ and $\delta > 0$ is any small number. This asymptotic formula holds uniformly for $u \in T_R := \{z \in \mathbb{C}\colon R^{-1} \leqslant |z| \leqslant R\}$, where $R > 0$ is any large real number.

Proof. For $t \geqslant 2$, we apply Theorem 4 to (5.6) with

$$a_k = \frac{(-1)^k}{(q;q)_k}, \qquad f_n(k) = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad g_n(k) = k^2, \qquad z = u^{-1} q^{n(t-2)},$$

$$c_0 = 1, \qquad M = \frac{1}{(q;q)_\infty}, \qquad A_\delta = \delta^2, \qquad \eta_n(\delta) = \frac{q^{n(1-\delta)}}{(q;q)_\infty (1-q)}.$$

For $1 \leqslant t < 2$, we apply Theorem 3 to (5.6) with

$$l = 1 - \frac{t}{2}, \qquad m = \lfloor nl \rfloor, \qquad f_n(k) = (q;q)_\infty \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad g_n(k) = k^2 - 2nlk + m(2nl - m),$$

$$z = -u^{-1}, \qquad c_0 = 1, \qquad M = 1, \qquad b_n = 2(m - nl), \qquad L = 2,$$

$$A_\delta = \delta^2(1-\delta), \qquad \eta_n(\delta) = \frac{2q^{n(l-\delta)}}{1-q}.$$

The arguments for verifying the conditions in Theorems 3 and 4 are the same as those used in the proof of Corollary 1. □

Corollary 3. *Assume that α is real and $\alpha > -1$. Let $x_n(t,u) := q^{-nt}u$ with $u \neq 0$ and $t \geqslant 1$. When $t \geqslant 2$, we have*

$$L_n^\alpha(x_n(t,u);q) = \frac{(-uq^\alpha)^n q^{n^2(1-\delta)}}{(q;q)_n} \{A_q(u^{-1}q^{n(t-2)-\alpha}) + O(q^{n(1-\delta)})\} \tag{5.17}$$

uniformly for $u^{-1} \in D_R := \{z \in \mathbb{C}\colon |z| \leqslant R\}$, where $R > 0$ is any large real number. When $1 \leqslant t < 2$, we have

$$L_n^\alpha(x_n(t;u);q) = \frac{(-uq^\alpha)^{n-m} q^{n^2(1-t)-m[n(2-t)-m]}}{(q;q)_n (q;q)_\infty} \{\Theta_q(-u^{-1}q^{2m-n(2-t)-\alpha}) + O(q^{n(l-\delta)})\}, \tag{5.18}$$

where $l := 1 - \frac{t}{2}, m := \lfloor nl \rfloor$ and $\delta > 0$ is any small number. The asymptotic formula holds uniformly for $u \in T_R := \{z \in \mathbb{C}\colon R^{-1} \leqslant |z| \leqslant R\}$, where $R > 0$ is any large real number.

Proof. For $t \geqslant 2$, we apply Theorem 4 to (5.7) with

$$a_k = \frac{(-1)^k}{(q;q)_k}, \qquad f_n(k) = (-1)^k (q^{\alpha+1+n-k};q)_k \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad g_n(k) = k^2,$$

$$z = u^{-1}q^{n(t-2)-\alpha}, \qquad c_0 = 1, \qquad M = \frac{1}{(q;q)_\infty}, \qquad A_\delta = \delta^2, \qquad \eta_n(\delta) = \frac{2q^{n(1-\delta)}}{(q;q)_\infty(1-q)}.$$

Simple calculation gives

$$|f_n(k) - a_k| = \frac{1 - (q^{\alpha+1+n-k};q)_k (q^{n-k+1};q)_k}{(q;q)_k}.$$

As in Corollary 1, by using the inequality $1-a+1-b>1-ab$ for $0<a,b<1$, we obtain

$$\left|f_n(k)-a_k\right| \leqslant \frac{1-(q^{\alpha+1+n-k};q)_k+1-(q^{n-k+1};q)_k}{(q;q)_k}.$$

Since $\alpha>-1$, it follows from (5.10) that

$$\left|f_n(k)-a_k\right| \leqslant \frac{q^{\alpha+1+n-k}+q^{n-k+1}}{(1-q)(q;q)_k} \leqslant \frac{2q^{n-k}}{(1-q)(q;q)_k}.$$

Since $(q;q)_k \geqslant (q;q)_\infty$, for $0\leqslant k\leqslant n\delta$ we have

$$\left|f_n(k)-a_k\right| \leqslant \frac{2q^{n(1-\delta)}}{(1-q)(q;q)_\infty} = \eta_n(\delta).$$

When $1\leqslant t<2$, we apply Theorem 3 to (5.7) with

$$l=1-\frac{1}{2}t, \qquad m=\lfloor nl\rfloor, \qquad f_n(k)=(q;q)_\infty\left(q^{\alpha+1+n-k};q\right)_k\begin{bmatrix} n\\ k\end{bmatrix}_q,$$

$$g_n(k)=k^2-2nlk+m(2nl-m), \qquad z=-u^{-1}q^{-\alpha},$$

$$c_0=1, \qquad M=1, \qquad b_n=2(m-nl), \qquad L=2,$$

$$A_\delta=\delta^2(1-\delta), \qquad \eta_n(\delta)=\frac{3q^{n(l-\delta)}}{1-q}.$$

The verification of condition (iv′) in Theorem 3 proceeds along the same lines as that given in Corollary 1. In particular, since

$$\left|f_n(k)-1\right| = 1-\left(q^{\alpha+1+n-k};q\right)_k\left(q^{n-k+1};q\right)_k\left(q^{k+1};q\right)_\infty$$

and $1-a+1-b+1-c>1-ab+1-c>1-abc$ for a,b and $c\in(0,1)$, the right-hand side of the last equality is less than or equal to

$$1-\left(q^{\alpha+1+n-k};q\right)_k+1-\left(q^{n-k+1};q\right)_k+1-\left(q^{k+1};q\right)_\infty$$

and we have by (5.10)

$$\left|f_n(k)-1\right| \leqslant \frac{q^{\alpha+1+n-k}+q^{n-k+1}+q^{k+1}}{1-q} \leqslant \frac{3q^{n(l-\delta)}}{1-q} = \eta_n(\delta)$$

for $k\in[n(l-\delta),n(l+\delta)]$. $\square$

When $t<1$, asymptotic formulas of the Stieltjes–Wigert polynomials and the q-Laguerre polynomials can be obtained by applying Theorems 3 and 4 to the two sums

$$S_n(x;q)=\frac{1}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{k^2}\left(-q^{-nt}u\right)^k, \tag{5.19}$$

$$L_n^\alpha(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k^2+\alpha k}}{(q^{\alpha+1};q)_k} \bigl(-q^{-nt}u\bigr)^k, \tag{5.20}$$

where again $x = q^{-nt}u$; see (5.2) and (5.3).

Indeed, let $-t = \widetilde{t} - 2$ and $u = \widetilde{u}^{-1}$ in (5.19). Then this equation becomes

$$S_n(x;q) = \frac{1}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \bigl(-q^{n(\widetilde{t}-2)}\widetilde{u}^{-1}\bigr)^k.$$

When $\widetilde{t} \geqslant 2$ (i.e., $t \leqslant 0$), we have by the argument used for Corollary 2

$$\begin{aligned} S_n(x;q) &= \frac{1}{(q;q)_n} \bigl\{A_q\bigl(\widetilde{u}^{-1}q^{n(\widetilde{t}-2)}\bigr) + \mathrm{O}\bigl(q^{n(1-\delta)}\bigr)\bigr\} \\ &= \frac{1}{(q;q)_n} \bigl\{A_q\bigl(uq^{-nt}\bigr) + \mathrm{O}\bigl(q^{n(1-\delta)}\bigr)\bigr\}, \end{aligned} \tag{5.21}$$

where δ is any small number; see (5.15). This result holds uniformly for $u = \widetilde{u}^{-1} \in D_R = \{z \in \mathbb{C}\colon |z| \leqslant R\}$ with $R > 0$ being any large real number.

When $1 < \widetilde{t} < 2$ (i.e., $0 < t < 1$), again by the argument used for Corollary 2 we obtain

$$\begin{aligned} S_n(x;q) &= \frac{(-\widetilde{u})^{-\widetilde{m}} q^{-\widetilde{m}[n(2-\widetilde{t})-\widetilde{m}]}}{(q;q)_n (q;q)_\infty} \bigl\{\Theta_q\bigl(-\widetilde{u}^{-1}q^{2\widetilde{m}-n(2-\widetilde{t})}\bigr) + \mathrm{O}\bigl(q^{n(\widetilde{l}-\delta)}\bigr)\bigr\} \\ &= \frac{(-u)^{\widetilde{m}} q^{-\widetilde{m}(nt-\widetilde{m})}}{(q;q)_n (q;q)_\infty} \bigl\{\Theta_q\bigl(-uq^{2\widetilde{m}-nt}\bigr) + \mathrm{O}\bigl(q^{n(\widetilde{l}-\delta)}\bigr)\bigr\}, \end{aligned} \tag{5.22}$$

where $\widetilde{l} = 1 - \frac{1}{2}\widetilde{t} = \frac{1}{2}t, \widetilde{m} = \lfloor n\widetilde{l}\rfloor$ and $\delta > 0$ is any small number; see (5.16). This formula holds uniformly for $u^{-1} = \widetilde{u} \in T_R = \{z \in \mathbb{C} : R^{-1} \leqslant |z| \leqslant R\}$, where R is any large real number.

Remark 3. Note that when $t \leqslant 0$, $u^{-1}q^{n(t-2)}$ is unbounded. Hence, Theorem 4 can not be applied to the representation of $S_n(x;q)$ in (5.6) with $z = u^{-1}q^{n(t-2)}$. However, if we use the alternative representation of $S_n(x;q)$ given in (5.19), Theorem 4 becomes applicable since uq^{-nt} is now uniformly bounded for large n. Also note that as in the proof of Corollary 1, (5.13) is needed in the proof of Corollary 2 for the case $1 \leqslant t < 2$. If $0 < t < 1$, then $l := 1 - t/2 \in (\frac{1}{2}, 1)$ and hence (5.13) fails to hold. There are two approaches to resolve this matter. The first approach is to choose $\eta_n(\delta) = 2q^{n(1-l-\delta)}/(1-q)$, instead of $\eta_n(\delta) = 2q^{n(l-\delta)}/(1-q)$ as was done in the proof of Corollary 2. With the new choice of $\eta_n(\delta)$, (5.13) continues to hold, and we have approximation (5.16) with the error estimate replaced by $\mathrm{O}(q^{n(1-l-\delta)})$. The second approach is to use the representation of $S_n(x;q)$ in (5.19), and by applying Theorem 3 we get the approximation (5.22). A careful calculation shows that these two approximations are exactly the same. In view of the symmetry of $S_n(x;q)$ at $t = 1$, it is preferable to use the representation of $S_n(x;q)$ in (5.6) when $t \geqslant 1$, and the representation of $S_n(x;q)$ in (5.19) for $t < 1$. Results corresponding to (5.21) and (5.22) can be obtained for the q-Laguerre polynomials.

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