Advances in Mathematics 343 (2019) 157–192

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Doubly infinite Jacobi matrices revisited: Resolvent and spectral measure

Dan Dai [a], Mourad E.H. Ismail [b], Xiang-Sheng Wang [c,*]

[a] *Department of Mathematics, City University of Hong Kong, Hong Kong Special Administrative Region*
[b] *Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA*
[c] *Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70503, USA*

A R T I C L E   I N F O

Article history:
Received 30 August 2017
Received in revised form 10 June 2018
Accepted 19 November 2018
Available online xxxx
Communicated by George Andrews

MSC:
47B36
33C45

Keywords:
Doubly infinite Jacobi matrix
Spectral analysis
Lommel polynomials
Associated ultraspherical polynomials
Al-Salam–Ismail polynomials

A B S T R A C T

We study the resolvent and spectral measure of certain doubly infinite Jacobi matrices via asymptotic solutions of two-sided difference equations. By finding the minimal (or subdominant) solutions or calculating the continued fractions for the difference equations, we derive explicit formulas for the matrix entries of resolvent of doubly infinite Jacobi matrices corresponding to Lommel polynomials, associated ultraspherical polynomials, and Al-Salam–Ismail polynomials. The spectral measures are then obtained by inverting Stieltjes transformations.

* Corresponding author.
E-mail addresses: dandai@cityu.edu.hk (D. Dai), mourad.eh.ismail@gmail.com (M.E.H. Ismail), xswang@louisiana.edu (X.-S. Wang).

https://doi.org/10.1016/j.aim.2018.11.017

1. Introduction

The theory of special functions and orthogonal polynomials have seen major developments since the late 1960's. Among the achievements were the discoveries of the Askey–Wilson polynomials and orthogonal polynomials in several variables, advances in the general theory of orthogonal polynomials in one variable, emergence of a combinatorial theory and number theoretic applications, developments of multiple and matrix-valued orthogonal polynomials, and the development of techniques in the qualitative and quantitative theory of orthogonal polynomials.

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials with positive leading terms and $\deg p_n(x) = n$. It is well-known that these polynomials satisfy a three-term recurrence relation of the form

$$xp_n(x) = b_{n+1}p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x) \qquad \text{for } n \geq 1, \tag{1.1}$$

with the initial conditions $p_0(x) = 1, p_1(x) = (x - a_0)/b_1$, where $a_n \in \mathbb{R}$ for $n \geq 0$ and $b_n > 0$ for $n > 0$. When the recurrence coefficients are rational functions of n or of q^n, where q is a fixed real number, one can formally replace n by $n+\gamma$. If $a_{n+\gamma} \in \mathbb{R}, n \geq 0$ and $b_{n+\gamma} > 0, n > 0$, then (1.1) generates a set of *associated orthogonal polynomials*. From the connection with the theory of continued J-fractions, it is sufficient to consider the case $\gamma \in [0, 1)$ to determine the orthogonality measure of the new polynomials. One important early paper on the development of associated polynomials systems is the memoir [26] by Felix Pollaczek who treated a general class of polynomials orthogonal on $[-1, 1]$ including what is now known as the Pollaczek polynomials. Askey and Wimp [3] studied the associated Laguerre and Hermite polynomials. Ismail, Lettesier and Valent [16] gave another treatment of the associated Laguerre and Hermite polynomials and pointed out a second family of associated orthogonal polynomials which comes from interpreting the recurrence coefficients as birth and death rates of birth and death chains. This hierarchy of associated classical orthogonal polynomials can be put in a scheme similar to the Askey scheme [19] with the associated Askey–Wilson polynomials [17] at the top. Masson [23] studied recurrence relations for associated Wilson polynomials and provided spectral analysis on the corresponding Jacobi matrix. Groenevelt [11,12] investigated the Wilson function transforms.

Masson and Repka [24] considered the case when the recurrence relation (1.1) is defined for all integer values of n. When such recursion is written in matrix form, the corresponding tridiagonal Jacobi matrix is doubly infinite, that is,

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & b_{-1} & a_{-1} & b_0 & & & \\ & & b_0 & a_0 & b_1 & & \\ & & & b_1 & a_1 & b_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad a_n \in \mathbb{R},\ b_n > 0 \quad \text{for all } n \in \mathbb{Z}. \tag{1.2}$$

They proved that if

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty \quad \text{and} \quad \sum_{n=-\infty}^{0} \frac{1}{b_n} = \infty, \tag{1.3}$$

then the doubly infinite Jacobi matrix A is self-adjoint and its spectral measure is related to a four-element matrix of measures

$$d\mu(x) = \begin{pmatrix} d\mu_{00}(x) & d\mu_{01}(x) \\ d\mu_{10}(x) & d\mu_{11}(x) \end{pmatrix} \tag{1.4}$$

with $d\mu_{01}(x) = d\mu_{10}(x)$. It is noted that both $d\mu_{00}$ and $d\mu_{11}$ are positive probability measures but $d\mu_{01} = d\mu_{10}$ is a signed measure; see (1.8) below. Two sequences of polynomials $\{P_n^{(0)}(x)\}_{n\in\mathbb{Z}}$ and $\{P_n^{(1)}(x)\}_{n\in\mathbb{Z}}$ are generated by the recurrence relation

$$xp_n(x) = b_{n+1}p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x) \qquad \text{for } n \in \mathbb{Z}, \tag{1.5}$$

and the initial conditions:

$$P_0^{(0)}(x) = 1, P_1^{(0)}(x) = 0, \qquad P_0^{(1)}(x) = 0, P_1^{(1)}(x) = 1. \tag{1.6}$$

For any non-real z, the matrix elements of the resolvent $(zI - A)^{-1}$ are given by

$$\langle e_m, (zI - A)^{-1} e_n \rangle = \int_{\mathbb{R}} \frac{1}{z-x} \begin{pmatrix} P_m^{(0)}(x) & P_m^{(1)}(x) \end{pmatrix} \begin{pmatrix} d\mu_{00}(x) & d\mu_{01}(x) \\ d\mu_{10}(x) & d\mu_{11}(x) \end{pmatrix} \begin{pmatrix} P_n^{(0)}(x) \\ P_n^{(1)}(x) \end{pmatrix}, \tag{1.7}$$

where $e_n := (\delta_{mn})_{m\in\mathbb{Z}}$ is the standard orthonormal basis in $l^2(\mathbb{Z})$; see Masson and Repka [24, Theorem 2.3]. Considering the leading term of the above formula as $z \to \infty$, one obtains the following four-term orthogonality for the polynomials $P_n^{(0)}(x)$ and $P_n^{(1)}(x)$:

$$\int_{\mathbb{R}} \sum_{i,j=0}^{1} P_m^{(i)}(x) P_n^{(j)}(x) d\mu_{ij}(x) = \delta_{m,n}. \tag{1.8}$$

By choosing $m = n = 0$ or $m = n = 1$, we obtain that $d\mu_{00}$ and $d\mu_{11}$ are probability measures. If $m = 1$ and $n = 0$, or $m = 0$ and $n = 1$, we observe that

$$\int_{\mathbb{R}} d\mu_{10}(x) = \int_{\mathbb{R}} d\mu_{01}(x) = 0,$$

which implies that $d\mu_{10}$ and $d\mu_{01}$ are signed measures.

The theory of matrix-valued orthogonal polynomials has been developed by Durán, Grünbaum, and their collaborators; see [7–9]. Damanik, Pushnitski and Simon [6] observed a close connection between doubly infinite Jacobi matrices with certain block structure and matrix-valued orthogonal polynomials. Indeed, they pointed out that the Mason–Repka theory is essentially equivalent to the theory of 2×2 matrix orthogonal polynomials. For $n \geq -1$, let us define the matrix of polynomials

$$\mathcal{P}_n(x) := \begin{pmatrix} P^{(0)}_{-n-1}(x) & P^{(1)}_{-n-1}(x) \\ P^{(0)}_n(x) & P^{(1)}_n(x) \end{pmatrix}.$$

It is readily seen from (1.5) and (1.6) that $\mathcal{P}_n(x)$ satisfies the following matrix three-term recurrence relation:

$$\begin{aligned} &\begin{pmatrix} b_{-n-1} & 0 \\ 0 & b_{n+1} \end{pmatrix} \mathcal{P}_{n+1}(x) + \begin{pmatrix} b_{-n} & 0 \\ 0 & b_n \end{pmatrix} \mathcal{P}_{n-1}(x) \\ &= \left[xI - \begin{pmatrix} a_{-n-1} & 0 \\ 0 & a_n \end{pmatrix} \right] \mathcal{P}_n(x) \quad \text{for } n \geq 0, \end{aligned}$$

with initial conditions:

$$\mathcal{P}_0(x) = \begin{pmatrix} (x-a_0)/b_0 & -b_1/b_0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{P}_{-1}(x) = \begin{pmatrix} 1 & 0 \\ (x-a_0)/b_0 & -b_1/b_0 \end{pmatrix}.$$

Moreover, from (1.8), one obtains the following orthogonality relation of $\mathcal{P}_n(x)$:

$$\int_{\mathbb{R}} \mathcal{P}_m(x) d\mu(x) \mathcal{P}^*_n(x) = \delta_{mn} I, \qquad \text{for } m, n \geq 0,$$

where I is the identity matrix and $d\mu(x)$ is the matrix measure given in (1.4). We refer to [4,6,13,14,20] and references therein for previous works on matrix-valued orthogonal polynomials.

It is worth mentioning that the spectral theory of the two-sided Jacobi matrices is very different from the corresponding theory of the semi-infinite Jacobi matrix. For instance, Štampach and Šťovíček [29] studied the indeterminate moment problem associated with the one-sided Jacobi matrix for the Lommel polynomials when $q > 1$. Ismail and Štampach [18] investigated the special two-sided Lommel polynomials when the coefficients in the recurrence relation are linear polynomials in q^n. It is noted that, for the two moment problems associated with the upper-half and lower-half of the Jacobi matrix respectively, one is determinate while the other one is indeterminate.

The objective of this paper is to study the resolvent of the doubly-infinite Jacobi matrix and the orthogonality measure for the corresponding matrix orthogonal polynomials for three specific models. In Section 2, several general properties for the resolvent and its spectral measure are discussed. As some special applications in the discrete, continuous and q-orthogonal polynomials, we will investigate the doubly infinite Jacobi matrix for Lommel polynomials, associated ultraspherical polynomials, Al-Salam–Ismail polynomials in Sections 3–6, respectively.

It must be noted that specific models where their representations and spectral measures are explicit are important for two reasons. First, there is usually more structure to systems with explicit simple looking exact formulas. Second, good models can guide us in developing a general qualitative for polynomials associated with bilateral Jacobi matrices.

2. Resolvent and spectral measure

In this section, we provide some properties of the resolvent operator and spectral measure for the doubly infinite Jacobi matrix. Throughout this section, we assume that $a_n \in \mathbb{R}$, $b_n > 0$ for all $n \in \mathbb{Z}$ and (1.3) are satisfied. Note that (1.7) and (1.8) indicate that the Stieltjes transform of the spectral measure $d\mu_{ij}(x)$ is simply a matrix element of the resolvent $(zI - A)^{-1}$:

$$S_{ij}(z) := \int_{\mathbb{R}} \frac{d\mu_{ij}(x)}{z - x} = \langle e_i, (zI - A)^{-1} e_j \rangle \qquad \text{for } i, j = 0, 1. \tag{2.1}$$

Moreover, Masson and Repka [24, Theorem 2.5, Eq. (4.3) and (4.4)] gave a continued fraction representation for the matrix elements of the resolvent:

$$\langle e_n, (zI - A)^{-1} e_n \rangle = \frac{1}{z - a_n + K_{k=n+1}^{\infty}\left[\dfrac{-b_k^2}{z - a_k}\right] + K_{k=1-n}^{\infty}\left[\dfrac{-b_{1-k}^2}{z - a_{-k}}\right]} \tag{2.2}$$

and

$$\begin{aligned}&\langle e_0, (zI - A)^{-1} e_1 \rangle \\ &= \frac{b_1}{(z - a_1 + K_{k=2}^{\infty}[-b_k^2/(z - a_k)])(z - a_0 + K_{k=1}^{\infty}[-b_{1-k}^2/(z - a_{-k})]) - b_1^2}.\end{aligned} \tag{2.3}$$

Here, $K_{k=1}^{\infty}[u_k/v_k]$ is the continued fraction defined as

$$K_{k=1}^{\infty}\left[\frac{u_k}{v_k}\right] = \cfrac{u_1}{v_1 + \cfrac{u_2}{v_2 + \cfrac{u_3}{\ddots}}}.$$

For convenience, we introduce two types of continued fractions:

$$K_j^+ := z - a_j + K_{k=1}^{\infty}\left[\frac{-b_{j+k}^2}{z - a_{j+k}}\right], \tag{2.4}$$

$$K_j^- := z - a_j + K_{k=1}^{\infty}\left[\frac{-b_{j-k+1}^2}{z - a_{j-k}}\right]. \tag{2.5}$$

It is readily seen that

$$S_{ii}(z) = \frac{1}{K_i^+ + K_i^- - (z - a_i)}, \quad i = 0, 1, \tag{2.6}$$

$$S_{01}(z) = S_{10}(z) = \frac{b_1}{K_1^+ K_0^- - b_1^2}. \tag{2.7}$$

Remark 1. Equations (2.1)–(2.3) show that both $d\mu_{00}$ and $d\mu_{11}$ are probability measures, while $d\mu_{01}$ is not. Actually, by comparing leading terms on both sides of (2.1) when $z \to \infty$, we obtain

$$\int_{\mathbb{R}} d\mu_{00}(x) = \int_{\mathbb{R}} d\mu_{11}(x) = 1 \quad \text{and} \quad \int_{\mathbb{R}} d\mu_{01}(x) = 0.$$

In the following theorem, we provide a sufficient condition on $\{a_n, b_n\}$ such that the spectral measure for the doubly infinite Jacobi matrix A is discrete.

Theorem 1. *Let $a_n \in \mathbb{R}$, $b_n > 0$ for all $n \in \mathbb{Z}$. If (1.3) holds and the coefficients satisfy the following conditions:*

$$\lim_{n\to\pm\infty} a_n = \infty \text{ or } -\infty, \quad \text{and} \quad \limsup_{n\to\pm\infty} \frac{b_n^2}{a_n a_{n-1}} = L < \frac{1}{4}, \tag{2.8}$$

then $d\mu_{ij}(y)$, $i, j = 0, 1$ in (1.4) are discrete measures.

Proof. By [22, Theorem 3.2], the condition (2.8) implies that all of the continued fractions in (2.2) and (2.3) are meromorphic functions. Therefore, the matrix entries $\langle e_i, (zI - A)^{-1} e_j\rangle$ with $i, j = 0, 1$ are also meromorphic functions. From (2.1), one can see that the measures $d\mu_{ij}(y)$, $i, j = 0, 1$, are all discrete. □

Since the continued fractions in the right-hand sides of (2.2)–(2.3) can be expressed as a limit of the ratio of two orthogonal polynomials satisfying the same difference equation, we can also rewrite the left-hand sides (i.e., the matrix elements of the resolvent) as a limit of a rational function given in terms of orthogonal polynomials. More precisely, we have the following result.

Proposition 1. *Let $\{P_n^{\pm}(z)\}_{n=-1}^{\infty}$ be two sequences of monic orthogonal polynomials satisfying the following three-term recurrence relations*

$$P_{n+1}^+(z) = (z - a_n)P_n^+(z) - b_n^2 P_{n-1}^+(z), \qquad n \geq 0, \tag{2.9}$$

and

$$P^-_{n+1}(z) = (z - a_{-n})P^-_n(z) - b^2_{1-n}P^-_{n-1}(z), \qquad n \geq 0, \tag{2.10}$$

with $P^\pm_0(z) = 1$ *and* $P^\pm_{-1}(z) = 0$*; respectively. Let* $\{Q^\pm_n(z)\}^\infty_{n=-1}$ *be the numerator polynomials corresponding to* $P^\pm_n(z)$*; namely, they satisfy the same difference equations* (2.9)–(2.10) *with initial conditions* $Q^\pm_0(z) = 0$ *and* $Q^\pm_1(z) = 1$*. Then, we have*

$$\langle e_0, (zI - A)^{-1}e_0\rangle = \lim_{n\to\infty} R_n(z), \tag{2.11}$$

where $R_n(z)$ *is a rational function defined as*

$$R_n(z) := \frac{Q^+_n(z)Q^-_n(z)}{P^+_n(z)Q^-_n(z) + P^-_n(z)Q^+_n(z) - (z - a_0)Q^+_n(z)Q^-_n(z)}. \tag{2.12}$$

Proof. From Ismail [15, Sec. 2.6], we have

$$\lim_{n\to\infty} \frac{Q^+_n(z)}{P^+_n(z)} = \frac{1}{z - a_0 + K^\infty_{k=1}[-b^2_k/(z - a_k)]}, \tag{2.13}$$

and

$$\lim_{n\to\infty} \frac{Q^-_n(z)}{P^-_n(z)} = \frac{1}{z - a_0 + K^\infty_{k=1}[-b^2_{1-k}/(z - a_{-k})]}. \tag{2.14}$$

Substituting the above formulas into (2.2) gives (2.11). □

Remark 2. A similar formula for $\langle e_1, (zI - A)^{-1}e_1\rangle$ can be obtained by changing the recurrence coefficients in (2.9) and (2.10) from $\{a_n, b_n\}$ and $\{a_{-n}, b_{1-n}\}$ to $\{a_{n+1}, b_{n+1}\}$ and $\{a_{1-n}, b_{2-n}\}$, respectively. The formula for $\langle e_0, (zI - A)^{-1}e_1\rangle$ has a slightly different expression, but can be still derived from a similar approach as given in the above proof.

It is interesting to note that the rational function in (2.11) has interlacing zeros and poles on the real axis.

Proposition 2. *Let* $P^\pm_n(z)$ *be any two sets of monic orthogonal polynomials and* $Q^\pm_n(z)$ *the corresponding numerator polynomials. Then, the rational function* $R_n(z)$ *defined in* (2.12) *has interlacing zeros and poles on the real axis.*

Proof. Note that $Q^\pm_n(z)$ and $P^\pm_n(z)$ are monic polynomials with $\deg Q^\pm_n(z) = n - 1$ and $\deg P^\pm_n(z) = n$. Let $\xi^\pm_k, k = 1, \cdots, n-1$ and $\eta^\pm_k, k = 1, \cdots, n$ denote the zeros of $Q^\pm_n(z)$ and $P^\pm_n(z)$, respectively. They are all real simple zeros and satisfy the following interlacing property

$$\eta_1^{\pm} < \xi_1^{\pm} < \cdots < \eta_{n-1}^{\pm} < \xi_{n-1}^{\pm} < \eta_n^{\pm}.$$

Moreover, we have

$$Q_n^{+\prime}(\xi_k^+)P_n^+(\xi_k^+) < 0 \quad \text{for all } k, \qquad Q_n^{-\prime}(\xi_j^-)P_n^-(\xi_j^-) < 0 \quad \text{for all } j. \tag{2.15}$$

We consider the consecutive zeros of the numerator $Q_n^+(z)Q_n^-(z)$. First, let us assume that $Q_n^+(z)$ and $Q_n^-(z)$ have no common zeros, that is $\xi_j^- \neq \xi_k^+$ for any j and k. Then, there are four cases for the consecutive zeros, namely,

$$\text{I.}\quad \xi_k^+ < \xi_{k+1}^+; \quad \text{II.}\quad \xi_j^- < \xi_k^+; \quad \text{III.}\quad \xi_k^+ < \xi_j^-; \quad \text{IV.}\quad \xi_j^- < \xi_{j+1}^-.$$

For simplicity, we only provide the details for the first case $\xi_k^+ < \xi_{k+1}^+$; the other three cases can be investigated in a similar manner. The values of the denominator at these points are $P_n^+(\xi_k^+)Q_n^-(\xi_k^+)$ and $P_n^+(\xi_{k+1}^+)Q_n^-(\xi_{k+1}^+)$ and we will show that they have different signs. As ξ_k^+ and ξ_{k+1}^+ are consecutive zeros of $Q_n^+(z)Q_n^-(z)$, we have

$$\begin{aligned} &[Q_n^+(z)Q_n^-(z)]'|_{z=\xi_k^+}[Q_n^+(z)Q_n^-(z)]'|_{z=\xi_{k+1}^+} < 0 \\ &\Longrightarrow Q_n^{+\prime}(\xi_k^+)Q_n^-(\xi_k^+)Q_n^{+\prime}(\xi_{k+1}^+)Q_n^-(\xi_{k+1}^+) < 0. \end{aligned}$$

Note that, (2.15) gives us

$$Q_n^{+\prime}(\xi_k^+)P_n^+(\xi_k^+)Q_n^{+\prime}(\xi_{k+1}^+)P_n^+(\xi_{k+1}^+) > 0.$$

Multiplying the above two formulas gives us $P_n^+(\xi_k^+)Q_n^-(\xi_k^+)P_n^+(\xi_{k+1}^+)Q_n^-(\xi_{k+1}^+) < 0$. Then, there is at least one zero of the denominator in (ξ_k^+, ξ_{k+1}^+). Similarly, one can also prove that the denominator has at least one zero in the other three cases.

If $Q_n^+(z)$ and $Q_n^-(z)$ have one common zero (that is $\xi_j^- = \xi_k^+$ for certain j and k), the situation is similar. Note that there is one common factor $z - \xi_j^-$ or $z - \xi_k^+$ in both the numerator and denominator. Canceling this common factor, we find the value of the denominator at ξ_j^-:

$$P_n^+(\xi_k^+)Q_n^{-\prime}(\xi_j^-) + P_n^-(\xi_j^-)Q_n^{+\prime}(\xi_k^+).$$

Its value at ξ_{j+1}^- is $P_n^-(\xi_{j+1}^-)Q_n^+(\xi_{j+1}^-)$ divided by a positive number $\xi_{j+1}^- - \xi_k^+$. We may assume ξ_{j+1}^- is the next zero on the right, that is $\xi_{j+1}^- < \xi_{k+1}^+$ (the critical case $\xi_{j+1}^- = \xi_{k+1}^+$ will be considered later). This gives us $Q_n^{+\prime}(\xi_k^+)Q_n^+(\xi_{j+1}^-) > 0$. Combining (2.15), we have

$$P_n^+(\xi_k^+)Q_n^+(\xi_{j+1}^-) < 0.$$

As $Q_n^{-\prime}(\xi_j^-)Q_n^{-\prime}(\xi_{j+1}^-) < 0$, using (2.15) again, we have

$$P_n^-(\xi_j^-)P_n^-(\xi_{j+1}^-) < 0 \quad \text{and} \quad P_n^-(\xi_{j+1}^-){Q_n^-}'(\xi_j^-) > 0.$$

The above two formulas give us

$$\left[P_n^+(\xi_k^+){Q_n^-}'(\xi_j^-) + P_n^-(\xi_j^-){Q_n^+}'(\xi_k^+)\right]P_n^-(\xi_{j+1}^-)Q_n^+(\xi_{j+1}^-) < 0,$$

which means the denominator has at least one zero in (ξ_j^-, ξ_{j+1}^-).

Now, we consider the critical case when $Q_n^+(z)$ and $Q_n^-(z)$ have two consecutive common zeros $\xi_j^- = \xi_k^+$ and $\xi_{j+1}^- = \xi_{k+1}^+$. To show that the rational function $R_n(z)$ has at least one pole between these two zeros, we need to cancel out the common factor $(z-\xi_j^-)(z-\xi_{j+1}^-)$ in the fraction and show that the simplified denominator has different signs at these two zeros. We take the values of the denominator of the reduced rational function $R_n(z)$ at ξ_j^- and ξ_{j+1}^-, respectively, and obtain

$$[P_n^+(\xi_k^+){Q_n^-}'(\xi_j^-) + P_n^-(\xi_j^-){Q_n^+}'(\xi_k^+)]/(\xi_j^- - \xi_{j+1}^-)$$

and

$$[P_n^+(\xi_{k+1}^+){Q_n^-}'(\xi_{j+1}^-) + P_n^-(\xi_{j+1}^-){Q_n^+}'(\xi_{k+1}^+)]/(\xi_{j+1}^- - \xi_j^-).$$

It remains to prove that

$$[P_n^+(\xi_k^+){Q_n^-}'(\xi_j^-) + P_n^-(\xi_j^-){Q_n^+}'(\xi_k^+)][P_n^+(\xi_{k+1}^+){Q_n^-}'(\xi_{j+1}^-) + P_n^-(\xi_{j+1}^-){Q_n^+}'(\xi_{k+1}^+)]$$

is positive. Expanding the above product gives a sum of four products. It is clear from the interlacing property of orthogonal polynomials that

$$P_n^+(\xi_k^+){Q_n^-}'(\xi_j^-)P_n^+(\xi_{k+1}^+){Q_n^-}'(\xi_{j+1}^-) > 0$$

and

$$P_n^-(\xi_j^-){Q_n^+}'(\xi_k^+)P_n^-(\xi_{j+1}^-){Q_n^+}'(\xi_{k+1}^+) > 0.$$

The remaining two products are also positive on account of (2.15) and the interlacing property of orthogonal polynomials. Thus, we have proved that, between any two consecutive zeros of $R_n(z)$, there exists at least one pole of $R_n(z)$.

To conclude the proof, we shall find two more poles which lie to the right of the largest zero and to the left of the smallest zero, respectively. Without loss of generality, let ξ_{n-1}^+ and ξ_1^- be the largest and smallest zero of $Q_n^+(z)Q_n^-(z)$. Note that $P_n^+(\xi_{n-1}^+) < 0$, $Q_n^-(\xi_{n-1}^+) > 0$,

$$P_n^-(\xi_1^-)\begin{cases} > 0, & \text{for odd } n \\ < 0, & \text{for even } n \end{cases} \quad \text{and} \quad Q_n^+(\xi_1^-)\begin{cases} > 0, & \text{for odd } n \\ < 0, & \text{for even } n \end{cases}$$

and

$$P_n^+(z)Q_n^-(z) + P_n^-(z)Q_n^+(z) - (z-a_0)Q_n^+(z)Q_n^-(z) \sim z^{2n-1} \qquad \text{as } z \to \pm\infty.$$

The above three formulas yield that there are at least two more zeros in the intervals $(\xi_{n-1}^+, +\infty)$ and $(-\infty, \xi_1^-)$. As the total degree of the denominator is $2n-1$, then all the zeros are simple and interlace with the zeros of $Q_n^+(z)Q_n^-(z)$. $\square$

We then have the following result.

Theorem 2. *Let the coefficients $\{a_n, b_n\}$ with $n \in \mathbb{Z}$ satisfy the conditions in Theorem 1. The meromorphic function $\langle e_0, (zI-A)^{-1}e_0\rangle$ maps the upper half-plane onto the upper half-plane, and its poles are all real and simple. The same result holds for $\langle e_1, (zI - A)^{-1}e_1\rangle$.*

Proof. It follows from Propositions 1 and 2 that the function $\langle e_0, (zI-A)^{-1}e_0\rangle$ (or $\langle e_1, (zI-A)^{-1}e_1\rangle$) is uniformly approximated in a bounded region by the rational function in (2.12) with real interlacing roots and poles. According to Levin [21, p. 310], this function maps the upper half-plane onto the upper half-plane, and its poles are all real and simple. $\square$

Now, we introduce the truncated Jacobi matrix:

$$A_N := \begin{pmatrix} a_{-N} & b_{1-N} & & & & & \\ b_{1-N} & a_{1-N} & b_{2-N} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & b_0 & a_0 & b_1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & b_{N-1} & a_{N-1} & b_N \\ & & & & & b_N & a_N \end{pmatrix}. \tag{2.16}$$

Let $P_n(x)$ be the solution to (1.5) with initial conditions $P_{-N-1}(x) = 0$ and $P_{-N}(x) = 1$. It is readily seen that $P_{N+1}(x)$ is a polynomial of degree $2N+1$ whose zeros coincide with the eigenvalues of A_N. By a theorem of Chihara [5], we have the upper and lower bounds of the eigenvalues for A_N; see also [15, Theorem 7.2.3].

Proposition 3. *All eigenvalues of A_N (namely, all zeros of $P_{N+1}(x)$) belong to (a,b) if and only if (i) $a_j \in (a,b)$ for $-N \le j \le N$; and (ii) the sequence $b_j^2/[(x-a_j)(x-a_{j-1})]$ with $-N < j \le N$ is a chain sequence at $x = a$ and $x = b$.*

Here, we say $\{c_j : -N < j \le N\}$ is a chain sequence if there exists another sequence $\{g_j : -N \le j \le N\}$ such that $c_n = g_n(1-g_{n-1})$ for $-N < j \le N$, where $g_{-N} \in [0,1)$ and $g_j \in (0,1)$ for $-N < j \le N$. Especially, if $0 < c_j \le 1/4$ for all $-N < j \le N$, then c_j with $-N < j \le N$ is a chain sequence.

3. Lommel polynomials

It is well-known that the Bessel functions satisfy the following recurrence relation

$$J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1}(z). \tag{3.1}$$

By iterating the above formula, we can express $J_{\nu+n}(z)$ as a linear combination of $J_\nu(z)$ and $J_{\nu-1}(z)$:

$$J_{\nu+n}(z) = R_{n,\nu}(z) J_\nu(z) - R_{n-1,\nu+1}(z) J_{\nu-1}(z), \tag{3.2}$$

where $\{R_{n,\nu}(z)\}_{n=-1}^{\infty}$ are obtained from the recurrence relation

$$R_{n+1,\nu}(z) = \frac{2(n+\nu)}{z} R_{n,\nu}(z) - R_{n-1,\nu}(z) \tag{3.3}$$

with initial conditions $R_{-1,\nu}(z) = 0$ and $R_{0,\nu}(z) = 1$. It is readily seen that $R_{n,\nu}(z)$ with $n \ge 0$ is a polynomial of degree n with respect to $1/z$. Moreover, we have

$$R_{n,\nu}(z) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r \, (n-r)! \, (\nu)_{n-r}}{r! \, (n-2r)! \, (\nu)_r} \left(\frac{2}{z}\right)^{n-2r}. \tag{3.4}$$

In the literature, $R_{n,\nu}(z)$ is named as the Lommel polynomial. Hurwitz proved that the limit

$$\lim_{n\to\infty} \frac{(z/2)^{n+\nu} \, R_{n,\nu+1}(z)}{\Gamma(n+\nu+1)} = J_\nu(z), \tag{3.5}$$

which holds uniformly on compact subsets of $\mathbb{C}$. This formula establishes the validity of

$$\frac{J_\nu(z)}{J_{\nu-1}(z)} = -K_{k=1}^{\infty} \left[\frac{-1}{(2\nu+k-1)z^{-1}} \right] \tag{3.6}$$

for all finite z when $J_{\nu-1}(z) \neq 0$, and the continued fraction converges uniformly over all compact subsets of $\mathbb{C}$ not containing $z = 0$ or any zero of $z^{1-\nu} J_{\nu-1}(z)$. The case $\nu = 1/2$ of formula (3.6) was known to Lambert in 1761 who used it to prove the irrationality of π because the continued fraction (3.6) becomes a continued fraction for $\tan z$. According to Wallisser [30], Lambert gave explicit formulas for the polynomials $R_{n,1/2}(z)$ and $R_{n,3/2}(z)$ from which he established Hurwitz's theorem in the cases $\nu = -1/2, 1/2$ then proved (3.6) for $\nu = 1/2$. This is remarkable since Lambert studied the polynomials without free parameters and parameter-dependent explicit formulas are actually much easier to prove.

The modified Lommel polynomials [31] is defined as

$$h_{n,\nu}(z) := R_{n,\nu}(1/z) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r\,(n-r)!\,(\nu)_{n-r}}{r!\,(n-2r)!\,(\nu)_r}\,(2z)^{n-2r}. \tag{3.7}$$

Rewriting (3.3) in terms of $h_{n,\nu}(z)$, we have

$$2z(n+\nu)h_{n,\nu}(z) = h_{n+1,\nu}(z) + h_{n-1,\nu}(z)$$

with initial conditions $h_{-1,\nu}(z) = 0$ and $h_{0,\nu}(z) = 1$. It is evident that $\{h_{n,\nu}(z)\}_{n=0}^{\infty}$ is a system of orthogonal polynomials when $\nu > 0$. We also note that $h_{n,\nu}(z)$ with $n \geq 0$ is actually a polynomial in both variables ν and z of the same degree n. Now, we treat z as a parameter and ν as a variable. To be more specific, we define a sequence of polynomials:

$$Y_n(x) = h_{n,-x}(-\frac{1}{c}), \qquad n \geq 0.$$

It is readily seen that $Y_n(x)$ satisfies the following simple difference equation:

$$xY_n(x) = \frac{c}{2}Y_{n+1}(x) + nY_n(x) + \frac{c}{2}Y_{n-1}(x)$$

for $n \geq 1$. This motivates us to consider the doubly infinite Jacobi matrix

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & c/2 & -1 & c/2 & & & \\ & & c/2 & 0 & c/2 & & \\ & & & c/2 & +1 & c/2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3.8}$$

The corresponding difference equation is

$$xp_n(x) = \frac{c}{2}p_{n+1}(x) + np_n(x) + \frac{c}{2}p_{n-1}(x) \qquad \text{for } n \in \mathbb{Z}. \tag{3.9}$$

By symmetry, if $p_n(x)$ is a solution of the above difference equation, so is $(-1)^n p_{-n}(-x)$. Using the notations in (1.5), we have

$$a_n = n \quad \text{and} \quad b_n \equiv \frac{c}{2}. \tag{3.10}$$

By Theorem 1, the spectral measure of A is discrete. Actually, the formulas for $d\mu_{ij}(x)$ in (1.8) can be derived explicitly. To achieve this goal, we use (3.1) and symmetry property to find two linearly independent solutions for the difference equation (3.9):

$$Y_n^+(x) = (-1)^n J_{n-x}(c), \qquad Y_n^-(x) = J_{x-n}(c).$$

The solutions $Y_n^{\pm}(x)$ are also called right and left minimal (or subdominant) solutions, respectively, in the sense that they satisfy the following properties: for any $z \notin \sigma(A)$,

$$Y_n^+(z) \ll Y_n^-(z) \quad \text{as } n \to \infty, \qquad \text{and} \qquad Y_n^-(z) \ll Y_n^+(z) \quad \text{as } n \to -\infty.$$

Actually, the minimal (or subdominant) solutions $Y_n^{\pm}(z)$ are square summable (as a doubly infinite sequence) in $\mathbb{Z}$ and uniquely determined upon a constant multiplication; see [24]. Using the properties of the Bessel functions (cf. [25, (10.5.1)] or [31, §3.2 and §3.12]), we calculate the Wronskian of the above two solutions as

$$\mathcal{W} = \frac{c}{2}[Y_n^+(x)Y_{n+1}^-(x) - Y_{n+1}^+(x)Y_n^-(x)] = \frac{(-1)^{n+1}\sin[(n-x)\pi]}{\pi} = \frac{\sin(x\pi)}{\pi}.$$

Note that $\mathcal{W}$ is a constant independent of n. The spectral measures $d\mu_{ij}(x)$ are given in the following result.

Theorem 3. *For the doubly infinite Jacobi matrix A given in (3.8) and $c > 0$, the measures $\mu_{00}(x)$, $\mu_{01}(x)$ and $\mu_{11}(x)$ are all discrete measures supported on $\mathbb{Z}$. Their masses at $x = k, k \in \mathbb{Z}$ are given by*

$$m_{00}(k) = [J_k(c)]^2, \qquad m_{01}(k) = J_k(c)J_{k-1}(c), \qquad m_{11}(k) = [J_{k-1}(c)]^2, \tag{3.11}$$

where $J_k(x)$ are the Bessel functions.

Proof. By [24, Theorem 2.4], the entries of the resolvent of the two-sided Jacobi matrix A in (3.8) are given by

$$\begin{aligned} \langle e_m, (zI-A)^{-1}e_n\rangle &= \frac{Y_m^-(z)Y_n^+(z)}{\mathcal{W}} \\ &= \frac{(-1)^n J_{z-m}(c)J_{n-z}(c)}{\sin(z\pi)/\pi} \qquad \text{for } m \le n. \end{aligned} \tag{3.12}$$

From the above formula, we find that

$$\int_{\mathbb{R}} \frac{d\mu_{00}(x)}{z-x} = \langle e_0, (zI-A)^{-1}e_0\rangle = \frac{J_z(c)J_{-z}(c)}{\sin(z\pi)/\pi} = \sum_{k\in\mathbb{Z}} \frac{[J_k(c)]^2}{z-k}, \tag{3.13}$$

$$\int_{\mathbb{R}} \frac{d\mu_{01}(x)}{z-x} = \langle e_0, (zI-A)^{-1}e_1\rangle = \frac{-J_z(c)J_{1-z}(c)}{\sin(z\pi)/\pi} = \sum_{k\in\mathbb{Z}} \frac{J_k(c)J_{k-1}(c)}{z-k}, \tag{3.14}$$

$$\int_{\mathbb{R}} \frac{d\mu_{11}(x)}{z-x} = \langle e_1, (zI-A)^{-1}e_1\rangle = \frac{-J_{z-1}(c)J_{1-z}(c)}{\sin(z\pi)/\pi} = \sum_{k\in\mathbb{Z}} \frac{[J_{k-1}(c)]^2}{z-k}. \tag{3.15}$$

This establishes (3.11). □

Note from (3.11) that the determinant of the 2×2 mass matrix

$$m(k) = \begin{pmatrix} m_{00}(k) & m_{01}(k) \\ m_{10}(k) & m_{11}(k) \end{pmatrix}$$

vanishes. It then follows from [6, Theorems 2.27–2.28] that the eigenvalues of the doubly infinite Jacobi matrix A are all simple.

Remark 3. As we have seen in (2.2) and (2.3) in Section 2, it is also possible to compute the resolvent and spectral measure by studying the corresponding continued fractions. From (3.6), we have

$$\frac{-2J_{-z}(c)}{cJ_{-z-1}(c)} = \frac{1}{z + K_{k=1}^{\infty}[(-c^2/4)/(z-k)]}; \tag{3.16}$$

see also Schwartz [27] and Watson [31]. Replacing z by $-z$ in the above formula gives

$$\frac{2J_{z}(c)}{cJ_{z-1}(c)} = \frac{1}{z + K_{k=1}^{\infty}[(-c^2/4)/(z+k)]}. \tag{3.17}$$

Then, combining (2.2), (3.16) and (3.17), we obtain

$$\langle e_0, (zI-A)^{-1}e_0\rangle = \frac{1}{\frac{cJ_{-z-1}(c)}{-2J_{-z}(c)} + \frac{cJ_{z-1}(c)}{2J_z(c)} - z} = \frac{J_{-z}(c)J_z(c)}{\sin(z\pi)/\pi},$$

which coincides with (3.13). Here we have used the property of the Bessel functions in (3.1). On account of (2.2) and (3.10), we replace z by $z-1$ in the formula of $\langle e_0, (zI - A)^{-1}e_0\rangle$ to obtain

$$\langle e_1, (zI-A)^{-1}e_1\rangle = \frac{J_{1-z}(c)J_{z-1}(c)}{-\sin(z\pi)/\pi}.$$

This is the same as (3.15). Finally, in view of (3.16) and (3.17), we have

$$K_j^{+} := z - j + K_{k=1}^{\infty}[(-c^2/4)/(z-j-k)] = \frac{cJ_{-z-1+j}(c)}{-2J_{-z+j}(c)};$$
$$K_j^{-} := z - j + K_{k=1}^{\infty}[(-c^2/4)/(z-j+k)] = \frac{cJ_{z-1-j}(c)}{2J_{z-j}(c)}.$$

Substituting the above two formulas into (2.3) yields

$$\langle e_0, (zI-A)^{-1}e_1\rangle = \frac{c}{2}\left[K_1^{+}K_0^{-} - \frac{c^2}{4}\right]^{-1} = \frac{-J_z(c)J_{-z+1}(c)}{\sin(z\pi)/\pi},$$

which agrees with (3.14).

4. Associated ultraspherical polynomials

The associated ultraspherical polynomials with associated parameter $\gamma \in [0,1)$ are determined by the recurrence relation:

$$(n+\gamma+1)C_{n+1}^{\gamma}(x;\beta) = 2x(n+\beta+\gamma)C_n^{\gamma}(x;\beta) - (2\beta+n+\gamma-1)C_{n-1}^{\gamma}(x;\beta).$$

Let

$$p_n(x) = \sqrt{\frac{(\gamma+\beta+n)(\gamma+1)_n}{(\gamma+\beta)(\gamma+2\beta)_n}}C_n^{\gamma}(x;\beta) \tag{4.1}$$

be the orthonormal polynomials. We have

$$xp_n(x) = b_{n+1}p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x) \tag{4.2}$$

with

$$a_n \equiv 0, \qquad b_n = \sqrt{\frac{(n+\gamma)(n+\gamma+2\beta-1)}{4(n+\gamma+\beta)(n+\gamma+\beta-1)}}. \tag{4.3}$$

To study the doubly infinite Jacobi matrix, we have to ensure $b_n^2 > 0$ for all $n \in \mathbb{Z}$. This requires $\gamma \neq 0$ and

$$-\gamma/2 < \beta < (1-\gamma)/2 \qquad \text{or} \qquad (2-\gamma)/2 < \beta < (3-\gamma)/2. \tag{4.4}$$

Indeed, the above condition for $b_n^2 > 0$ is a necessary and sufficient one.

Lemma 1. *Let $\gamma \in (0,1)$. We have $b_n^2 > 0$ for all $n \in \mathbb{Z}$ if and only if $2\beta+\gamma \in (0,1)\cup(2,3)$.*

Proof. If $\gamma \in (0,1)$ and $2\beta+\gamma \in (0,1)$, then $\beta+\gamma = (2\beta+\gamma)/2+\gamma/2 \in (0,1)$. We have $(n+\gamma)/(n+\gamma+\beta) > 0$ and $(n-1+2\beta+\gamma)/(n-1+\beta+\gamma) > 0$ for all $n \in \mathbb{Z}$. This implies $b_n^2 > 0$ for all $n \in \mathbb{Z}$.

If $\gamma \in (0,1)$ and $2\beta+\gamma \in (2,3)$, then $\beta+\gamma = (2\beta+\gamma)/2+\gamma/2 \in (1,2)$. Thus, $(n+\gamma)/(n+\beta+\gamma-1) > 0$ and $(n+2\beta+\gamma-1)/(n+\beta+\gamma) > 0$ for all $n \in \mathbb{Z}$, which again, implies that $b_n^2 > 0$ for all $n \in \mathbb{Z}$.

On the other hand, if $2\beta+\gamma \in \mathbb{N}$, then b_n^2 cannot be positive at $n = 1-(2\beta+\gamma)$.

If $2\beta+\gamma \in (1,2)$, then $(n+\gamma)(n+2\beta+\gamma-1) > 0$ for all $n \in \mathbb{Z}$. However, $(n+\gamma+\beta)(n+\gamma+\beta-1)$ cannot be positive when $n \in [-\gamma-\beta, -\gamma-\beta+1]$. Thus, b_n^2 is not positive for some integer $n \in [-\gamma-\beta, -\gamma-\beta+1]$. Such integer exists because the length of the interval is one.

If $2\beta+\gamma > 3$, then $\beta > (3-\gamma)/2 > 1$ and $n+\gamma < n+\gamma+\beta-1 < n+\gamma+\beta < n+\gamma+2\beta-1$. We claim that either b_{-1}^2 or b_{-2}^2 is not positive. Otherwise, if $b_{-1}^2 > 0$, we have $\gamma+\beta < 2$, which together with $b_{-2}^2 > 0$ implies that $\gamma+2\beta < 3$, a contradiction.

Finally, we consider the case when $2\beta+\gamma<0$. Since $\gamma\in(0,1)$, we have $\beta<0$ and $n+\gamma+2\beta-1<n+\gamma+\beta-1<n+\gamma+\beta<n+\gamma$. We will show that either b_0^2 or b_1^2 is not positive. Otherwise, from $b_0^2>0$ we have $\gamma+\beta>0$, which together with $b_1^2>0$ implies that $\gamma+2\beta>0$, a contradiction.

Therefore, we have verified that $2\beta+\gamma\in(0,1)\cup(2,3)$ is a necessary and sufficient condition to guarantee the positiveness of b_n^2 for all $n\in\mathbb{Z}$. This completes the proof. □

Let us go back to the difference equation (4.2). It possesses two linearly independent solutions given explicitly as follows

$$Y_n^-(x)=\sqrt{\frac{(\gamma+\beta+1)_n(\gamma+1)_n}{(\gamma+\beta)_n(\gamma+2\beta)_n}}R^{\beta-1/2}_{-n-\gamma-\beta-1/2}(x), \tag{4.5}$$

$$Y_n^+(x)=\sqrt{\frac{(\gamma+\beta+1)_n(\gamma+1)_n}{(\gamma+\beta)_n(\gamma+2\beta)_n}}R^{\beta-1/2}_{n+\gamma+\beta-1/2}(x), \tag{4.6}$$

where

$$R^\mu_\nu(x):=\frac{\sqrt{\pi}\Gamma(\nu+\mu+1)(x^2-1)^{\mu/2}}{2^{\nu+1}x^{\nu+\mu+1}} \times\frac{{}_2F_1(\nu/2+\mu/2+1,\nu/2+\mu/2+1/2;\nu+3/2;1/x^2)}{\Gamma(\nu+3/2)}$$

is related to the associated Legendre function of second kind [25, (14.3.7) and (14.3.10)]. Moreover, in view of the asymptotic behavior of associated Legendre function of second kind [25, (14.15.14)], the solutions $Y_n^\pm(x)$ are one-sided subdominant solutions, i.e., $Y_n^+\ll Y_n^-$ as $n\to\infty$ and $Y_n^-\ll Y_n^+$ as $n\to-\infty$. The Wronskian of these two solutions is

$$\mathcal{W}=b_{n+1}[Y_n^+(x)Y_{n+1}^-(x)-Y_{n+1}^+(x)Y_n^-(x)], \tag{4.7}$$

which can be computed explicitly as follows.

Proposition 4. *We have*

$$\mathcal{W}=\frac{-\pi\cos[\pi(\gamma+\beta-1/2)]\Gamma(\gamma+2\beta)}{2(\gamma+\beta)\sin(\pi\gamma)\Gamma(\gamma+1)}. \tag{4.8}$$

Proof. To calculate $\mathcal{W}$, we shall use some properties of associated Legendre functions. Let

$$P^\mu_\nu(x):=\left(\frac{x+1}{x-1}\right)^{\mu/2}\frac{{}_2F_1(-\nu,\nu+1;1-\mu;1/2-x/2)}{\Gamma(1-\mu)},$$

$$Q_\nu^\mu(x) := e^{i\mu\pi} R_\nu^\mu(x) = \frac{e^{i\mu\pi}\sqrt{\pi}\Gamma(\nu+\mu+1)(x^2-1)^{\mu/2}}{2^{\nu+1}x^{\nu+\mu+1}} \\ \times \frac{{}_2F_1(\nu/2+\mu/2+1, \nu/2+\mu/2+1/2; \nu+3/2; 1/x^2)}{\Gamma(\nu+3/2)}.$$

It follows from [1, (8.1.8)] that

$$(x^2-1)[P_\nu^\mu(x)\frac{dQ_\nu^\mu(x)}{dx} - \frac{dP_\nu^\mu(x)}{dx}Q_\nu^\mu(x)] \\ = \frac{-e^{i\mu\pi}2^{2\mu}\Gamma(\nu/2+\mu/2+1)\Gamma(\nu/2+\mu/2+1/2)}{\Gamma(\nu/2-\mu/2+1)\Gamma(\nu/2-\mu/2+1/2)}.$$

Since

$$Q_{-\nu-1}^\mu(x) = \frac{-\pi e^{i\mu\pi}\cos(\nu\pi)P_\nu^\mu(x) + \sin[\pi(\nu+\mu)]Q_\nu^\mu(x)}{\sin[\pi(\nu-\mu)]}$$

by [1, (8.2.2)], we obtain

$$(x^2-1)[Q_{-\nu-1}^\mu(x)\frac{dQ_\nu^\mu(x)}{dx} - \frac{dQ_{-\nu-1}^\mu(x)}{dx}Q_\nu^\mu(x)] \\ = \frac{\pi e^{2i\mu\pi}\cos(\nu\pi)2^{2\mu}\Gamma(\nu/2+\mu/2+1)\Gamma(\nu/2+\mu/2+1/2)}{\sin[\pi(\nu-\mu)]\Gamma(\nu/2-\mu/2+1)\Gamma(\nu/2-\mu/2+1/2)}.$$

Furthermore, we have from [1, (8.5.3) and (8.5.4)] that

$$(x^2-1)\frac{dQ_\nu^\mu(x)}{dx} = (\nu-\mu+1)Q_{\nu+1}^\mu(x) - (\nu+1)xQ_\nu^\mu(x),$$
$$(x^2-1)\frac{dQ_{-\nu-1}^\mu(x)}{dx} = -(\nu+1)xQ_{-\nu-1}^\mu(x) + (\nu-\mu+1)Q_{-\nu-2}^\mu(x);$$

the first is obtained by subtracting [1, (8.5.3)] from [1, (8.5.4)], while the second replacing ν by $-\nu-1$ in [1, (8.5.4)]. It is readily seen that

$$Q_{-\nu-1}^\mu(x)Q_{\nu+1}^\mu(x) - Q_{-\nu-2}^\mu(x)Q_\nu^\mu(x) \\ = \frac{\pi e^{2i\mu\pi}\cos(\nu\pi)2^{2\mu}\Gamma(\nu/2+\mu/2+1)\Gamma(\nu/2+\mu/2+1/2)}{(\nu-\mu+1)\sin[\pi(\nu-\mu)]\Gamma(\nu/2-\mu/2+1)\Gamma(\nu/2-\mu/2+1/2)}.$$

Note that $R_\nu^\mu(x) = e^{-i\mu\pi}Q_\nu^\mu(x)$, we then have

$$R_{-\nu-1}^\mu(x)R_{\nu+1}^\mu(x) - R_{-\nu-2}^\mu(x)R_\nu^\mu(x) \\ = \frac{\pi\cos(\nu\pi)2^{2\mu}\Gamma(\nu/2+\mu/2+1)\Gamma(\nu/2+\mu/2+1/2)}{(\nu-\mu+1)\sin[\pi(\nu-\mu)]\Gamma(\nu/2-\mu/2+1)\Gamma(\nu/2-\mu/2+1/2)}.$$

In view of $\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z)/2^{2z-1}$, we obtain

$$R^{\mu}_{-\nu-1}(x)R^{\mu}_{\nu+1}(x) - R^{\mu}_{-\nu-2}(x)R^{\mu}_{\nu}(x) = \frac{\pi\cos(\nu\pi)\Gamma(\nu+\mu+1)}{\sin[\pi(\nu-\mu)]\Gamma(\nu-\mu+2)}.$$

A combination of (4.5)–(4.7) and the above formula yields

$$\mathcal{W} = \sqrt{\frac{(n+\gamma+1)(n+\gamma+2\beta)}{4(n+\gamma+\beta+1)(n+\gamma+\beta)}}\sqrt{\frac{(\gamma+\beta+n)(\gamma+1)_n}{(\gamma+\beta)(\gamma+2\beta)_n}} \times \sqrt{\frac{(\gamma+\beta+n+1)(\gamma+1)_{n+1}}{(\gamma+\beta)(\gamma+2\beta)_{n+1}}}\frac{-\pi\cos(\nu\pi)\Gamma(\nu+\mu+1)}{\sin[\pi(\nu-\mu)]\Gamma(\nu-\mu+2)},$$

where $\mu = \beta - 1/2$ and $\nu = n + \gamma + \beta - 1/2$. A simple calculation gives us

$$\mathcal{W} = \frac{(\gamma+1)_{n+1}}{2(\gamma+\beta)(\gamma+2\beta)_n} \cdot \frac{-\pi\cos[\pi(n+\gamma+\beta-1/2)]\Gamma(n+\gamma+2\beta)}{\sin[\pi(n+\gamma)]\Gamma(n+\gamma+2)}.$$

Finally, (4.8) follows from the above formula. □

Similar to Theorem 3, with the exact formula for the Wronskian given in the above proposition, we are ready to derive the spectral measures.

Theorem 4. *Let* $\gamma \in (0,1)$ *and* $2\beta+\gamma \in (0,1)\cup(2,3)$. *For the coefficients* a_n *and* b_n *given in* (4.3), *the spectral measures* $d\mu_{ij}(x)$, $i,j = 0,1$, *are continuous measures supported on* $[-1,1]$ *with the following exact expressions:*

$$(1-x^2)^{1/2-\beta}\frac{d\mu_{00}(x)}{dx} = \frac{-4x^2 f_1^2\Gamma(\gamma+\beta+1)\Gamma(-\gamma-\beta+1)}{\Gamma(\gamma/2+\beta)\Gamma(\gamma/2+1/2)\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)} + \frac{f_2^2\Gamma(\gamma+\beta+1)\Gamma(-\gamma-\beta+1)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)}, \tag{4.9}$$

$$(1-x^2)^{1/2-\beta}\frac{d\mu_{01}(x)}{dx}\sqrt{\frac{(\gamma+\beta)(\gamma+\beta+1)}{(\gamma+1)(\gamma+2\beta)}} = \frac{xf_1f_4\Gamma(-\gamma-\beta+1)\Gamma(\beta+\gamma+2)}{\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)\Gamma(\gamma/2+\beta+1)\Gamma(\gamma/2+3/2)} + \frac{xf_2f_3\Gamma(-\gamma-\beta+1)\Gamma(\beta+\gamma+2)}{\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)}, \tag{4.10}$$

and

$$(1-x^2)^{1/2-\beta}\frac{d\mu_{11}(x)}{dx} = \frac{-4x^2 f_3^2\Gamma(\gamma+\beta+2)\Gamma(-\gamma-\beta)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)\Gamma(-\gamma/2-1/2)\Gamma(-\gamma/2-\beta)} + \frac{f_4^2\Gamma(\gamma+\beta+2)\Gamma(-\gamma-\beta)}{\Gamma(\gamma/2+\beta+1)\Gamma(\gamma/2+3/2)\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)}, \tag{4.11}$$

where f_i with $i = 1, \cdots 4$, are ${}_2F_1$ functions defined as follows:

$$f_1 := {}_2F_1(\gamma/2 + \beta + 1/2, -\gamma/2 + 1/2; 3/2; x^2), \tag{4.12}$$

$$f_2 := {}_2F_1(\gamma/2 + \beta, -\gamma/2; 1/2; x^2), \tag{4.13}$$

$$f_3 := {}_2F_1(\gamma/2 + \beta + 1, -\gamma/2; 3/2; x^2), \tag{4.14}$$

$$f_4 := {}_2F_1(\gamma/2 + \beta + 1/2, -\gamma/2 - 1/2; 1/2; x^2). \tag{4.15}$$

Proof. By [24, Theorem 2.4], we have for $m \le n$,

$$\langle e_m, (zI - A)^{-1} e_n \rangle = \frac{Y_m^-(z) Y_n^+(z)}{\mathcal{W}} = \frac{2(\gamma + \beta)\sin(\pi\gamma)\Gamma(\gamma + 1)}{-\pi \sin[\pi(\gamma + \beta)]\Gamma(\gamma + 2\beta)} Y_m^-(z) Y_n^+(z).$$

When $m, n = 0, 1$, we obtain, for $z \in \mathbb{C} \setminus (-\infty, 1]$,

$$\begin{aligned} S_{00}(z) = \int_{\mathbb{R}} \frac{d\mu_{00}(y)}{z - y} &= \frac{(z^2 - 1)^{\beta - 1/2}}{z^{2\beta}} {}_2F_1(\gamma/2 + \beta + 1/2, \gamma/2 + \beta; \gamma + \beta + 1; 1/z^2) \\ &\quad \times {}_2F_1(-\gamma/2 + 1/2, -\gamma/2; -\gamma - \beta + 1; 1/z^2), \end{aligned}$$

and

$$\begin{aligned} S_{01}(z) = \int_{\mathbb{R}} \frac{d\mu_{01}(y)}{z - y} &= \frac{(z^2 - 1)^{\beta - 1/2}}{2 z^{2\beta + 1}} \sqrt{\frac{(\gamma + 1)(\gamma + 2\beta)}{(\gamma + \beta)(\gamma + \beta + 1)}} \\ &\quad \times {}_2F_1(\gamma/2 + \beta + 1, \gamma/2 + \beta + 1/2; \gamma + \beta + 2; 1/z^2) \\ &\quad \times {}_2F_1(-\gamma/2 + 1/2, -\gamma/2; -\gamma - \beta + 1; 1/z^2), \end{aligned}$$

and

$$\begin{aligned} S_{11}(z) = \int_{\mathbb{R}} \frac{d\mu_{11}(y)}{z - y} &= \frac{(z^2 - 1)^{\beta - 1/2}}{z^{2\beta}} {}_2F_1(\gamma/2 + \beta + 1, \gamma/2 + \beta + 1/2; \gamma + \beta + 2; 1/z^2) \\ &\quad \times {}_2F_1(-\gamma/2, -\gamma/2 - 1/2; -\gamma - \beta; 1/z^2), \end{aligned}$$

where $(z^2 - 1)^{\beta - 1/2}$ with $z \notin (-\infty, 1]$ and $z^{2\beta}$ with $z \notin (-\infty, 0]$ take their principal branches. To calculate the spectral measures, we shall make use of the following identity for hypergeometric functions [1, (15.3.7)]:

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} (-z)^{-a} {}_2F_1(a, 1 - c + a; 1 - b + a; 1/z) \\ &\quad + \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} (-z)^{-b} {}_2F_1(b, 1 - c + b; 1 - a + b; 1/z). \end{aligned}$$

For $x \in (-1,1)$, we take the limit from upper-half and lower-half planes respectively. It then follows that

$$\begin{aligned}
&{}_2F_1(\gamma/2+\beta+1/2, \gamma/2+\beta; \gamma+\beta+1; 1/x^2) \\
&\quad = \frac{\Gamma(\gamma+\beta+1)\Gamma(-1/2)}{\Gamma(\gamma/2+\beta)\Gamma(\gamma/2+1/2)} e^{\mp i\pi(\gamma/2+\beta+1/2)} x^{\gamma+2\beta+1} f_1 \\
&\qquad + \frac{\Gamma(\gamma+\beta+1)\Gamma(1/2)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)} e^{\mp i\pi(\gamma/2+\beta)} x^{\gamma+2\beta} f_2,
\end{aligned}$$

and

$$\begin{aligned}
&{}_2F_1(-\gamma/2+1/2, -\gamma/2; -\gamma-\beta+1; 1/x^2) \\
&\quad = \frac{\Gamma(-\gamma-\beta+1)\Gamma(-1/2)}{\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)} e^{\mp i\pi(-\gamma/2+1/2)} x^{-\gamma+1} f_1 \\
&\qquad + \frac{\Gamma(-\gamma-\beta+1)\Gamma(1/2)}{\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)} e^{\mp i\pi(-\gamma/2)} x^{-\gamma} f_2,
\end{aligned}$$

where f_1 and f_2 are the ${}_2F_1$ functions given in (4.12) and (4.13). For $i,j=0,1$ and $x \in (-1,1)$, we denote $S_{ij}^{\pm}(x) := \lim_{\varepsilon\to 0^+} S_{ij}(x \pm i\varepsilon)$. Since

$$\lim_{z\to x}(z^2-1)_{\pm}^{\beta-1/2} = (1-x^2)^{\beta-1/2} e^{\pm i\pi(\beta-1/2)} \qquad \text{for } x \in (-1,1),$$

we obtain

$$\begin{aligned}
&\frac{S_{00}^{\pm}(x)}{(1-x^2)^{\beta-1/2}} \\
&\quad = \left[\frac{\Gamma(\gamma+\beta+1)\Gamma(-1/2)}{\Gamma(\gamma/2+\beta)\Gamma(\gamma/2+1/2)}(-x)f_1 + \frac{\Gamma(\gamma+\beta+1)\Gamma(1/2)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)}(\mp i)f_2\right] \\
&\qquad \times \left[\frac{\Gamma(-\gamma-\beta+1)\Gamma(-1/2)}{\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)}(\mp ix)f_1 + \frac{\Gamma(-\gamma-\beta+1)\Gamma(1/2)}{\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)}f_2\right] \\
&\quad = \frac{2(\gamma+\beta)\sin(\beta\pi)xf_1f_2}{\sin[(\beta+\gamma)\pi]} \pm ix^2f_1^2 \frac{4\pi\Gamma(\gamma+\beta+1)\Gamma(-\gamma-\beta+1)}{\Gamma(\gamma/2+\beta)\Gamma(\gamma/2+1/2)\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)} \\
&\qquad \mp if_2^2 \frac{\pi\Gamma(\gamma+\beta+1)\Gamma(-\gamma-\beta+1)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)}.
\end{aligned}$$

Recall the Plemelj formula, if $\Phi(z) = \dfrac{1}{2\pi i}\displaystyle\int_I \frac{\varphi(t)}{t-z}dt$, then

$$\Phi_+(x) - \Phi_-(x) = \varphi(x) \qquad \text{for } x \in I. \tag{4.16}$$

Therefore, the above two formulas give us (4.9).

Similarly, we rewrite

$$\begin{aligned}
&{}_2F_1(\gamma/2+\beta+1,\gamma/2+\beta+1/2;\gamma+\beta+2;1/x^2)\\
&\quad=\frac{\Gamma(\gamma+\beta+2)\Gamma(-1/2)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)}e^{\mp i\pi(\gamma/2+\beta+1)}x^{\gamma+2\beta+2}f_3\\
&\qquad+\frac{\Gamma(\gamma+\beta+2)\Gamma(1/2)}{\Gamma(\gamma/2+\beta+1)\Gamma(\gamma/2+3/2)}e^{\mp i\pi(\gamma/2+\beta+1/2)}x^{\gamma+2\beta+1}f_4,
\end{aligned}$$

and

$$\begin{aligned}
&{}_2F_1(-\gamma/2,-\gamma/2-1/2;-\gamma-\beta;1/x^2)\\
&\quad=\frac{\Gamma(-\gamma-\beta)\Gamma(-1/2)}{\Gamma(-\gamma/2-1/2)\Gamma(-\gamma/2-\beta)}e^{\mp i\pi(-\gamma/2)}x^{-\gamma}f_3\\
&\qquad+\frac{\Gamma(-\gamma-\beta)\Gamma(1/2)}{\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)}e^{\mp i\pi(-\gamma/2-1/2)}x^{-\gamma-1}f_4,
\end{aligned}$$

where f_3 and f_4 are the ${}_2F_1$ functions given in (4.14) and (4.15). It then follows that

$$\begin{aligned}
&\frac{2S_{01}^{\pm}(x)}{(1-x^2)^{\beta-1/2}}\sqrt{\frac{(\gamma+\beta)(\gamma+\beta+1)}{(\gamma+1)(\gamma+2\beta)}}\\
&\quad=\left[\mp ixf_1\frac{\Gamma(-\gamma-\beta+1)\Gamma(-1/2)}{\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)}+f_2\frac{\Gamma(-\gamma-\beta+1)\Gamma(1/2)}{\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)}\right]\\
&\qquad\times\left[\pm ixf_3\frac{\Gamma(\gamma+\beta+2)\Gamma(-1/2)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)}-f_4\frac{\Gamma(\gamma+\beta+2)\Gamma(1/2)}{\Gamma(\gamma/2+\beta+1)\Gamma(\gamma/2+3/2)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{S_{11}^{\pm}(x)}{(1-x^2)^{\beta-1/2}}\\
&\quad=\left[\pm ixf_3\frac{\Gamma(\gamma+\beta+2)\Gamma(-1/2)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)}-f_4\frac{\Gamma(\gamma+\beta+2)\Gamma(1/2)}{\Gamma(\gamma/2+\beta+1)\Gamma(\gamma/2+3/2)}\right]\\
&\qquad\times\left[xf_3\frac{\Gamma(-\gamma-\beta)\Gamma(-1/2)}{\Gamma(-\gamma/2-1/2)\Gamma(-\gamma/2-\beta)}\pm if_4\frac{\Gamma(-\gamma-\beta)\Gamma(1/2)}{\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)}\right].
\end{aligned}$$

Combining the Plemelj formula in (4.16) and the above formulas, we have (4.10) and (4.11). □

For simplicity, we denote two constants:

$$g_1:=\frac{\Gamma(\gamma+\beta+1)\Gamma(-\gamma-\beta)}{\Gamma(\gamma/2+\beta+1/2)\Gamma(\gamma/2+1)\Gamma(-\gamma/2+1/2)\Gamma(-\gamma/2-\beta+1)},\tag{4.17}$$

$$g_2 := \frac{\Gamma(\gamma+\beta+1)\Gamma(-\gamma-\beta)}{\Gamma(\gamma/2+\beta+1)\Gamma(\gamma/2+3/2)\Gamma(-\gamma/2)\Gamma(-\gamma/2-\beta+1/2)}. \tag{4.18}$$

The spectral measures (4.9)–(4.11) can be rewritten as

$$\begin{aligned}
\frac{(1-x^2)^{1/2-\beta}}{\gamma+\beta}\frac{d\mu_{00}(x)}{dx} &= (\gamma+2\beta)(\gamma+1)x^2 f_1^2 g_2 - f_2^2 g_1,\\
\frac{(1-x^2)^{1/2-\beta}}{\sqrt{(\gamma+\beta)(\gamma+\beta+1)}}\frac{d\mu_{01}(x)}{dx} &= -\sqrt{(\gamma+2\beta)(\gamma+1)}x(f_1 f_4 g_2 + f_2 f_3 g_1),\\
\frac{(1-x^2)^{1/2-\beta}}{\gamma+\beta+1}\frac{d\mu_{11}(x)}{dx} &= -(\gamma+2\beta)(\gamma+1)x^2 f_3^2 g_1 + f_4^2 g_2.
\end{aligned}$$

Recall that $d\mu_{00}$ and $d\mu_{11}$ are probability measures, while $d\mu_{01} = d\mu_{10}$ has zero total integral. This gives us some nontrivial inequalities and identities related to the hypergeometric functions f_1, f_2, f_3, f_4 defined in (4.12)–(4.15).

Corollary 1. *Let the constants g_1 and g_2 be given in (4.17) and (4.18). We have the following inequalities*

$$(\gamma+2\beta)(\gamma+1)g_2\, x^2 f_1^2(x) > g_1 f_2^2(x) \qquad \text{for } x \in (-1,1) \tag{4.19}$$

and

$$g_2 f_4^2(x) > (\gamma+2\beta)(\gamma+1)g_1\, x^2 f_3^2(x) \qquad \text{for } x \in (-1,1). \tag{4.20}$$

Moreover, we have

$$\int_{-1}^{1} \frac{(\gamma+2\beta)(\gamma+1)g_2\, x^2 f_1^2(x) - g_1 f_2^2(x)}{(1-x^2)^{1/2-\beta}}dx = \frac{1}{\gamma+\beta}, \tag{4.21}$$

$$\int_{-1}^{1} \frac{(\gamma+2\beta)(\gamma+1)g_1\, x^2 f_3^2(x) - g_2 f_4^2(x)}{(1-x^2)^{1/2-\beta}}dx = -\frac{1}{\gamma+\beta+1}, \tag{4.22}$$

$$\int_{-1}^{1} \frac{x[g_2 f_1(x) f_4(x) + g_1 f_2(x) f_3(x)]}{(1-x^2)^{1/2-\beta}}dx = 0. \tag{4.23}$$

Now, we calculate the determinant:

$$\frac{(1-x^2)^{1-2\beta}}{(\gamma+\beta)(\gamma+\beta+1)}\begin{vmatrix} \frac{d\mu_{00}(x)}{dx} & \frac{d\mu_{01}(x)}{dx} \\ \frac{d\mu_{10}(x)}{dx} & \frac{d\mu_{11}(x)}{dx}\end{vmatrix} = -g_1 g_2\Big[(\gamma+2\beta)(\gamma+1)x^2 f_1 f_3 + f_2 f_4\Big]^2. \tag{4.24}$$

It is interesting to note that, the quantity involving hypergeometric functions on the right hand side of the above formula reduces to the following elementary function

$$(\gamma+2\beta)(\gamma+1)x^2 f_1 f_3 + f_2 f_4 = (1-x^2)^{1/2-\beta}. \tag{4.25}$$

To see it, we make use of (4.12)–(4.15) and calculate the coefficient of x^{2n} in the series expansion of left-hand side of (4.25) as

$$\begin{aligned}
&(\gamma+2\beta)(\gamma+1)\sum_{k=0}^{n-1}\frac{(\gamma/2+\beta+1)_k(-\gamma/2)_k(\gamma/2+\beta+1/2)_{n-k-1}(-\gamma/2+1/2)_{n-k-1}}{(3/2)_k(3/2)_{n-k-1}k!(n-k-1)!}\\
&\quad+\sum_{k=0}^{n}\frac{(\gamma/2+\beta)_k(-\gamma/2)_k(\gamma/2+\beta+1/2)_{n-k}(-\gamma/2-1/2)_{n-k}}{(1/2)_k(1/2)_{n-k}k!(n-k)!}\\
&\quad=\sum_{k=0}^{n}\Bigg[\frac{(\gamma/2+\beta)_k(-\gamma/2)_k(\gamma/2+\beta+1/2)_{n-k}(-\gamma/2-1/2)_{n-k}}{(1/2)_k(1/2)_{n-k}k!(n-k)!}\\
&\qquad\times\frac{(2k-n+1/2)(\gamma/2+\beta-1/2)}{(k+1/2)(\gamma/2+\beta+n-k-1/2)}\Bigg]\\
&\quad=\frac{(\gamma/2+\beta-1/2)_n(-g/2-1/2)_n}{(-1/2)_n n!}\\
&\qquad\times{}_5F_4\left(\left.\begin{matrix}-n,-n+1/2,-n/2+5/4,\gamma/2+\beta,-\gamma/2\\ -n/2+1/4,-n+\gamma/2+3/2,-n-\beta-\gamma/2+3/2,3/2\end{matrix}\right|1\right)\\
&=\frac{(\beta-1/2)_n}{n!},
\end{aligned}$$

which is the same as the coefficient of x^{2n} in the series expansion of $(1-x^2)^{1/2-\beta}$. Here, in the last step of the above equation, we have made use of Dixon's identity; see [28, (III.13)].

Now, combining (4.24) and (4.25), we obtain

$$\begin{aligned}
\begin{vmatrix}\frac{d\mu_{00}(x)}{dx} & \frac{d\mu_{01}(x)}{dx}\\ \frac{d\mu_{10}(x)}{dx} & \frac{d\mu_{11}(x)}{dx}\end{vmatrix} &= -(\gamma+\beta)(\gamma+\beta+1)g_1g_2\\
&= \frac{(\gamma+\beta)(\gamma+\beta+1)\sin(\pi\gamma)\sin[\pi(\gamma+2\beta)]}{\pi^2(\gamma+1)(\gamma+2\beta)\sin^2[\pi(\gamma+\beta)]}.
\end{aligned} \tag{4.26}$$

As $\gamma\in(0,1)$ and $2\beta+\gamma\in(0,1)\cup(2,3)$, one can see that the above determinant is positive constant. This means the multiplicity of the spectrum $[-1,1]$ is equal to 2.

It is well-known that Chebyshev polynomials of the first and second kinds $T_n(x)$ and $U_n(x)$ satisfy the same recurrence relation:

$$\frac{1}{2}[p_{n+1}(x)+p_{n-1}(x)] = xp_n(x) \qquad \text{for } n\geq 1.$$

We consider the doubly infinite Jacobi matrix corresponding to the two-sided difference equation:

$$\frac{1}{2}[p_{n+1}(x)+p_{n-1}(x)] = xp_n(x) \qquad \text{for } n\in\mathbb{Z}.$$

Two one-sided subdominant solutions to the above equation are given by

$$Y_n^+(x) = (x-\sqrt{x^2-1})^n, \qquad Y_n^-(x) = (x+\sqrt{x^2-1})^n$$

such that $Y_n^+ \ll Y_n^-$ as $n\to\infty$ and $Y_n^- \ll Y_n^+$ as $n\to-\infty$. The Wronskian of these two solutions is

$$\mathcal{W} = \mathcal{W}\{Y_n^+(x), Y_n^-(x)\} = Y_n^+(x)Y_{n+1}^-(x) - Y_{n+1}^+(x)Y_n^-(x) = 2\sqrt{x^2-1}.$$

By a similar calculation as in the proof of Theorem 4, we have

$$\int_{\mathbb{R}} \frac{d\mu_{00}(y)}{z-y} = \langle e_0, (zI-A)^{-1}e_0\rangle = \frac{1}{\sqrt{z^2-1}},$$

and

$$\int_{\mathbb{R}} \frac{d\mu_{01}(y)}{z-y} = \langle e_0, (zI-A)^{-1}e_1\rangle = \frac{z-\sqrt{z^2-1}}{\sqrt{z^2-1}}$$

and

$$\int_{\mathbb{R}} \frac{d\mu_{11}(y)}{z-y} = \langle e_1, (zI-A)^{-1}e_1\rangle = \frac{1}{\sqrt{z^2-1}},$$

for $z\in\mathbb{C}\setminus[-1,1]$. Using the Plemelj formula in (4.16) again, we obtain

$$\frac{d\mu_{00}}{dx} = \frac{d\mu_{11}}{dx} = -\frac{1}{2\pi i}\left[\left(\frac{1}{\sqrt{x^2-1}}\right)_+ - \left(\frac{1}{\sqrt{x^2-1}}\right)_-\right] = \frac{1}{\pi\sqrt{1-x^2}} \tag{4.27}$$

and

$$\frac{d\mu_{01}}{dx} = -\frac{1}{2\pi i}\left[\left(\frac{x-\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)_+ - \left(\frac{x-\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)_-\right] = \frac{x}{\pi\sqrt{1-x^2}} \tag{4.28}$$

for $x\in(-1,1)$.

Remark 4. Note that Chebyshev polynomials are special case of associated ultraspherical polynomials with $\beta = 0$. We shall demonstrate that (4.9), (4.10) and (4.11) can be reduced to (4.27) and (4.28). First, we recall the following formulas for hypergeometric functions with special parameters:

$$ {}_2F_1\left(a, -a; \tfrac{1}{2}; \sin^2\theta\right) = \cos(2a\theta), \qquad {}_2F_1\left(a, 1-a; \tfrac{3}{2}; \sin^2\theta\right) = \frac{\sin[(2a-1)\theta]}{(2a-1)\sin\theta}; $$

see [25, (15.4.12) and (15.4.16)]. Let $x = \sin\theta$ in (4.12), (4.13), (4.14) and (4.15) and make use of the above formulas, then the functions f_i, $i = 1, \cdots 4$, are rewritten as

$$ f_1 = \frac{\sin(\gamma\theta)}{\gamma\sin\theta}, \quad f_2 = \cos(\gamma\theta), \quad f_3 = \frac{\sin[(\gamma+1)\theta]}{(\gamma+1)\sin\theta}, \quad f_4 = \cos[(\gamma+1)\theta]. $$

Note that $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, then we have

$$ \frac{\Gamma(\gamma+1)\Gamma(-\gamma)}{\Gamma(\gamma/2+1/2)\Gamma(-\gamma/2+1/2)\Gamma(\gamma/2+1)\Gamma(-\gamma/2)} = \frac{1}{2\pi}. $$

Substituting the above formulas into (4.9), (4.10) and (4.11), we have

$$ \frac{d\mu_{00}(x)}{dx} = \frac{d\mu_{11}(x)}{dx} = \frac{1}{\pi\sqrt{1-x^2}}, \qquad \frac{d\mu_{01}(x)}{dx} = \frac{x}{\pi\sqrt{1-x^2}}, \qquad x \in (-1,1). $$

5. Al-Salam–Ismail polynomials

Al-Salam and Ismail [2] considered the monic orthogonal polynomials satisfying the difference equation

$$ \pi_{n+1}(x;a,b) = x\pi_n(x;a,b) - \frac{bq^{n-1}}{(1+aq^n)(1+aq^{n-1})}\pi_{n-1}(x;a,b), \quad n \geq 1 \tag{5.1} $$

with initial conditions $\pi_0(x;a,b) = 1$ and $\pi_1(x;a,b) = x$, where $q \in (0,1)$, $a > -1$ and $b > 0$. It was shown that the measure $d\mu(x)$ for Al-Salam–Ismail polynomials is purely discrete. Moreover, its Stieltjes transform has following explicit expression:

$$ \int_{\mathbb{R}} \frac{d\mu(x)}{z-x} = \frac{F(qbz^{-2};qa)}{zF(bz^{-2};a)}, \quad z \notin \operatorname{supp}(d\mu), \tag{5.2} $$

where $F(x;a)$ is an entire function defined as

$$ F(x;a) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k(-a;q)_k}(-x)^k. \tag{5.3} $$

In this section, we study the doubly infinite Jacobi matrix associated with the difference equation

$$xp_n(x;a,b) = b_{n+1}p_{n+1}(x;a,b) + a_np_n(x;a,b) + b_np_{n-1}(x;a,b), \quad n \in \mathbb{Z}, \tag{5.4}$$

where

$$a_n \equiv 0, \quad b_n = \sqrt{\frac{bq^{n-1}}{(1+aq^n)(1+aq^{n-1})}}.$$

Here, we require positiveness of both a and b so that $b_n > 0$ for all $n \in \mathbb{Z}$. On account of (5.2), we have the following continued fraction representation:

$$K_0^+ := z + K_{k=1}^{\infty}\left[\frac{-b_k^2}{z}\right] = \frac{zF(bz^{-2};a)}{F(qbz^{-2};qa)}. \tag{5.5}$$

Note that b_{1-n} has the same expression as b_n by replacing a and b with a^{-1} and ba^{-2}, respectively. Thus, we have

$$K_0^- := z + K_{k=1}^{\infty}\left[\frac{-b_{1-k}^2}{z}\right] = \frac{zF(b(az)^{-2};a^{-1})}{F(qb(az)^{-2};qa^{-1})}. \tag{5.6}$$

Substituting the above two formulas into (2.2) gives

$$\langle e_0, (zI-A)^{-1}e_0\rangle = \frac{1}{K_0^+ + K_0^- - z} = \frac{F(qbz^{-2};qa)F(qb(az)^{-2};qa^{-1})}{zG(z^{-2};a,b)}, \tag{5.7}$$

where

$$\begin{aligned} G(t;a,b) := {} & F(tb;a)F(tqba^{-2};qa^{-1}) + F(tqb;qa)F(tba^{-2};a^{-1}) \\ & - F(tqb;qa)F(tqba^{-2};qa^{-1}). \end{aligned}$$

Let $G(t;a,b) = \sum_{n=0}^{\infty} G_n t^n$. Making use of (5.3), the coefficient G_n is given explicitly as follows:

$$\begin{aligned} G_n = {} & \sum_{k+l=n}\left[\frac{(-b)^k q^{k(k-1)}(-qb/a^2)^l q^{l(l-1)}}{(q;q)_k(-a;q)_k(q;q)_l(-q/a;q)_l} + \frac{(-qb)^k q^{k(k-1)}(-b/a^2)^l q^{l(l-1)}}{(q;q)_k(-qa;q)_k(q;q)_l(-1/a;q)_l}\right. \\ & \left. - \frac{(-qb)^k q^{k(k-1)}(-qb/a^2)^l q^{l(l-1)}}{(q;q)_k(-qa;q)_k(q;q)_l(-q/a;q)_l}\right] \\ = {} & \sum_{k+l=n}\frac{(-b)^n a^{-2l} q^{k^2+l^2-n}(q^l+q^k a)}{(q;q)_k(-qa;q)_k(q;q)_l(-q/a;q)_l(1+a)} \\ = {} & q^{n(n-1)}(-b/a)^n \sum_{k=0}^{n}\frac{a^{2k-n}q^{2k(k-n)}(q^{n-k}+q^k a)}{(q;q)_k(-qa;q)_k(q;q)_{n-k}(-q/a;q)_{n-k}(1+a)}. \end{aligned}$$

To simplify the above formula for G_n, we make use of the following lemma.

Lemma 2. *For any* $0 < q < 1$, $a > 0$ *and* $n \in \mathbb{N}$, *we have*

$$S := \sum_{k=0}^{n} \frac{a^{2k-n} q^{2k(k-n)} (q^{n-k} + q^k a)}{(q;q)_k (-qa;q)_k (q;q)_{n-k} (-q/a;q)_{n-k} (1+a)} = \frac{q^{-n(n-1)/2}}{(q;q)_n}.$$

Proof. Note that

$$\frac{(q;q)_n}{(q;q)_{n-k}} = (-1)^k q^{k(2n-k+1)/2} (q^{-n};q)_k, \tag{5.8}$$

$$\frac{(-q/a;q)_n}{(-q/a;q)_{n-k}} = a^{-k} q^{k(2n-k+1)/2} (-aq^{-n};q)_k, \tag{5.9}$$

$$(-q/a;q)_n (1+a) = (-1/a;q)_n (1+q^{-n}a) q^n, \tag{5.10}$$

$$\frac{1+q^{2k-n}a}{1+q^{-n}a} = \frac{(iq^{1-n/2}\sqrt{a};q)_k (-iq^{1-n/2}\sqrt{a};q)_k}{(iq^{-n/2}\sqrt{a};q)_k (-iq^{-n/2}\sqrt{a};q)_k}. \tag{5.11}$$

Then, the above formulas give us

$$\begin{aligned} S &= \sum_{k=0}^{n} \frac{a^{k-n} q^{2k(k-n)+n-k} (1+q^{2k-n}a)(-1)^k q^{k(2n-k+1)} (q^{-n};q)_k (-aq^{-n};q)_k}{(q;q)_k (-qa;q)_k (q;q)_n (-1/a;q)_n (1+q^{-n}a) q^n} \\ &= \frac{a^{-n}}{(q;q)_n (-1/a;q)_n} \sum_{k=0}^{n} \frac{(-aq^{-n};q)_k (q^{-n};q)_k (iq^{1-n/2}\sqrt{a};q)_k (-iq^{1-n/2}\sqrt{a};q)_k}{(q;q)_k (-qa;q)_k (iq^{-n/2}\sqrt{a};q)_k (-iq^{-n/2}\sqrt{a};q)_k} (-a)^k q^{k^2}. \end{aligned}$$

Next, we make use of the limit

$$\lim_{\varepsilon \to 0} (1/\varepsilon;q)_k (1/\varepsilon;q)_k \varepsilon^{2k} = q^{k^2-k}$$

to obtain another expression

$$\begin{aligned} S = \lim_{\varepsilon \to 0} & \frac{a^{-n}}{(q;q)_n (-1/a;q)_n} \\ & \times {}_6\phi_5 \left(\begin{matrix} -aq^{-n} & q^{-n} & iq^{1-n/2}\sqrt{a} & -iq^{1-n/2}\sqrt{a} & 1/\varepsilon & 1/\varepsilon \\ & -qa & iq^{-n/2}\sqrt{a} & -iq^{-n/2}\sqrt{a} & -\varepsilon a q^{1-n} & -\varepsilon a q^{1-n} \end{matrix} \middle| -qa\varepsilon^2 \right). \end{aligned}$$

By [10, p. 238, (II.21)], we have

$$S = \lim_{\varepsilon \to 0} \frac{a^{-n}}{(q;q)_n (-1/a;q)_n} \frac{(-aq^{1-n};q)_n (-\varepsilon a q^{1-n};q)_n}{(-\varepsilon^2 a q^{1-n};q)_n (-\varepsilon^2 a q^{1-n};q)_n} = \frac{q^{-n(n-1)/2}}{(q;q)_n}.$$

This completes the proof. □

From the above lemma, we have $G_n = (-b/a)^n q^{n(n-1)/2}/(q;q)_n$, and consequently,

$$G(t;a,b) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} (-bt/a)^n = (bt/a;q)_\infty, \tag{5.12}$$

where we have used the Euler's theorem [15, (12.2.25)].

To calculate the other matrix entries of the resolvent, we need to find K_1^+ and K_1^-. Note that b_{k+1} has the same expression as b_k with a and b multiplied by q. Consequently, we obtain

$$K_1^+ := z + K_{k=1}^{\infty} \left[\frac{-b_{k+1}^2}{z} \right] = \frac{zF(qbz^{-2};qa)}{F(q^2bz^{-2};q^2a)},$$
$$K_1^- := z + K_{k=1}^{\infty} \left[\frac{-b_{2-k}^2}{z} \right] = \frac{zF(q^{-1}b(az)^{-2};(qa)^{-1})}{F(b(az)^{-2};a^{-1})}.$$

It follows from (2.2) and (2.3) that

$$\langle e_1, (zI-A)^{-1} e_1 \rangle = \frac{1}{K_1^+ + K_1^- - z} = \frac{F(q^2bz^{-2};q^2a)F(b(az)^{-2};a^{-1})}{zG(z^{-2};a,b)},$$
$$\langle e_0, (zI-A)^{-1} e_1 \rangle = \frac{b_1}{K_1^+ K_0^- - b_1^2} = \frac{F(q^2bz^{-2};q^2a)F(qb(az)^{-2};qa^{-1})b_1}{z^2 H(z^{-2};a,b)},$$

where $G(t;a,b)$ is given in (5.12), and

$$H(t;a,b) := F(tqb;qa)F(tba^{-2};a^{-1}) - \frac{tF(tq^2b;q^2a)F(tqba^{-2};qa^{-1})b}{(1+a)(1+qa)}. \tag{5.13}$$

Again let $H(t;a,b) = \sum_{n=0}^{\infty} H_n t^n$. Adopting the similar analysis used in deriving $G(t;a,b)$ in (5.12), we find the explicit formula for H_n by using (5.3):

$$\begin{aligned} H_n &= \sum_{k+l=n} \frac{(-qb)^k q^{k(k-1)} (-b/a^2)^l q^{l(l-1)}}{(q;q)_k (-qa;q)_k (q;q)_l (-1/a;q)_l} \\ &\quad - \sum_{k+l=n-1} \frac{(-q^2b)^k q^{k(k-1)} (-qb/a^2)^l q^{l(l-1)} b}{(q;q)_k (-q^2a;q)_k (q;q)_l (-q/a;q)_l (1+a)(1+qa)} \\ &= q^{n^2-n} (-b/a)^n \sum_{k=0}^{n} \frac{a^{2k-n} q^{2k^2-2kn+k} (1+q^{2k-n+1}a)}{(q;q)_k (-q^2a;q)_k (q;q)_{n-k} (-1/a;q)_{n-k} (1+qa)} \\ &= q^{n(n-1)} (-b/a)^n \sum_{k=0}^{n} \frac{(qa)^{2k-n} q^{2k(k-n)} (q^{n-k} + q^{k+1}a)}{(q;q)_k (-q^2a;q)_k (q;q)_{n-k} (-1/a;q)_{n-k} (1+qa)}. \end{aligned}$$

Note that the sum on the right-hand side is the same as the sum expression for S in Lemma 2 where a is replaced by qa. We obtain $H_n = G_n$ and thus

$$H(t; a, b) = G(t; a, b) = (bt/a; q)_\infty. \tag{5.14}$$

Summarizing the above results, we have the following results.

Theorem 5. *Let $a, b > 0$, the spectral measures for the doubly infinite Jacobi matrix with*

$$a_n \equiv 0, \quad b_n = \sqrt{\frac{bq^{n-1}}{(1 + aq^n)(1 + aq^{n-1})}}$$

have discrete mass points at $\pm x_k = \pm\sqrt{bq^k/a}$ for all $k = 0, 1, \cdots$. Moreover, the masses at $\pm x_k$ are given by

$$m_{00}(\pm x_k) = \frac{F(q^{1-k}a; qa)F(q^{1-k}a^{-1}; qa^{-1})}{2(q^{-k}; q)_k(q; q)_\infty}, \tag{5.15}$$

$$m_{11}(\pm x_k) = \frac{F(q^{2-k}a; q^2a)F(q^{-k}a^{-1}; a^{-1})}{2(q^{-k}; q)_k(q; q)_\infty}, \tag{5.16}$$

$$m_{01}(\pm x_k) = \frac{F(q^{2-k}a; q^2a)F(q^{1-k}a^{-1}; qa^{-1})}{2(q^{-k}; q)_k(q; q)_\infty\sqrt{(1 + a)(1 + qa)q^k/a}}. \tag{5.17}$$

Proof. The Stieltjes transforms of the spectral measures have the explicit expressions:

$$\int_{\mathbb{R}} \frac{d\mu_{00}(x)}{z - x} = \frac{F(qbz^{-2}; qa)F(qb(az)^{-2}; qa^{-1})}{z(bz^{-2}/a; q)_\infty}, \tag{5.18}$$

$$\int_{\mathbb{R}} \frac{d\mu_{11}(x)}{z - x} = \frac{F(q^2bz^{-2}; q^2a)F(b(az)^{-2}; a^{-1})}{z(bz^{-2}/a; q)_\infty}, \tag{5.19}$$

$$\int_{\mathbb{R}} \frac{d\mu_{01}(x)}{z - x} = \frac{F(q^2bz^{-2}; q^2a)F(qb(az)^{-2}; qa^{-1})b_1}{z^2(bz^{-2}/a; q)_\infty}, \tag{5.20}$$

where $b_1 = \sqrt{b/[(1 + a)(1 + qa)]}$. Taking the residues at x_k gives the desired results. □

From (5.15)–(5.17) and $m_{10} = m_{01}$, we calculate the determinant

$$\begin{vmatrix} m_{00}(\pm x_k) & m_{01}(\pm x_k) \\ m_{10}(\pm x_k) & m_{11}(\pm x_k) \end{vmatrix} = \frac{F(q^{2-k}a; q^2a)F(q^{1-k}a^{-1}; qa^{-1})}{[2(q^{-k}; q)_k(q; q)_\infty]^2} H(q^{-k}; a, a),$$

where H is the function defined in (5.13). In view of (5.14), we have $H(q^{-k}; a, a) = (q^{-k}, q)_\infty = 0$, and consequently,

$$\begin{vmatrix} m_{00}(\pm x_k) & m_{01}(\pm x_k) \\ m_{10}(\pm x_k) & m_{11}(\pm x_k) \end{vmatrix} = 0.$$

This together with [6, Theorems 2.27–2.28] implies that any point spectrum of the doubly infinite Jacobi matrix is a simple eigenvalue (i.e., with multiplicity one).

We note that the point $z = 0$ is the limit of the discrete mass points $\pm x_k$, hence is in the spectrum. However, the point $z = 0$ can not be a mass point. Otherwise, let $\{v_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be the eigenvector associated with zero. It follows from (5.4) that

$$\frac{v_{n+1}}{v_{n-1}} = -\frac{b_n}{b_{n+1}} = -\sqrt{\frac{1+aq^{n+1}}{q(1+aq^{n-1})}}.$$

From the above relation, we obtain the explicit expressions for v_n as follows:

$$v_{2n} = v_0(-1)^n\sqrt{\frac{1+aq^{2n}}{q^n(1+a)}},$$

and

$$v_{2n+1} = v_1(-1)^n\sqrt{\frac{1+aq^{2n+1}}{q^n(1+aq)}}.$$

Since v_0 and v_1 can not be identically zero, we have either $|v_{2n}| \to \infty$ or $|v_{2n+1}| \to \infty$ as $n \to \infty$, which contradicts to the square summability of v_n. Applying Theorem 5 gives the following identity

$$\frac{F(qbz^{-2};qa)F(qb(az)^{-2};qa^{-1})}{z(bz^{-2}/a;q)_\infty} = \sum_{k=0}^{\infty}\frac{zF(q^{1-k}a;qa)F(q^{1-k}a^{-1};qa^{-1})}{(q^{-k};q)_k(q;q)_\infty(z^2-bq^k/a)}. \tag{5.21}$$

6. Non-symmetric Al-Salam–Ismail polynomials

Let $\pi_n(x;a,b)$ be the monic Al-Salam–Ismail polynomials defined from the difference equation (5.1). The polynomials $U_n(x;a,b) = (-a;q)_n\pi_n(x;a,b)$ satisfy the following difference equation

$$U_{n+1}(x;a,b) = x(1+aq^n)U_n(x;a,b) - bq^{n-1}U_{n-1}(x;a,b) \tag{6.1}$$

for $n \geq 1$, and the initial values are $U_0(x;a,b) = 1$ and $U_1(x;a,b) = x(1+a)$. Note that, if we replace a by $a + c/x$ for $c \in \mathbb{R}$, we still obtain a sequence of orthogonal polynomials

$$\widetilde{U}_n(x;a,b,c) := U_n(x;a+c/x,b), \tag{6.2}$$

which satisfy the following difference equation

$$\widetilde{U}_{n+1}(x;a,b,c) = [x(1+aq^n)+cq^n]\widetilde{U}_n(x;a,b) - bq^{n-1}\widetilde{U}_{n-1}(x;a,b) \tag{6.3}$$

for $n \geq 1$, and the initial values are $\widetilde{U}_0(x;a,b,c) = 1$ and $\widetilde{U}_1(x;a,b,c) = x(1+a)+c$. We refer $\widetilde{U}_n(x;a,b,c)$ as the *non-symmetric Al-Salam–Ismail polynomials*. It is easy to see that

$$\widetilde{U}_n^*(x;a,b,c) = (1+a)\widetilde{U}_{n-1}(x;aq,bq,cq), \tag{6.4}$$

where $\widetilde{U}_n^*(x;a,b,c)$ is corresponding numerator polynomial with the initial values

$$\widetilde{U}_0^*(x;a,b,c) = 0, \quad \widetilde{U}_1^*(x;a,b,c) = 1+a.$$

As in the previous sections, let us study the doubly infinite Jacobi matrix associated with the difference equation

$$xp_n(x) = \widetilde{b}_{n+1}p_{n+1}(x) + \widetilde{a}_n p_n(x) + \widetilde{b}_n p_{n-1}(x), \quad n \in \mathbb{Z}, \tag{6.5}$$

with

$$\widetilde{a}_n = -\frac{cq^n}{1+aq^n}, \quad \widetilde{b}_n = \sqrt{\frac{bq^{n-1}}{(1+aq^n)(1+aq^{n-1})}},$$

where $a,b>0$ and $c \in \mathbb{R}$. To calculate the continued fractions $\widetilde{K}_0^+$ and $\widetilde{K}_0^-$, we recall their relation with orthogonal polynomials in (2.13) and (2.14). As $\widetilde{U}_n(z;a,b,c)$ and $\widetilde{U}_n^*(z;a,b,c)$ have the same leading coefficients when $n>1$, we obtain from (2.13):

$$\widetilde{K}_0^+ := z - \widetilde{a}_0 + K_{k=1}^{\infty}\left[\frac{-\widetilde{b}_k^2}{z-\widetilde{a}_k}\right] = \lim_{n\to\infty}\frac{\widetilde{U}_n(z;a,b,c)}{\widetilde{U}_n^*(z;a,b,c)}. \tag{6.6}$$

By (6.2) and (6.4), we have

$$\widetilde{K}_0^+ = \lim_{n\to\infty}\frac{\widetilde{U}_n(z;a,b,c)}{(1+a)\widetilde{U}_{n-1}(z;aq,bq,cq)} = \lim_{n\to\infty}\frac{U_n(z;a+c/z,b)}{(1+a)U_{n-1}(z;aq+cq/z,bq)}. \tag{6.7}$$

A similar argument as above gives us that

$$K_0^+ = \lim_{n\to\infty}\frac{U_n(z;a,b)}{(1+a)U_{n-1}(z;a,b)}, \tag{6.8}$$

where K_0^+ is the continued fraction associated with the symmetric Al-Salam–Ismail polynomials $U_n(z;a,b)$ in (5.5). Then, combining the above two formulas and (5.5), we have

$$\widetilde{K}_0^+ = \frac{[z+c/(1+a)]F(bz^{-2};a+c/z)}{F(qbz^{-2};qa+qc/z)}. \tag{6.9}$$

The computation for $\widetilde{K}_0^-$ is similar. Now the associated orthogonal polynomial $\widetilde{U}_n^-(x;a,b,c)$ satisfies the following difference equation

$$\widetilde{U}_{n+1}^-(x;a,b,c) = [x(a+q^n)+c]\widetilde{U}_n^-(x;a,b) - bq^{n-1}\widetilde{U}_{n-1}^-(x;a,b) \tag{6.10}$$

for $n \geq 1$, and the initial values are $\widetilde{U}_0^-(x;a,b,c) = 1$ and $\widetilde{U}_1^-(x;a,b,c) = x(a+1)+c$. Let $\widetilde{U}_n^{-,*}(x;a,b,c)$ be corresponding numerator polynomial with the initial values $\widetilde{U}_0^{-,*}(x;a,b,c) = 0$ and $\widetilde{U}_1^{-,*}(x;a,b,c) = a+1$. Then, similar to (6.4), we have

$$\widetilde{U}_n^{-,*}(x;a,b,c) = (1+a)q^{n-1}\widetilde{U}_{n-1}^-(x;a/q,b/q,c/q). \tag{6.11}$$

Again, since $\widetilde{U}_n^-(z;a,b,c)$ and $\widetilde{U}_n^{-,*}(z;a,b,c)$ have the same leading coefficients when $n > 1$, it follows from (2.14) that

$$\begin{aligned}\widetilde{K}_0^- &= \lim_{n\to\infty} \frac{\widetilde{U}_n^-(z;a,b,c)}{\widetilde{U}_n^{-,*}(z;a,b,c)} = \lim_{n\to\infty} \frac{\widetilde{U}_n^-(z;a,b,c)}{(1+a)q^{n-1}\widetilde{U}_{n-1}^-(z;a/q,b/q,c/q)} \\ &= \lim_{n\to\infty} \frac{U_n^-(z;a+c/z,b)}{(1+a)q^{n-1}U_{n-1}^-(z;a/q+c/(qz),bq)}.\end{aligned} \tag{6.12}$$

Note that

$$K_0^- = \lim_{n\to\infty} \frac{U_n^-(z;a,b)}{(1+a)q^{n-1}U_{n-1}^-(z;a,b)}, \tag{6.13}$$

where K_0^- is the continued fraction given in (5.6), we have from the above two formulas

$$\widetilde{K}_0^- = \frac{(z+\frac{c}{1+a})F(\frac{b}{(az+c)^2};\frac{1}{a+c/z})}{F(\frac{qb}{(az+c)^2};\frac{q}{a+c/z})}. \tag{6.14}$$

Coupling the formulas for $\widetilde{K}_0^+$ and $\widetilde{K}_0^-$ in (6.9) and (6.14), we obtain

$$\widetilde{S}_{00}(z) = \frac{1}{\widetilde{K}_0^+ + \widetilde{K}_0^- - (z+\frac{c}{1+a})} = \frac{F(\frac{qb}{z^2};qa+\frac{qc}{z})F(\frac{qb}{(az+c)^2};\frac{q}{a+c/z})}{(z+\frac{c}{1+a})G(\frac{1}{z^2};a+\frac{c}{z},b)},$$

where $G(t;a+c/z,b)$ takes the same form of the $G(t;a,b)$ in the previous section with a replaced by $a+c/z$. To be more specific, $G(t;a+c/z,b) = (bt/(a+c/z);q)_\infty$.

Next, we derive the formulas for $\widetilde{K}_1^\pm$ from those of $\widetilde{K}_0^\pm$ where the parameters a,b,c are multiplied by a common factor of q.

$$\widetilde{K}_1^+ = z - \widetilde{a}_1 + K_{k=1}^\infty \left[\frac{-\widetilde{b}_{k+1}}{z-\widetilde{a}_{k+1}}\right] = \frac{(z+\frac{qc}{1+qa})F(\frac{qb}{z^2};qa+\frac{qc}{z})}{F(\frac{q^2b}{z^2};q^2a+\frac{q^2c}{z})}, \tag{6.15}$$

$$\widetilde{K}_1^- = z - \widetilde{a}_1 + K_{k=1}^{\infty}\left[\frac{-\widetilde{b}_{2-k}}{z-\widetilde{a}_{1-k}}\right] = \frac{(z+\frac{qc}{1+qa})F(\frac{b}{q(az+c)^2};\frac{1}{qa+qc/z})}{F(\frac{b}{(az+c)^2};\frac{1}{a+c/z})}. \tag{6.16}$$

Using a similar argument as in the previous section, we obtain

$$\widetilde{S}_{11}(z) = \frac{1}{\widetilde{K}_1^+ + \widetilde{K}_1^- - (z-\widetilde{a}_1)} = \frac{F(\frac{q^2b}{z^2};q^2a+\frac{q^2c}{z})F(\frac{b}{(az+c)^2};\frac{1}{a+c/z})}{(z+\frac{qc}{1+qa})G(\frac{1}{z^2};a+\frac{c}{z},b)},$$

$$\widetilde{S}_{01}(z) = \frac{\widetilde{b}_1}{\widetilde{K}_1^+\widetilde{K}_0^- - \widetilde{b}_1^2} = \frac{F(\frac{q^2b}{z^2};q^2a+\frac{q^2c}{z})F(\frac{qb}{(az+c)^2};\frac{q}{a+c/z})\widetilde{b}_1}{(z+\frac{c}{1+a})(z+\frac{qc}{1+qa})G(\frac{1}{z^2};a+\frac{c}{z},b)},$$

where in the last equation we have made use of the identities $\widetilde{b}_1^2 = b/[(1+a)(1+aq)]$ and

$$(z+\frac{c}{1+a})(z+\frac{qc}{1+qa})(1+a)(1+aq) = z^2(1+a+\frac{c}{z})(1+qa+\frac{qc}{z}).$$

Summarizing the above formulas for the Stieltjes transformations of the spectral measures, we obtain the following theorem.

Theorem 6. *Let $a, b > 0$ and $c \in \mathbb{R}$, the spectral measures for the doubly infinite Jacobi matrix with*

$$\widetilde{a}_n = -\frac{cq^n}{1+aq^n}, \quad \widetilde{b}_n = \sqrt{\frac{bq^{n-1}}{(1+aq^n)(1+aq^{n-1})}}$$

have discrete mass points at $x_k^{\pm} = (-c \pm \sqrt{c^2+4abq^k})/(2a)$ for all $k = 0, 1, \cdots$. Moreover, the masses at $x_k^{\pm}$ are given by

$$m_{00}(x_k^{\pm}) = \frac{bq^kF(\frac{qb}{(x_k^{\pm})^2};\frac{q^{1+k}b}{(x_k^{\pm})^2})F(\frac{(x_k^{\pm})^2}{q^{2k-1}b};\frac{(x_k^{\pm})^2}{q^{k-1}b})}{(2ax_k^{\pm}+c)(x_k^{\pm}+\frac{c}{1+a})(q^{-k};q)_k(q;q)_\infty}, \tag{6.17}$$

$$m_{11}(x_k^{\pm}) = \frac{bq^kF(\frac{q^2b}{(x_k^{\pm})^2};\frac{q^{2+k}b}{(x_k^{\pm})^2})F(\frac{(x_k^{\pm})^2}{q^{2k}b};\frac{(x_k^{\pm})^2}{q^kb})}{(2ax_k^{\pm}+c)(x_k^{\pm}+\frac{qc}{1+a})(q^{-k};q)_k(q;q)_\infty}, \tag{6.18}$$

$$m_{01}(x_k^{\pm}) = \frac{bq^k\sqrt{\frac{b}{(1+a)(1+aq)}}F(\frac{q^2b}{(x_k^{\pm})^2};\frac{q^{2+k}b}{(x_k^{\pm})^2})F(\frac{(x_k^{\pm})^2}{q^{2k-1}b};\frac{(x_k^{\pm})^2}{q^{k-1}b})}{(2ax_k^{\pm}+c)(x_k^{\pm}+\frac{c}{1+a})(x_k^{\pm}+\frac{qc}{1+a})(q^{-k};q)_k(q;q)_\infty}. \tag{6.19}$$

Proof. Note that the zeros of

$$G(\frac{1}{z^2};a+\frac{c}{z},b) = (\frac{b}{az^2+cz};q)_\infty$$

satisfy the equation

$$\frac{bq^k}{az^2+cz}=1$$

for some $k=0,1,\cdots$. Solving the above quadratic equation gives

$$z=x_k^{\pm}=\frac{-c\pm\sqrt{c^2+4bq^k}}{2a}.$$

The masses at $x_k^{\pm}$ are obtained by simple application of residue theorem. □

On account of $\tilde{a}:=a+c/x_k^{\pm}=bq^k/(x_k^{\pm})^2$, we calculate the determinant

$$\begin{vmatrix} m_{00}(x_k^{\pm}) & m_{01}(x_k^{\pm}) \\ m_{10}(x_k^{\pm}) & m_{11}(x_k^{\pm}) \end{vmatrix}=\frac{(bq^k)^2F(\frac{q^2b}{(x_k^{\pm})^2};\frac{q^{2+k}b}{(x_k^{\pm})^2})F(\frac{(x_k^{\pm})^2}{q^{2k-1}b};\frac{(x_k^{\pm})^2}{q^{k-1}b})H(\frac{1}{(x_k^{\pm})^2};\tilde{a},b)}{(x_k^{\pm}+\frac{c}{1+a})(x_k^{\pm}+\frac{qc}{1+a})[(2ax_k^{\pm}+c)(q^{-k};q)_k(q;q)_\infty]^2},$$

where H is the function defined in (5.13). In view of (5.14), we have $H(\frac{1}{(x_k^{\pm})^2};\tilde{a},b)=(q^{-k},q)_\infty=0$, and consequently,

$$\begin{vmatrix} m_{00}(x_k^{\pm}) & m_{01}(x_k^{\pm}) \\ m_{10}(x_k^{\pm}) & m_{11}(x_k^{\pm}) \end{vmatrix}=0.$$

This together with [6, Theorems 2.27–2.28] implies that any element of the point spectrum of the doubly infinite Jacobi matrix is a simple eigenvalue (i.e., with multiplicity one).

Remark 5. If $x_k^{\pm}$ are the only discrete mass points of the spectral measure $d\mu_{00}(x)$, we then have for $z\neq x_k^{\pm}$,

$$\begin{aligned}&\frac{F(\frac{qb}{z^2};qa+\frac{qc}{z})F(\frac{qb}{(az+c)^2};\frac{q}{a+c/z})}{(z+\frac{c}{1+a})G(\frac{1}{z^2};a+\frac{c}{z},b)}\\&=\sum_{k=0}^{\infty}\frac{bq^kF(\frac{qb}{(x_k^{+})^2};\frac{q^{1+k}b}{(x_k^{+})^2})F(\frac{(x_k^{+})^2}{q^{2k-1}b};\frac{(x_k^{+})^2}{q^{k-1}b})}{(2ax_k^{+}+c)(x_k^{+}+\frac{c}{1+a})(q^{-k};q)_k(q;q)_\infty(z-x_k^{+})}\\&\quad+\sum_{k=0}^{\infty}\frac{bq^kF(\frac{qb}{(x_k^{-})^2};\frac{q^{1+k}b}{(x_k^{-})^2})F(\frac{(x_k^{-})^2}{q^{2k-1}b};\frac{(x_k^{-})^2}{q^{k-1}b})}{(2ax_k^{-}+c)(x_k^{-}+\frac{c}{1+a})(q^{-k};q)_k(q;q)_\infty(z-x_k^{-})},\end{aligned}\tag{6.20}$$

where $x_k^{\pm}=(-c\pm\sqrt{c^2+4abq^k})/(2a)$. We leave the proof of the above equality as an open problem.

Acknowledgments

Dan Dai is partially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 11300814, CityU

11300115, CityU 11303016). We are grateful to the anonymous referees for constructive suggestions which have led to a significant improvement on the manuscript.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U.S. Government Printing Office, Washington, D.C., 1964.
[2] W.A. Al-Salam, M.E.H. Ismail, Orthogonal polynomials associated with the Rogers–Ramanujan continued fraction, Pacific J. Math. 104 (1983) 269–283.
[3] R.A. Askey, J. Wimp, Associated Laguerre and Hermite polynomials, Proc. Roy. Soc. Edinburgh Sect. A 96 (1984) 15–37.
[4] Ju.M. Berezanskii, Expansions in Eigenfunctions of Selfadjoint Operators, Translations of Mathematical Monographs, vol. 17, American Mathematical Society, Providence, R.I., 1968.
[5] T.S. Chihara, Chain sequences and orthogonal polynomials, Trans. Amer. Math. Soc. 104 (1962) 1–16.
[6] D. Damanik, A. Pushnitski, B. Simon, The analytic theory of matrix orthogonal polynomials, Surv. Approx. Theory 4 (2008) 1–85.
[7] A.J. Duran, Ratio asymptotics for orthogonal matrix polynomials, J. Approx. Theory 100 (1999) 304–344.
[8] A.J. Durán, F.A. Grünbaum, A survey on orthogonal matrix polynomials satisfying second order differential equations, J. Comput. Appl. Math. 178 (2005) 169–190.
[9] A.J. Duran, P. Lopez-Rodriguez, E.B. Saff, Zero asymptotic behaviour for orthogonal matrix polynomials, J. Anal. Math. 78 (1999) 37–60.
[10] G. Gasper, M. Rahman, Basic Hypergeometric Series, second edition, Cambridge University Press, Cambridge, 2004.
[11] W. Groenevelt, The Wilson function transform, Int. Math. Res. Not. 52 (2003) 2779–2817.
[12] W. Groenevelt, Wilson function transforms related to Racah coefficients, Acta Appl. Math. 91 (2006) 133–191.
[13] W. Groenevelt, M.E.H. Ismail, E. Koelink, Spectral decomposition and matrix-valued orthogonal polynomials, Adv. Math. 244 (2013) 91–105.
[14] W. Groenevelt, E. Koelink, A hypergeometric function transform and matrix-valued orthogonal polynomials, Constr. Approx. 38 (2013) 277–309.
[15] M.E.H. Ismail, Classical and Quantum Orthogonal Polynomials in one Variable, Cambridge University Press, Cambridge, 2005.
[16] M.E.H. Ismail, J. Letessier, G. Valent, Linear birth and death models and associated Laguerre and Meixner polynomials, J. Approx. Theory 55 (1988) 337–348.
[17] M.E.H. Ismail, M. Rahman, Associated Askey–Wilson polynomials, Trans. Amer. Math. Soc. 328 (1991) 201–237.
[18] M.E.H. Ismail, F. Štampach, Spectral analysis of two doubly infinite Jacobi matrices with exponential entries, under review.
[19] R. Koekoek, P.A. Lesky, R.F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.
[20] E. Koelink, Spectral theory and special functions, in: Laredo Lectures on Orthogonal Polynomials and Special Functions, in: Adv. Theory Spec. Funct. Orthogonal Polynomials, Nova Sci. Publ., Hauppauge, NY, 2004, pp. 45–84.
[21] B.Ja. Levin, Distribution of Zeros of Entire Functions, Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980.
[22] D. Maki, On constructing distribution functions: with applications to Lommel polynomials and Bessel functions, Trans. Amer. Math. Soc. 130 (1968) 281–297.
[23] D.R. Masson, Associated Wilson polynomials, Constr. Approx. 7 (1991) 521–534.
[24] D.R. Masson, J. Repka, Spectral theory of Jacobi matrices in $l^2(Z)$ and the $su(1,1)$ Lie algebra, SIAM J. Math. Anal. 22 (1991) 1131–1146.
[25] F.W.J. Olver, D. Lozier, R. Boisvert, C. Clark, NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
[26] F. Pollaczek, Sur une généralisation des polynômes de Jacobi, Memorial des Sciences Mathematique, vol. 131, Gauthier-Villars, Paris, 1956.
[27] H.M. Schwartz, A class of continued fractions, Duke Math. J. 6 (1940) 48–65.

[28] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.
[29] F. Štampach, P. Šťovíček, The Nevanlinna parametrization for q-Lommel polynomials in the indeterminate case, J. Approx. Theory 201 (2016) 48–72.
[30] R.V. Wallisser, On entire functions assuming integer values in a geometric sequence, in: Théorie des Nombres, Quebec, PQ, 1987, de Gruyter, Berlin, 1989, pp. 981–989.
[31] G.N. Watson, A Treatise on the Theory of Bessel Functions, second edition, Cambridge University Press, Cambridge, 1944.