APPROXIMATE UNIQUENESS FOR MAPS FROM $C(X)$ INTO SIMPLE REAL RANK ZERO C*-ALGEBRAS

P. W. NG

ABSTRACT. Let X be a finite CW complex and let $\mathcal{A}$ be a unital separable simple finite $\mathcal{Z}$-stable C*-algebra with real rank zero.

We prove an approximate uniqueness theorem for almost multiplicative contractive completely positive linear maps from $C(X)$ into $\mathcal{A}$.

We also give conditions for when such a map can, within a certain "error", be approximated by a finite dimensional *-homomorphism.

1. INTRODUCTION

A basic result in linear algebra says the following: Let A, B be two n by n normal matrices, then A and B are unitarily equivalent (i.e., there exists an n by n unitary matrix U such that $A = UBU^*$) if and only if A and B have the same spectrum, counting multiplicities. This fundamental result is a starting point for spectral theory with vast generalizations and implications, especially infinite dimensional ones.

In C*-algebra theory, one important class of generalizations is the class of uniqueness theorems which dot the landscape of extension theory as well as the Elliott classification program. One important such generalization was the work of Brown–Douglas–Fillmore (BDF) who classified the essentially normal operators using Fredholm indices. One of the items that they proved was the following: Let X be a compact metric space, and $\phi, \psi : C(X) \rightarrow \mathbb{B}(l^2)/\mathcal{K}$ be two unital *-monomorphisms. Then ϕ and ψ are unitarily equivalent if and only if $[\phi] = [\psi]$ in $KK(C(X), \mathbb{B}(l^2)/\mathcal{K})$ ([1]).

The work of BDF resulted in very intense interest in extension theory for C*-algebras with implications to K-theory, KK-theory and noncommutative topology. Their work, together with Elliott's, were also the starting point of the powerful uniqueness and stable uniqueness theorems of classification theory (with connections to many interesting subjects including the theory of absorbing extensions).

Since the literature on this subject is too big, we mention only a few uniqueness theorems, with an emphasis on those which are most closely related to the contents of this paper.

Let X be a compact metric space and $\mathcal{A}$ be a unital C*-algebra. Recall that two maps $\phi, \psi : C(X) \rightarrow \mathcal{A}$ are *approximately unitarily equivalent* if there exists a sequence $\{u_n\}$ of unitaries in $\mathcal{A}$ such that for all $f \in C(X)$, $u_n \phi(f) u_n^* \rightarrow \psi(f)$ in the norm topology.

While there are many precedents, perhaps a good place to begin is the following uniqueness result of Gong and Lin: Let X be a compact metric space and let $\mathcal{A}$ be a unital simple C*-algebra with real rank zero, stable rank one, weakly unperforated K_0 group, and with a unique tracial state τ. Suppose that $\phi, \psi : C(X) \rightarrow \mathcal{A}$ are two unital *-monomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(C(X), \mathcal{A})$ and $\tau \circ \phi = \tau \circ \psi$ ([4]).

In [14], the above restriction on the tracial simplex was removed on the assumption that $\mathcal{A}$ is TAF. We note that Lin used this result to generalize a result of Kishimoto. Specifically, Lin showed in [14] that if $\mathcal{A}$ is a simple unital AH-algebra with real rank zero and bounded dimension growth, and if α is an approximately inner *-automorphism of $\mathcal{A}$ with the tracial Rokhlin property, then the crossed product $\mathcal{A} \times_\alpha \mathbb{Z}$ is a unital simple AH-algebra with bounded dimension growth and real rank zero.

In [18], the restriction on the tracial simplex in the result of [4] was again removed with the additional assumption that $\mathcal{A}$ is $\mathcal{Z}$-stable. (Here, $\mathcal{Z}$ is the Jiang–Su algebra.)

Finally, we briefly mention that, in recent years, with the many breakthroughs in classification theory, the real rank zero condition in the codomain algebra $\mathcal{A}$ was removed, though with additional assumptions like rationally TAI and $\mathcal{Z}$-stability. This has allowed for interesting uniqueness results like in the case where the codomain algebra is the projectionless algebra $\mathcal{Z}$. (See, for example, [15] and the references therein.)

In this paper, we prove an approximate, almost multiplicative version of the result in [18]. In fact, approximate, almost multiplicative versions of the corresponding uniqueness result are often present in the papers cited above. That is because such a result is often more useful in applications. We also prove a result which gives (natural K-theoretic and "injectivity"-type) conditions for when such an almost multiplicative map can be, within an "error", approximated by a finite dimensional *-homomorphism. This is actually part of a long line of results related to many issues, including, for example, Lin's famous result that almost commuting self-adjoint matrices are uniformly close to commuting self-adjoint matrices. (See, for example, [2], [4], [5], [6], [8], [10], [11], [16], and the references in these papers.)

The results in this short note will be used, in a future paper, to classify all extensions of the form

$$0 \to \mathcal{B} \to \mathcal{E} \to C(X) \to 0$$

where $\mathcal{B}$ is a simple, real rank zero, $\mathcal{Z}$-stable C*-algebra with continuous scale and X is a finite CW complex.

Some good basic references for the subject matter of this note are [6], [13], [14], [4], and the references in these papers. We refer to these references for basic results, definitions and notation that we use in this paper.

2. Main result

For a C*-algebra $\mathcal{C}$, we let $T(\mathcal{C})$ denote the *tracial state space* of $\mathcal{C}$.

Next, throughout this paper, whenever we have a stably finite simple unital separable C*-algebra $\mathcal{C}$, we will assume that every quasitrace of $\mathcal{C}$ is a trace.

Recall that $\mathcal{Z}$ denotes the Jiang–Su algebra. (See [7], [3].) Also, recall that a C*-algebra $\mathcal{C}$ is said to be *$\mathcal{Z}$-stable* if $\mathcal{C} \otimes \mathcal{Z} \cong \mathcal{C}$.

Proposition 2.1. *Let X be a compact metric space with finite covering dimension, and let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of unital simple separable finite $\mathcal{Z}$-stable real rank zero C*-algebras.*

*Suppose that $\phi : C(X) \to \prod_{k \in \mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k \in \mathbb{Z}_+} \mathcal{A}_k$ is a unital *-homomorphism and let $\epsilon > 0$ be given.*

Then, there are a unital commutative AF-subalgebra $\mathcal{D} \subseteq \prod_{k \in \mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k \in \mathbb{Z}_+} \mathcal{A}_k$*, a sequence* $(p_k)_{k \in \mathbb{Z}_+} \in \prod_{k \in \mathbb{Z}_+} \mathcal{A}_k$ *of projections lifting* $p =_{df} 1_{\mathcal{D}} \in \prod_{k \in \mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k \in \mathbb{Z}_+} \mathcal{A}_k$ *and an integer* $L \geq 1$ *such that*

(1) $p \in \phi(C(X))' \cap \left(\prod_{k \in \mathbb{Z}_+} \mathcal{A}_k / \bigoplus_{k \in \mathbb{Z}_+} \mathcal{A}_k \right)$,
(2) $p\phi(C(X)) \subseteq \mathcal{D}$,
(3) $\tau(p_k) > 1 - \epsilon$, *for all* $\tau \in T(\mathcal{A}_k)$ *and for all* $k \geq L$.

Proof. The proof is exactly the same as that of [18] Proposition 2.6. □

Let $\mathcal{C}, \mathcal{D}$ be C*-algebras, let $\mathcal{G} \subseteq \mathcal{C}$ and $\delta > 0$. Recall that a map $\phi : \mathcal{C} \to \mathcal{D}$ is said to be $\mathcal{G}$-δ-*multiplicative* if for all $a, b \in \mathcal{G}$,

$$\|\phi(ab) - \phi(a)\phi(b)\| < \delta.$$

Also, throughout this paper, we often use *"c.p.c."* to abbreviate "completely positive contractive". All our c.p.c. maps are assumed to be linear.

Lemma 2.2. *Let* X *be a compact metric space.*

Then for every $\epsilon > 0$*, for every finite subset* $\mathcal{F} \subset C(X)$*, there exist* $\delta > 0$ *and a finite subset* $\mathcal{G} \subset C(X)$ *such that the following is true:*

For any unital simple separable finite real rank zero $\mathcal{Z}$*-stable C*-algebra* $\mathcal{A}$*, for any* $\mathcal{G}$*-*δ*-multiplicative unital c.p.c. map* $\phi : C(X) \to \mathcal{A}$*, there exists a commutative finite dimensional C*-subalgebra* $\mathcal{B} \subseteq \mathcal{A}$ *where*

(1) $\|\phi(f)1_{\mathcal{B}} - 1_{\mathcal{B}}\phi(f)\| < \epsilon$ *for all* $f \in \mathcal{F}$
(2) $dist(1_{\mathcal{B}}\phi(f)1_{\mathcal{B}}, \mathcal{B}) < \epsilon$ *for all* $f \in \mathcal{F}$
(3) $\tau(1_{\mathcal{B}}) > 1 - \epsilon$ *for all* $\tau \in T(\mathcal{A})$.

Proof. Suppose, to the contrary, that X is a compact metric space, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ is a finite subset for which the statement fails.

We can replace $C(X)$ with the (unital) C*-subalgebra generated by $\mathcal{F} \cup \{1_X\}$. This C*-subalgebra is a unital commutative C*-algebra with spectrum having finite covering dimension. Hence, we may assume that X has finite covering dimension.

Hence, let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be an increasing sequence of finite subsets of $C(X)$, let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a sequence of unital simple separable finite real rank zero $\mathcal{Z}$-stable C*-algebras, and for each $n \geq 1$, let $\phi_n : C(X) \to \mathcal{A}_n$ be a unital c.p.c. $\mathcal{G}_n$-$\frac{1}{n}$-multiplicative map such that

(a) $\mathcal{F} \subseteq \mathcal{G}_1$,
(b) $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $C(X)$, and
(c) for all $n \geq 1$, there is no commutative finite dimensional C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}_n$ such that the following three conditions hold: (i.) $\|\phi_n(f)1_{\mathcal{B}} - 1_{\mathcal{B}}\phi_n(f)\| < \epsilon$ for all $f \in \mathcal{F}$, (ii.) $dist(1_{\mathcal{B}}\phi_n(f)1_{\mathcal{B}}, \mathcal{B}) < \epsilon$ for all $f \in \mathcal{F}$ and (iii.) $\tau(1_{\mathcal{B}}) > 1 - \epsilon$ for all $\tau \in T(\mathcal{A}_n)$.

We denote the above statements by "(*)".

The unital c.p.c. map $(\phi_n)_{n \in \mathbb{Z}_+} : C(X) \to \prod_{n \in \mathbb{Z}_+} \mathcal{A}_n$ naturally induces a unital *-homomorphism $\phi : C(X) \to \prod_{n \in \mathbb{Z}_+} \mathcal{A}_n / \bigoplus_{n \in \mathbb{Z}_+} \mathcal{A}_n$.

To simplify notation, let us denote $\mathcal{C} =_{df} \prod_{n \in \mathbb{Z}_+} \mathcal{A}_n / \bigoplus_{n \in \mathbb{Z}_+} \mathcal{A}_n$.

By Proposition 2.1, let $\mathcal{D} \subseteq \mathcal{C}$ be a unital commutative AF-subalgebra, $(p_n)_{n \in \mathbb{Z}_+} \in \prod_{n \in \mathbb{Z}_+} \mathcal{A}_n$ an ∞-tuple of projections lifting $p =_{df} 1_{\mathcal{D}} \in \mathcal{C}$, and $L_0 \geq 1$ such that

i. $p \in \phi(C(X))' \cap \mathcal{C}$
ii. $p\phi(C(X)) \subseteq \mathcal{D}$

iii. $\tau(p_k) > 1 - \epsilon/10$ for all $\tau \in T(\mathcal{A}_k)$ and for all $k \geq L_0$.

Hence, there exists a finite-dimensional unital C*-subalgebra $\mathcal{B}_0 \subseteq \mathcal{D}$ such that $dist(1_{\mathcal{B}_0}\phi(f)1_{\mathcal{B}_0}, \mathcal{B}_0) = dist(p\phi(f), \mathcal{B}_0) < \epsilon/2$ for all $f \in \mathcal{F}$.

Note that $\mathcal{B}_0$ is commutative and $1_{\mathcal{B}_0} = 1_{\mathcal{D}} = p$. Since $(p_n)_{n \in \mathbb{Z}_+}$ lifts $1_{\mathcal{D}} = 1_{\mathcal{B}_0}$, $1_{\mathcal{B}_0}\mathcal{C}1_{\mathcal{B}_0}$ is a quotient of $\prod_{n \in \mathbb{Z}_+} p_n \mathcal{A}_n p_n$. Since finite dimensional C*-algebras are semiprojective, there exists a *-homomorphism

$$\sigma : \mathcal{B}_0 \to \prod_{n \in \mathbb{Z}_+} p_n \mathcal{A}_n p_n$$

which lifts the natural inclusion map $\mathcal{B}_0 \hookrightarrow 1_{\mathcal{B}_0}\mathcal{C}1_{\mathcal{B}_0}$. Now $\sigma = (\sigma_n)_{n \in \mathbb{Z}_+}$ where for all n, $\sigma_n : \mathcal{B}_0 \to p_n \mathcal{A}_n p_n$ is a *-homomorphism. All but finitely many of the σ_ns are unital. We can find a large enough integer $L \geq L_0$ such that if $\mathcal{B} =_{df} \sigma_L(\mathcal{B}_0)$ then $\mathcal{B} \subseteq \mathcal{A}_L$ is a commutative finite dimensional C*-subalgebra such that $dist(p_L \phi_L(f) p_L, \mathcal{B}) < \epsilon$ for all $f \in \mathcal{F}$, $1_{\mathcal{B}} = p_L$, and $\|\phi_L(f)1_{\mathcal{B}} - 1_{\mathcal{B}}\phi_L(f)\| < \epsilon$ for all $f \in \mathcal{F}$. Since $\tau(1_{\mathcal{B}}) = \tau(p_L) > 1 - \epsilon$ for all $\tau \in T(\mathcal{A}_L)$, this contradicts part (c) of (*). $\square$

We need a lemma concerning c.p.c. almost multiplicative maps from $C(X)$ into a matrix algebra (cf. [13] Lemma 6.2.7):

Lemma 2.3. *Let X be a compact metric space. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given.*

Then there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following:

*For every positive integer $n \geq 1$, if $L : C(X) \to \mathbb{M}_n(\mathbb{C})$ is a $\mathcal{G}$-δ-multiplicative c.p.c. map then there exists a *-homomorphism $h : C(X) \to \mathbb{M}_n$ with $h(1_{C(X)}) = r$ such that*

$$tr(1 - r) < \epsilon$$

and

$$\|(1 - r)L(f)(1 - r) + h(f) - L(f)\| < \epsilon$$

for all $f \in \mathcal{F}$.
Moreover, the map $C(X) \to (1 - r)\mathbb{M}_n(1 - r) : f \mapsto (1 - r)L(f)(1 - r)$ is $\mathcal{F}$-ϵ-multiplicative.
(In the above, tr is the unique tracial state on $\mathbb{M}_n$.)

The proof of the next lemma is a variation on [18] Theorem 3.1 and also [14] Lemma 4.4. See also [17].

Proposition 2.4. *Let X be a compact metric space, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ a finite subset. Let $N \geq 1$ be an integer and $\eta > 0$ be such that for all $f \in \mathcal{F}$, $|f(x) - f(x')| < \frac{\epsilon}{8}$ if $dist(x, x') < 2\eta$. Then for any integer $s \geq 2$, any finite $\eta/2$-dense subset $\{x_1, x_2, ..., x_n\} \subset X$ for which $\overline{O_j} \cap \overline{O_k} = \emptyset$ for $j \neq k$, where for all j,*

$$O_j =_{df} B(x_j, \eta/s) = \{x \in X : dist(x, x_j) < \eta/s\},$$

and for any $1/(2s) > \sigma > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following:

For any unital separable simple finite $\mathcal{Z}$-stable C-algebra $\mathcal{A}$ with real rank zero, and any unital $\mathcal{G}$-δ-multiplicative c.p.c. map $\phi : C(X) \to \mathcal{A}$ with*

$$\mu_{\tau \circ \phi}(O'_j) > \sigma\eta$$

for all $\tau \in T(\mathcal{A})$ *and for all* $1 \leq j \leq n$*, where*

$$O'_j =_{df} B(x_j, \eta/(2s)) = \{x \in X : dist(x, x_j) < \eta/(2s)\}$$

for $1 \leq j \leq n$*, there exist a projection* $p \in \mathcal{A}$ *and an* $\mathcal{F}$*-*ϵ*-multiplicative c.p.c. map* $L_1 : C(X) \to p\mathcal{A}p$ *such that*

(1) *there exist pairwise orthogonal projections* $p_0, p_1, p_2, ..., p_n, t \in \mathcal{A}$ *with* $p_0 = p$*,* $\sum_{j=0}^{n} p_j + t = 1_{\mathcal{A}}$*, and* $Np \preceq p_j$ *for* $j \neq 0$[1]*, and*
(2) *there exists a finite dimensional *-homomorphism* $h_1 : C(X) \to t\mathcal{A}t$ *such that* $\phi(f)$ *is within* ϵ *of* $L_1(f) + \sum_{j=1}^{n} f(x_j)p_j + h_1(f)$ *for all* $f \in \mathcal{F}$*.*

Proof. Let X, ϵ, $\mathcal{F}$, $N \geq 1$, $\eta > 0$, $s \geq 2$, $\{x_1, x_2, ..., x_n\}$, $O_1, O_2, ..., O_n$, and σ be given as in the hypotheses. For simplicity, we may assume that $\epsilon < 1$ and all the elements of $\mathcal{F}$ have norm less than or equal to one.

For $1 \leq j \leq n$, let $f_j : X \to [0, 1]$ be a continuous function such that $f_j(x) = 1$ for all $x \in \overline{O'_j}$, $f_j(x) = 0$ for all $x \in X - O_j$, and for all $j \neq k$, $f_j f_k = 0$.

Let $\mathcal{G}_1 =_{df} \mathcal{F} \cup \{1_{C(X)}\} \cup \{f_j : 1 \leq j \leq n\}$.

Let

$$\epsilon_1 =_{df} \frac{1}{10(N+1)} \inf\{\epsilon, \sigma\eta\} > 0.$$

Plug X, ϵ_1 and $\mathcal{G}_1$ into Lemma 2.3 to get $\mathcal{G}_2$ and $\epsilon_2 > 0$. Contracting ϵ_2 if necessary, we may assume that all the elements of $\mathcal{G}_2$ have norm less than or equal to one, and also $\epsilon_2 < \epsilon_1$. We may also assume that $\mathcal{G}_1 \subset \mathcal{G}_2$.

Let

$$\mathcal{G}_3 =_{df} \mathcal{G}_2 \cup \{fg : f, g \in \mathcal{G}_2\}.$$

Let $\epsilon_3 > 0$ be such that if $\mathcal{C}$ is a C*-algebra, if $\psi_1 : C(X) \to \mathcal{C}$ is a c.p.c $\mathcal{G}_3$-ϵ_3 multiplicative map and $q \in \mathcal{A}$ is a nonzero projection such that $\|\psi_1(f)q - q\psi_1(f)\| < \epsilon_3$ for all $f \in \mathcal{G}_3$, then the map $C(X) \to q\mathcal{C}q : f \mapsto q\psi_1(f)q$ is a c.p.c. $\mathcal{G}_2$-ϵ_2-multiplicative map.

Contracting ϵ_3 if necessary, we may assume that if $\mathcal{C}$ is a C*-algebra, $\mathcal{C}_1 \subseteq \mathcal{C}$ is a finite-dimensional C*-subalgebra, and $\psi_2 : C(X) \to \mathcal{C}$ is a c.p.c. $\mathcal{G}_3$-ϵ_3-multiplicative map such that

$$\|1_{\mathcal{C}_1}\psi_2(f) - \psi_2(f)1_{\mathcal{C}_1}\| < \epsilon_3$$

and

$$dist(1_{\mathcal{C}_1}\psi_2(f)1_{\mathcal{C}_1}, \mathcal{C}_1) < \epsilon_3$$

for all $f \in \mathcal{G}_3$, then there exists a $\mathcal{G}_2$-ϵ_2-multiplicative c.p.c. map $\psi_3 : C(X) \to \mathcal{C}_1$ such that

$$\|\psi_3(f) - 1_{\mathcal{C}_1}\psi_2(f)1_{\mathcal{C}_1}\| < \epsilon_2$$

for all $f \in \mathcal{G}_2$.

Contracting ϵ_3 further if necessary, we may assume that $\epsilon_3 < \epsilon_1$.

Plug X, $\epsilon_3/2$ and $\mathcal{G}_3$ into Lemma 2.2 to get δ and a finite subset $\mathcal{G} \subset C(X)$. Again, contracting δ if necessary, we may assume that $\delta < \epsilon_3$ and that the elements of $\mathcal{G}$ have norm less than or equal to one. We may also assume that $\mathcal{G}_3 \subseteq \mathcal{G}$.

Now suppose that $\mathcal{A}$ is a unital separable simple finite real rank zero $\mathcal{Z}$-stable C*-algebra. Suppose that $\phi : C(X) \to \mathcal{A}$ is a unital $\mathcal{G}$-δ-multiplicative c.p.c. map such that

$$\mu_{\tau\circ\phi}(O'_j) > \sigma\eta \tag{2.5}$$

[1] "Np" abbreviates "$\bigoplus^N p$".

for all $\tau \in T(\mathcal{A})$ and for all $1 \leq j \leq n$.

By Lemma 2.2 and by the definition of $\mathcal{G}$ and δ, let $\mathcal{B} \subseteq \mathcal{A}$ be a finite dimensional C*-subalgebra where

i. $\|\phi(f)1_{\mathcal{B}} - 1_{\mathcal{B}}\phi(f)\| < \epsilon_3/2$ for all $f \in \mathcal{G}_3$,
ii. $dist(1_{\mathcal{B}}\phi(f)1_{\mathcal{B}}, \mathcal{B}) < \epsilon_3/2$ for all $f \in \mathcal{G}_3$,
iii. $\|1_{\mathcal{B}}\phi(f)1_{\mathcal{B}} + (1_{\mathcal{A}} - 1_{\mathcal{B}})\phi(f)(1_{\mathcal{A}} - 1_{\mathcal{B}}) - \phi(f)\| < \epsilon_3$ for all $f \in \mathcal{G}_3$, and
iv. $\tau(1_{\mathcal{B}}) > 1 - \epsilon_3/2$ for all $\tau \in T(\mathcal{A})$.

By the definition of ϵ_3, the map

$$L : C(X) \to (1_{\mathcal{A}} - 1_{\mathcal{B}})\mathcal{A}(1_{\mathcal{A}} - 1_{\mathcal{B}}) : f \mapsto (1_{\mathcal{A}} - 1_{\mathcal{B}})\phi(f)(1_{\mathcal{A}} - 1_{\mathcal{B}})$$

is a c.p.c. $\mathcal{G}_2$-ϵ_2-multiplicative map.

Also, by the definition of ϵ_3, there exists a c.p.c. $\mathcal{G}_2$-ϵ_2-multiplicative map

$$L' : C(X) \to \mathcal{B}$$

such that

$$\|L'(f) - 1_{\mathcal{B}}\phi(f)1_{\mathcal{B}}\|) < \epsilon_2$$

for all $f \in \mathcal{G}_2$.

Hence,

$$\|L(f) + L'(f) - \phi(f)\| < \epsilon_2 + \epsilon_3$$

for all $f \in \mathcal{G}_2$. Also,

$$\tau(L(1_{C(X)})) = \tau(1_{\mathcal{A}} - 1_{\mathcal{B}}) < \epsilon_3$$

for all $\tau \in T(\mathcal{A})$.

Since $L' : C(X) \to \mathcal{B}$ is $\mathcal{G}_2$-ϵ_2-multiplicative and c.p.c., by the definitions of $\mathcal{G}_2$ and ϵ_2 and by Lemma 2.3, there exists a projection $r \in \mathcal{B}$ and there exists a *-homomorphism $h : C(X) \to \mathcal{B}$ with $h(1_{C(X)}) = r$ such that

$$\nu(1_{\mathcal{B}} - r) < \epsilon_1$$

for all $\nu \in T(\mathcal{B})$, and

$$\|(1_{\mathcal{B}} - r)L'(f)(1_{\mathcal{B}} - r) + h(f) - L'(f)\| < \epsilon_1$$

for all $f \in \mathcal{G}_1$.

Note that from the above, for all $\tau \in T(\mathcal{A})$,

$$\tau(1_{\mathcal{B}} - r) < \epsilon_1\tau(1_{\mathcal{B}}) < \epsilon_1.$$

Moreover, we have that the map $L'' : C(X) \to (1_{\mathcal{B}} - r)\mathcal{B}(1_{\mathcal{B}} - r) : f \mapsto (1_{\mathcal{B}} - r)L'(f)(1_{\mathcal{B}} - r)$ is a c.p.c. $\mathcal{G}_1$-ϵ_1-multiplicative map.

Let $L_1 = L + L'' : C(X) \to (1_{\mathcal{A}} - r)\mathcal{A}(1_{\mathcal{A}} - r)$. Hence, L_1 is a unital c.p.c. $\mathcal{G}_1$-ϵ_1-multiplicative map. (Recall that $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and $\epsilon_2 < \epsilon_1$.)

Also,

$$\|L_1(f) + h(f) - \phi(f)\| < \epsilon_1 + \epsilon_2 + \epsilon_3 < 3\epsilon_1 \leq \frac{3\epsilon}{10(N+1)}$$

for all $f \in \mathcal{G}_1$ and

$$\tau(L_1(1_{C(X)})) = \tau(1_{\mathcal{A}} - r) = \tau(1_{\mathcal{A}} - 1_{\mathcal{B}}) + \tau(1_{\mathcal{B}} - r) < \epsilon_3 + \epsilon_1 < 2\epsilon_1 \leq \frac{2\epsilon}{10(N+1)}$$

for all $\tau \in T(\mathcal{A})$. We let

$$p =_{df} L_1(1_{C(X)}).$$

Now, for all $1 \leq j \leq n$, for all $\tau \in T(\mathcal{A})$,

$$\begin{aligned}\tau(h(f_j)) &= (\tau(h(f_j)) + \tau(L_1(f_j))) - \tau(L_1(f_j)) \\ &> \tau(\phi(f_j)) - 3\epsilon_1 - \tau(L_1(f_j)) \\ &> \tau(\phi(f_j)) - 5\epsilon_1 \\ &> \sigma\eta - 5\epsilon_1 \text{ (by (2.5))} \\ &> 2N\epsilon_1 \text{ (by the definition of } \epsilon_1) \\ &> N\tau(L_1(1_{C(X)})).\end{aligned}$$

Let $r_1, r_2, ..., r_m \in \mathcal{B}$ be projections and $y_1, y_2, ..., y_m \in X$ be such that for all $f \in \mathcal{G}_1$,

$$h(f) = \sum_{k=1}^{m} f(y_k) r_k.$$

Recall that for all $1 \leq j \leq n$, $f_j \in \mathcal{G}_1$. Hence, for all $1 \leq j \leq n$, for all $\tau \in T(\mathcal{A})$,

$$\sum_{k=1}^{m} f_j(y_k)\tau(r_k) > N\tau(L_1(1_{C(X)})) > 0.$$

For all $1 \leq j \leq n$, let

$$S_j =_{df} \{k : y_k \in O_j\} = \{k : dist(y_k, x_j) < \eta/s\} \supseteq \{k : f_j(y_k) > 0\}.$$

Let $p_j \in \mathcal{A}$ be the projection given by

$$p_j =_{df} \sum_{k \in S_j} r_k.$$

Also, let

$$t =_{df} 1_{\mathcal{B}} - \sum_{j=1}^{n} p_j,$$

and let $h_1 : C(X) \to \mathcal{B}$ be the *-homomorphism

$$h_1(f) =_{df} th(f)t$$

for all $f \in C(X)$.

Finally, let $p_0 =_{df} p$.

Then we have that

$$1_{\mathcal{A}} = t + \sum_{j=0}^{n} p_j,$$

$$\|L_1(f) + \sum_{j=1}^{n} f(x_j)p_j + h_1(f) - \phi(f)\| < \frac{3\epsilon}{10} + \frac{\epsilon}{8} < \epsilon,$$

for all $f \in \mathcal{F}$ (recall the definition of η in the hypotheses), and

$$\tau(p_j) > N\tau(L_1(1_{C(X)})) = N\tau(p)$$

for all $1 \leq j \leq n$ and for all $\tau \in T(\mathcal{A})$. $\square$

The next result is from [14] Theorem 4.6 (see also [6]).

Theorem 2.6. *Let X be a compact metric space, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. Let $\nu > 0$ be such that $|f(x) - f(y)| < \epsilon/8$ if $dist(x, y) < \nu$ for all $f \in \mathcal{F}$ and all $x, y \in X$. Then for any $s \geq 1$, any finite $\nu/2$-dense subset $\{x_1, x_2, ..., x_m\}$ of X for which $O_j \cap O_k = \emptyset$ for $j \neq k$, where*

$$O_k =_{df} \{x \in X : dist(x, x_k) < \nu/(2s)\},$$

and any $1/(2s) > \sigma > 0$, there exist $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$ and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ satisfying the following:

For any unital separable simple C-algebra $\mathcal{A}$ with tracial rank zero, and $\mathcal{G}$-δ-multiplicative unital c.p.c. maps $\phi, \psi : C(X) \to \mathcal{A}$ with $\tau \circ \phi(g)$ within γ of $\tau \circ \psi(g)$ for all $g \in \mathcal{G}$ and for all $\tau \in T(\mathcal{A})$, if*

i. $\mu_{\tau\circ\phi}(O_k), \mu_{\tau\circ\psi}(O_k) > \sigma\nu$ *for all k and for all $\tau \in T(\mathcal{A})$, and*
ii. $[\phi]|\mathcal{P} = [\psi]|\mathcal{P}$

then there exists a unitary $u \in \mathcal{A}$ such that $u\phi(f)u^*$ is within ϵ of $\psi(f)$ for all $f \in \mathcal{F}$.

Finally, if, in the above, the elements of $\mathcal{F}$ all have norm less than or equal to one, then we can choose $\mathcal{G}$ so that its elements all have norm less than or equal to one.

The next stable uniqueness theorem can be found in [5] Theorem 3.1 (see also [5] Remark 1.1, [4] and [14]).

Theorem 2.7. *Let X be a compact metric space. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$, $\eta > 0$, and integer $N \geq 1$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ satisfying the following:*

For any unital separable simple C-algebra $\mathcal{A}$ with real rank zero, stable rank one and weakly unperforated K_0 group, for any η-dense subset $\{x_1, x_2, ..., x_m\}$ in X, and any $\mathcal{G}$-δ-multiplicative c.p.c. maps $\phi, \psi : C(X) \to \mathcal{A}$, if*

$$[\phi]|\mathcal{P} = [\psi]|\mathcal{P}$$

then there exists a unitary $u \in \mathbb{M}_{Nm+1}(\mathcal{A})$ such that

$$u(\phi(f) \oplus f(x_1)1_N \oplus f(x_2)1_N \oplus ... \oplus f(x_m)1_N)u^* \approx_\epsilon \psi(f) \oplus f(x_1)1_N \oplus f(x_2)1_N \oplus ... \oplus f(x_m)1_N$$

for all $f \in \mathcal{F}$.

The next result follows from [13] Theorem 6.2.9.

Theorem 2.8. *Let X be a finite CW complex, and $\mathcal{A}$ a unital simple AH algebra with bounded dimension growth and real rank zero.*

Let $\mathcal{P} \subset \mathbb{P}(C(X))$ be a finite subset and $\alpha \in KL(C(X), \mathcal{A})$ such that $\alpha|_{K_0(C(X))}$ is positive and $\alpha([1_{C(X)}]) = [1_{\mathcal{A}}]$ (with equivalence classes in K_0).

Then there exists a sequence of unital c.p.c. maps $L_n : C(X) \to \mathcal{A}$ such that

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

and

$$\|L_n(f)L_n(g) - L_n(fg)\| \to 0$$

as $n \to \infty$, for all $f, g \in C(X)$.

Lemma 2.9. *Let X be a finite CW-complex, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ a finite subset.*

Then there exist $\nu_1, \nu_2 > 0$ *such that for all finite* $\nu_1/2$*-dense subset* $\{x_1, x_2, ..., x_m\} \subset X$*, for all finite* $\nu_2/2$*-dense subset* $\{y_1, y_2, ..., y_n\} \subset X$*, for all integers* $s, s' \geq 3$ *for which*

$$\overline{O^0_j} \cap \overline{O^0_k} = \emptyset$$

for all $j \neq k$*, and*

$$\overline{O^1_j} \cap \overline{O^1_k} = \emptyset$$

for all $j \neq k$ *(here,* $O^0_j =_{df} B(x_j, \nu_1/s)$ *and* $O^1_k =_{df} B(y_k, \nu_2/s')$*), for all* $\sigma, \sigma' > 0$ *with*

$$\frac{1}{4s} > \sigma > 0$$

and

$$\frac{1}{2s'} > \sigma' > 0,$$

there exist $\delta > 0$*,* $\gamma > 0$*, a finite subset* $\mathcal{G} \subset C(X)$*, a finite subset* $\mathcal{G}' \subset C(X)$ *and a finite subset* $\mathcal{P} \subset \mathbb{P}(C(X))$ *such that the following holds:*

For every unital separable simple finite real rank zero $\mathcal{Z}$*-stable C*-algebra* $\mathcal{A}$*, for all unital c.p.c.* $\mathcal{G}$*-*δ*-multiplicative maps* $\phi, \psi : C(X) \to \mathcal{A}$*,*

if

$$[\phi]|\mathcal{P} = [\psi]|\mathcal{P},$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \gamma$$

for all $g \in \mathcal{G}'$ *and for all* $\tau \in T(\mathcal{A})$*,*

$$\mu_{\tau\circ\phi}(O^2_j), \mu_{\tau\circ\psi}(O^2_j) > 2\sigma\nu_1$$

and

$$\mu_{\tau\circ\phi}(O^3_k), \mu_{\tau\circ\psi}(O^3_k) > \sigma'\nu_2$$

for all j, k*, for all* $\tau \in T(\mathcal{A})$ *(here,* $O^2_j =_{df} B(x_j, \frac{\nu_1}{2(s+2)})$ *and* $O^3_k =_{df} B(y_k, \frac{\nu_2}{2s'})$*),*

then there exists a unitary $u \in \mathcal{A}$ *such that*

$$u\phi(f)u^* \approx_\epsilon \psi(f)$$

for all $f \in \mathcal{F}$*.*

Proof. Let X be a finite CW complex, $\mathcal{F} \subset C(X)$ a finite subset and $\epsilon > 0$ be given. Let d be the metric for X. We may assume that all the elements of $\mathcal{F}$ have norm less than or equal to one. We may also assume that $1_{C(X)} \in \mathcal{F}$.

Let $\nu_1 > 0$ be such that for all $x, y \in X$, if $d(x, y) < 2\nu_1$ then for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \epsilon/80$. Let $s \geq 3$ be an arbitrary integer. Let $\{x_1, x_2, ..., x_m\} \subset X$ be a $\nu_1/2$-dense subset of X for which

$$\overline{O^a_j} \cap \overline{O^a_k} = \emptyset$$

for all $j \neq k$ and let $\frac{1}{4s} > \sigma > 0$ be arbitrary. (In the above, $O^a_j =_{df} B(x_j, \nu_1/s) = \{x \in X : d(x, x_j) < \nu_1/s\}$ for all j.)

Plug X, $\mathcal{F}$, $\epsilon/10$, ν_1, $\{x_1, x_2, ..., x_m\}$, σ and s into Theorem 2.6 to get $\gamma_1 > 0$, a finite subset $\mathcal{G}_1 \subset C(X)$, $\delta_1 > 0$ and a finite subset $\mathcal{P}_1 \subset \mathbb{P}(C(X))$.

We may assume that all the elements of $\mathcal{G}_1$ have norm less than or equal to one. We may also assume that $\delta_1 < \epsilon/10$, $\gamma_1 < \epsilon/10$ and $\mathcal{F} \subset \mathcal{G}_1$.

We denote the above statements by "(*)".

Let $\delta_2 > 0$ and a finite subset $\mathcal{G}_2 \subset C(X)$ be such that for every unital C*-algebra $\mathcal{C}$, for all unital c.p.c. $\mathcal{G}_2$-δ_2-multiplicative maps $\alpha_1, \alpha_2 : C(X) \to \mathcal{C}$, if $\|\alpha_1(f) - \alpha_2(f)\| < \delta_2$ for all $f \in \mathcal{G}_2$ and $\mu_{\tau \circ \alpha_1}(O^b_j) > \frac{5}{4}\sigma\nu_1$ for all $\tau \in T(\mathcal{C})$ and for all j,

then α_1, α_2 are well-defined on $\mathcal{P}_1$,

$$[\alpha_1]|\mathcal{P}_1 = [\alpha_2]|\mathcal{P}_1$$

$$|\tau \circ \alpha_1(g) - \tau \circ \alpha_2(g)| < \frac{\gamma_1}{10}$$

for all $g \in \mathcal{G}_1$ and

$$\mu_{\tau \circ \alpha_1}(O^a_j), \mu_{\tau \circ \alpha_2}(O^a_j) > \sigma\nu_1$$

for all $\tau \in T(\mathcal{C})$ and for all j. (Here, $O^b_j =_{df} B(x_j, \frac{\nu_1}{2(s+1)}) = \{x \in X : d(x, x_j) < \frac{\nu_1}{2(s+1)}\}$.)

We may assume that $\delta_2 < \frac{\delta_1}{10} < \frac{\epsilon}{100}$, $(\mathcal{F} \subseteq)\mathcal{G}_1 \subseteq \mathcal{G}_2$, and all the elements of $\mathcal{G}_2$ have norm less than or equal to one.

We denote the above statements by "(**)".

Plug X, $\delta_2/10$ and $\mathcal{G}_2$ into Theorem 2.7 to get $\delta_3 > 0$, $\eta_3 > 0$, an integer $N_1 \geq 1$, a finite subset $\mathcal{G}_3 \subset C(X)$, and a finite subset $\mathcal{P}_3 \subset \mathbb{P}(C(X))$.

Expanding $\mathcal{P}_3$ if necessary, we may assume that for all unital separable simple real rank zero C*-algebra $\mathcal{D}$ with weak unperforation and stable rank one, if a c.p.c. map $\alpha_1 : C(X) \to \mathcal{D}$ is multiplicative enough to induce a well-defined map from $\mathcal{P}_3$ into $\underline{K}(\mathcal{D})$ then α_1 induces a well-defined element $[\alpha_1] \in KL(C(X), \mathcal{D})$ and $[\alpha_1]|_{K_0(C(X))}$ is positive. (See the discussion in [9] page 1274.)

We may also assume that $\delta_3 < \frac{\delta_2}{10} < \frac{\epsilon}{1000}$, $\eta_3 < \frac{\epsilon}{10}$, $\mathcal{G}_2 \subseteq \mathcal{G}_3$, $\mathcal{P}_1 \subseteq \mathcal{P}_3$, and all the elements of $\mathcal{G}_3$ have norm less than or equal to one.

We denote the above statements by "(***)".

Let $\delta_4 > 0$ and $\mathcal{G}_4 \subset C(X)$ a finite subset be such that for every unital C*-algebra $\mathcal{C}$,

if $\beta_1, \beta_2 : C(X) \to \mathcal{C}$ are unital c.p.c. $\mathcal{G}_4$-δ_4-multiplicative maps for which

$$\|\beta_1(f) - \beta_2(f)\| < \delta_4$$

for all $f \in \mathcal{G}_4$ and

$$\mu_{\tau \circ \beta_1}(O^c_j), \mu_{\tau \circ \beta_2}(O^c_j) > 2\sigma\nu_1$$

and for all j (where $O^c_j =_{df} B(x_j, \frac{\nu_1}{2(s+2)}) = \{x \in X : d(x, x_j) < \frac{\nu_1}{2(s+2)}\}$),

then β_1 and β_2 are well-defined on $\mathcal{P}_3$,

$$[\beta_1]|\mathcal{P}_3 = [\beta_2]|\mathcal{P}_3,$$

$$|\tau \circ \beta_1(f) - \tau \circ \beta_2(f)| < \frac{\gamma_1}{10}$$

for all $f \in \mathcal{G}_3$, and

$$\mu_{\tau \circ \beta_1}(O^b_j), \mu_{\tau \circ \beta_2}(O^b_j) > \frac{3}{2}\sigma\nu_1$$

for all $\tau \in T(\mathcal{C})$ and for all j. We may assume that $\delta_4 < \frac{\delta_3}{10} < \frac{\epsilon}{10000}$, $\mathcal{F} \subseteq \mathcal{G}_3 \subseteq \mathcal{G}_4$, and that the elements of $\mathcal{G}_4$ all have norm less than or equal to one.

We denote the above statements by "(****)".

Let $\nu_2 > 0$ be such that $\nu_2 < \eta_3$ and for all $f \in \mathcal{G}_4$, for all $x, y \in X$, if $d(x, y) < 2\nu_2$ then $|f(x) - f(y)| < \epsilon/80$. Let $s' \geq 3$ and let $\{y_1, y_2, ..., y_n\} \subset X$ be a $\frac{\nu_2}{2}$-dense subset such that $\overline{O^d_j} \cap \overline{O^d_k} = \emptyset$ for all $j \neq k$ where $O^d_j =_{df} B(y_j, \frac{\nu_2}{s'})$.

Let $\frac{1}{2s'} > \sigma' > 0$ be arbitrary.

Plug $X, \delta_4, \mathcal{G}_4, N_1, \nu_2, \{y_1, y_2, ..., y_n\}, s'$ and σ' into Proposition 2.4 to get $\mathcal{G}_5$ and δ_5. We may assume that $\delta_5 < \frac{\delta_4}{10} < \frac{\epsilon}{100000}$, $\mathcal{G}_4 \subseteq \mathcal{G}_5$ and all the elements of $\mathcal{G}_5$ have norm less than or equal to one.

We denote the above statements by "(+)".

In the notation of the statement of Lemma 2.9, we take $\delta = \delta_5$, $\gamma = \frac{\gamma_1}{10}$, $\mathcal{G} = \mathcal{G}_5$, $\mathcal{G}' = \mathcal{G}_1$ and $\mathcal{P} = \mathcal{P}_3$.

Now suppose that $\mathcal{A}$ is an arbitrary unital simple separable finite real rank zero $\mathcal{Z}$-stable C*-algebra, and suppose that

$$\phi, \psi : C(X) \to \mathcal{A}$$

are both unital, c.p.c., $\mathcal{G}_5$-δ_5-multiplicative maps such that

$$[\phi]|\mathcal{P}_3 = [\psi]|\mathcal{P}_3,$$

$$|\tau \circ \phi(f) - \tau \circ \psi(f)| < \frac{\gamma_1}{10}$$

for all $f \in \mathcal{G}_1$,

$$\mu_{\tau\circ\phi}(O^e_j), \mu_{\tau\circ\psi}(O^e_j) > \sigma'\nu_2$$

for all $\tau \in T(\mathcal{A})$ and for all j (where $O^e_j =_{df} B(y_j, \frac{\nu_2}{2s'})$), and

$$\mu_{\tau\circ\phi}(O^c_j), \mu_{\tau\circ\psi}(O^c_j) > 2\sigma\nu_1$$

for all j and for all $\tau \in T(\mathcal{A})$.

Let us denote the above statements by "(++)".

By (++) and by Proposition 2.4, there exist projections $p, q \in \mathcal{A}$ and $\mathcal{G}_4$-δ_4-multiplicative c.p.c. maps

$$L_\phi : C(X) \to p\mathcal{A}p$$

and

$$L_\psi : C(X) \to q\mathcal{A}q$$

such that the following statements are true:

(1) There exist pairwise orthogonal projections $p_0, p_1, ..., p_n, t \in \mathcal{A}$ and pairwise orthogonal projections $q_0, q_1, ..., q_n, t_1 \in \mathcal{A}$ with $p_0 = p$, $q_0 = q$, $\sum_{j=0}^n p_j + t = 1_\mathcal{A} = \sum_{j=0}^n q_j + t_1$, $N_1[p] \leq p_j$ and $N_1[q] \leq q_j$ for all $j \neq 0$.

(2) There exist finite dimensional *-homomorphisms $h_1 : C(X) \to t\mathcal{A}t$ and $h_2 : C(X) \to t_1\mathcal{A}t_1$ such that

$$\phi(f) \approx_{\delta_4} \phi_1(f)$$

and

$$\psi(f) \approx_{\delta_4} \psi_1(f)$$

for all $f \in \mathcal{G}_4$, where

$$\phi_1(f) =_{df} L_\phi(f) + \sum_{j=1}^n f(y_j)p_j + h_1(f)$$

and

$$\psi_1(f) =_{df} L_\psi(f) + \sum_{j=1}^n f(y_j)q_j + h_2(f).$$

Note that $\delta_4 < \epsilon/10000$, and that both ϕ_1 and ψ_1 are c.p.c. $\mathcal{G}_4$-δ_4-multiplicative maps.

We denote the above statements by "(+++)".

By (****), (++) and (+++), we also have that ϕ_1 and ψ_1 are well-defined on $\mathcal{P}_3$,

$$[\phi_1]|\mathcal{P}_3 = [\psi_1]|\mathcal{P}_3,$$

$$|\tau \circ \phi_1(f) - \tau \circ \psi_1(f)| < \frac{3\gamma_1}{10}$$

for $f \in \mathcal{G}_3$, and

$$\mu_{\tau\circ\phi_1}(O_j^b), \mu_{\tau\circ\psi_1}(O_j^b) > \frac{3}{2}\sigma\nu_1$$

for all $\tau \in T(\mathcal{A})$ and for all j.

We denote the above statements by "(++++)".

By [12] Theorem 4.5, there exists a unital simple AH algebra $\mathcal{A}_0$ with bounded dimension growth and real rank zero, and there exists a unital *-homomorphism $\Phi : \mathcal{A}_0 \to \mathcal{A}$ which induces an order isomorphism (which, of course, respects the Bockstein operations) between the full K groups $\underline{K}(\mathcal{A}_0)$ and $\underline{K}(\mathcal{A})$. And since $\mathcal{A}_0$ and $\mathcal{A}$ are both real rank zero, this induces an isomorphism between the tracial simplexes $T(\mathcal{A})$ and $T(\mathcal{A}_0)$. Henceforth, to simplify notation, we view $\mathcal{A}_0$ as a *-subalgebra of $\mathcal{A}$, we view Φ as the inclusion map, we identify $\underline{K}(\mathcal{A}_0)$, $T(\mathcal{A}_0)$ with $\underline{K}(\mathcal{A})$, $T(\mathcal{A})$, and we take the induced maps $\underline{K}(\Phi)$, $T(\Phi)$ to be identity maps.

Let $\{p'_j\}_{j=0}^n \cup \{t'\}$ be pairwise orthogonal projections in $\mathcal{A}_0$, and let $\{q'_j\}_{j=0}^n \cup \{t'_1\}$ be pairwise orthogonal projections in $\mathcal{A}_0$ such that $p'_j \sim p_j$, $t' \sim t$, $q'_j \sim q_j$, and $t'_1 \sim t_1$ for all j, and

$$\sum_{j=0}^n p'_j + t' = 1_{\mathcal{A}_0} = 1_{\mathcal{A}} = \sum_{j=0}^n q'_j + t'_1.$$

Let $h_{0,1} : C(X) \to t'\mathcal{A}_0 t'$ and $h_{0,2} : C(X) \to t'_1\mathcal{A}_0 t'_1$ be finite dimensional *-homomorphisms which are unitarily equivalent (in $\mathcal{A}$) to h_1, h_2 respectively.

By Theorem 2.8 and by (***), let $L_{0,\phi} : C(X) \to p'_0\mathcal{A}_0 p'_0$ and $L_{0,\psi} : C(X) \to q'_0\mathcal{A}_0 q'_0$ be unital c.p.c. $\mathcal{G}_3$-δ_3-multiplicative maps such that

$$[L_{0,\phi}]|\mathcal{P}_3 = [L_\phi]|\mathcal{P}_3$$

and

$$[L_{0,\psi}]|\mathcal{P}_3 = [L_\psi]|\mathcal{P}_3.$$

Let $\phi_2, \psi_2 : C(X) \to \mathcal{A}_0$ be the c.p.c. $\mathcal{G}_3$-δ_3-multiplicative maps given by:

$$\phi_2(f) =_{df} L_{0,\phi}(f) + \sum_{j=1}^n f(y_j)p'_j + h_{0,1}(f)$$

and

$$\psi_2(f) =_{df} L_{0,\psi}(f) + \sum_{j=1}^n f(y_j)q'_j + h_{0,2}(f)$$

for all $f \in C(X)$.

Therefore, by (***) and Theorem 2.7, let $u, v \in \mathcal{A}$ be unitaries such that

$$u\phi(f)u^* \approx_{\delta_4} u\phi_1(f)u^* \approx_{\delta_2/10} \phi_2(f)$$

and

$$v\psi(f)v^* \approx_{\delta_4} v\psi_1(f)v^* \approx_{\delta_2/10} \psi_2(f)$$

for all $f \in \mathcal{G}_2$. So

$$u\phi(f)u^* \approx_{\delta_4+\delta_2/10} \phi_2(f)$$

and

$$v\psi(f)v^* \approx_{\delta_4+\delta_2/10} \psi_2(f)$$

for all $f \in \mathcal{G}_2$.

Note that $\delta_4 < \delta_3/10 < \delta_2/100$. So $\delta_4 + \delta_2/10 < 11\delta_2/100$.

Hence, by (**),

$$[\phi_2]|\mathcal{P}_1 = [Ad(u)\phi]|\mathcal{P}_1 = [\phi]|\mathcal{P}_1 = [\psi]|\mathcal{P}_1 = [Ad(v)\psi]|\mathcal{P}_1 = [\psi_2]|\mathcal{P}_1,$$

$$|\tau \circ \phi_2(g) - \tau \circ \phi(g)| < \gamma_1/10$$

and

$$|\tau \circ \psi_2(g) - \tau \circ \psi(g)| < \gamma_1/10$$

for all $\tau \in T(\mathcal{A})$ and for all $g \in \mathcal{G}_1$.

Hence, by (++),

$$|\tau \circ \phi_2(g) - \tau \circ \psi_2(g)| < 3\gamma_1/10$$

for all $\tau \in T(\mathcal{A}_0)$ and for all $g \in \mathcal{G}_1$.

Again, by (++++) and (**),

$$\mu_{\tau\circ\phi_2}(O_j^a), \mu_{\tau\circ\psi_2}(O_j^a) > \sigma\nu_1$$

for all $\tau \in T(\mathcal{A}_0)$ and for all j.

Hence, by (*) and Theorem 2.6, let $w \in \mathcal{A}_0$ be a unitary such that

$$w\phi_2(f)w^* \approx_{\epsilon/10} \psi_2(f)$$

for all $f \in \mathcal{F}$.

Hence,

$$wu\phi(f)u^*w^* \approx_{\epsilon/10+2\delta_4+\delta_2/5} v\psi(f)v^*$$

for all $f \in \mathcal{F}$.

Now $2\delta_4 + \delta_2/5 < 11\delta_2/50 < 11\epsilon/500$.

Hence,

$$v^*wu\phi(f)u^*w^*v \approx_{\epsilon} \psi(f)$$

for all $f \in \mathcal{F}$.

□

Recall our standing hypotheses for this paper: (i.) For all unital simple separable stably finite C*-algebras appearing in this paper, all quasitraces are assumed to be traces. (ii.) All c.p.c. maps between C*-algebras in this paper are assumed to be linear.

Also, for a unital C*-algebra $\mathcal{C}$ and for $\tau \in T(\mathcal{C})$, recall that $d_\tau : \mathcal{C}_+ \to [0, \infty)$ is the map given by $d_\tau(c) =_{df} \limsup_{n\to\infty} \tau(c^{1/n})$ for all $c \in \mathcal{C}_+$.

Theorem 2.10. *Let X be a finite CW complex, $\epsilon > 0$, and $\mathcal{F} \subset C(X)$ a finite subset.*

Then there exists a nonempty finite subset $\mathcal{E} \subset C(X)_+ - \{0\}$ such that for all $\lambda > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, a finite subset $\mathcal{E}' \subset C(X)$, $\gamma > 0$ and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following hold:

For all unital separable simple finite real rank zero $\mathcal{Z}$-stable C-algebras $\mathcal{A}$, for all unital c.p.c. $\mathcal{G}$-δ-multiplicative maps $\phi, \psi : C(X) \to \mathcal{A}$,*

if

$$[\phi]|\mathcal{P} = [\psi]|\mathcal{P},$$

$$|\tau \circ \phi(f) - \tau \circ \psi(f)| < \gamma$$

for all $f \in \mathcal{E}'$, for all $\tau \in T(\mathcal{A})$, and

$$d_\tau(\phi(g)), d_\tau(\psi(g)) > \lambda$$

for all $\tau \in T(\mathcal{A})$ and all $g \in \mathcal{E}$,

then there exists a unitary $u \in \mathcal{A}$ such that

$$u\phi(f)u^* \approx_\epsilon \psi(f)$$

for all $f \in \mathcal{F}$.

Proof. This follows immediately from Lemma 2.9. □

Let X be a compact metric space and let $\mathcal{A}$ be a C*-algebra Recall that a *-homomorphism $\phi : C(X) \to \mathcal{A}$ is said to be *finite dimensional* if there exist $x_1, x_2, ..., x_n \in X$ and pairwise orthogonal projections $p_1, p_2, ..., p_n \in \mathcal{A}$ such that $\phi(f) = \sum_{j=1}^n f(x_j)p_j$ for all $f \in C(X)$.

Definition 2.11. *Let X be a compact metric space, and let $\mathcal{A}$ be a C*-algebra.*

*Let $\mathcal{P} \subseteq \mathbb{P}(C(X))$. Then $\mathfrak{N}_\mathcal{P}$ denotes the set of all maps $\alpha : \mathcal{P} \to \underline{K}(\mathcal{A})$ such that there exists a finite dimensional *-homomorphism $\phi : C(X) \to \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi]|_\mathcal{P} = \alpha$.*

*Let $\mathfrak{N}$ denote the set of $\alpha \in KL(C(X), \mathcal{A})$ such that there exists a finite dimensional *-homomorphism $\phi : C(X) \to \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi] = \alpha$.*

Theorem 2.12. *Let X be a finite CW complex, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ a finite subset.*

Then there exists a nonempty finite subset $\mathcal{E} \subset C(X)_+ - \{0\}$ such that for all $\lambda > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following hold:

For all unital separable simple finite real rank zero $\mathcal{Z}$-stable C-algebras $\mathcal{A}$, for every unital c.p.c. $\mathcal{G}$-δ-multiplicative map $\phi : C(X) \to \mathcal{A}$,*

if $[\phi]|_\mathcal{P} : \mathcal{P} \to \underline{K}(\mathcal{A})$ lies in $\mathfrak{N}_\mathcal{P}$, and

$$d_\tau(\phi(g)) > \lambda$$

for all $\tau \in T(\mathcal{A})$ and all $g \in \mathcal{E}$,

*then there exists a unital *-homomorphism $\psi : C(X) \to \mathcal{A}$ with finite dimensional range such that*

$$\|\phi(f) - \psi(f)\| < \epsilon$$

for all $f \in \mathcal{F}$.

Proof. By considering each path-connected component, we may assume that X is path-connected.

Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given.

Plug $\epsilon/10$, $\mathcal{F}$ into Theorem 2.7 to get δ_1, η_1, $N \geq 1$, $\mathcal{G}_1 \subset C(X)$ (finite subset) and $\mathcal{P} \subset \mathbb{P}(C(X))$ (finite subset).

Choose $\delta_2 > 0$ and a finite subset $\mathcal{G}_2 \subset C(X)$ such that for all unital C*-algebras $\mathcal{C}$, if $\psi_1, \psi_2 : C(X) \to \mathcal{C}$ are unital c.p.c. $\mathcal{G}_2$-δ_2-multiplicative maps such that

$$\psi_1(f) \approx_{\delta_2} \psi_2(f)$$

for all $f \in \mathcal{G}_2$ then

$$[\psi_1]|_{\mathcal{P}} = [\psi_2]|_{\mathcal{P}}.$$

We may assume that $\delta_2 < \frac{1}{100} \min\{\epsilon, \delta_1\}$ and $\mathcal{G}_1 \cup \mathcal{F} \subset \mathcal{G}_2$.

Choose $\eta_2 > 0$ with $\eta_2 < \eta_1$ such that for all $f \in \mathcal{G}_2$, $|f(x) - f(x')| < \delta_2/10$ if $dist(x, x') < 2\eta_2$.

Let $\{x_1, x_2, ..., x_n\} \subset X$ be an $\eta_2/2$-dense subset such that $\overline{B(x_j, \eta_2/4)} \cap \overline{B(x_k, \eta_2/4)} = \emptyset$ for all $j \neq k$.

Let $\mathcal{E} =_{df} \{f_j : 1 \leq j \leq n\}$ where for all j, $f_j \in C(X)$ is such that

$$f_j(x) \begin{cases} > 0 & x \in B(x_j, \eta_2/8) \\ = 0 & x \notin B(x_j, \eta_2/8). \end{cases}$$

Let $\lambda > 0$ be arbitrary.

Choose σ with $1/8 > \sigma > 0$ so that

$$\lambda/2 > \sigma\eta_2.$$

Plug $X, \delta_2, \mathcal{G}_2, N, \eta_2$, and σ into Proposition 2.4 to get δ and $\mathcal{G}$.

Now suppose that $\mathcal{A}$ is a unital separable simple finite real rank zero $\mathcal{Z}$-stable C*-algebra, and suppose that $\phi : C(X) \to \mathcal{A}$ is a c.p.c. $\mathcal{G}$-δ-multiplicative map such that

$$[\phi] \in \mathfrak{N}_{\mathcal{P}}$$

and

$$d_\tau(\phi(f)) > \lambda$$

for all $f \in \mathcal{E}$ and for all $\tau \in T(\mathcal{A})$.

By Proposition 2.4, there exist a projection $p \in \mathcal{A}$ and a unital c.p.c. $\mathcal{G}_2$-δ_2-multiplicative map $L_1 : C(X) \to p\mathcal{A}p$ such that the following statements are true:

1. There exist pairwise orthogonal projections $p_0, p_1, ..., p_n, t \in \mathcal{A}$ with $p_0 = p$, $\sum_{j=0}^n p_j + t = 1_{\mathcal{A}}$ and $Np \preceq p_j$ for all $j \neq 0$.
2. There exists a finite dimensional *-homomorphism $h_1 : C(X) \to t\mathcal{A}t$ such that for all $f \in \mathcal{G}_2$,

$$\phi(f) \approx_{\delta_2} L_1(f) + \sum_{j=1}^n f(x_j)p_j + h_1(f).$$

Let $h_2 : C(X) \to (1-p)\mathcal{A}(1-p)$ be the finite dimensional *-homomorphism given by

$$h_2(f) =_{df} \sum_{j=1}^n f(x_j)p_j + h_1(f)$$

for all $f \in C(X)$.

By the definition of δ_2 and $\mathcal{G}_2$,

$$[\phi]|_{\mathcal{P}} = [L_1 + h_2]|_{\mathcal{P}}.$$

Therefore, since $[\phi]|_{\mathcal{P}} \in \mathfrak{N}_{\mathcal{P}}$, $[L_1 + h_2]|_{\mathcal{P}} \in \mathfrak{N}_{\mathcal{P}}$.

Let $x_0 \in X$ and let $h_3 : C(X) \to \mathcal{A}$ be the finite dimensional *-homomorphism given by $h_3(f) =_{df} f(x_0)1_{\mathcal{A}}$ for all $f \in C(X)$.

Since X is path-connected, $[L_1 + h_2]|_{\mathcal{P}} = [h_3]|_{\mathcal{P}}$. Hence,

$$[L_1]|_{\mathcal{P}} + [h_2]|_{\mathcal{P}} = [ph_3]|_{\mathcal{P}} + [(1-p)h_3]|_{\mathcal{P}}.$$

(Note that $ph_3(f) = f(x_0)p$ and $(1-p)h_3(f) = f(x_0)(1-p)$ for all $f \in C(X)$.)

Clearly, $[h_2] = [(1-p)h_3]$. Hence,

$$[L_1]|_{\mathcal{P}} = [ph_3]|_{\mathcal{P}}.$$

Let $h_4 : C(X) \to \mathcal{A}$ be the finite dimensional unital *-homomorphism given by

$$h_4(f) =_{df} ph_3(f) + h_2(f)$$

for all $f \in C(X)$.

Therefore, by Theorem 2.7, there exists a unitary $u \in \mathcal{A}$ such that

$$L_1(f) + h_2(f) \approx_{\epsilon/10} uh_4(f)u^*$$

for all $f \in \mathcal{F}$. Hence,

$$\phi(f) \approx_{\epsilon} uh_4(f)u^*$$

for all $f \in C(X)$. □

We note that in our context (finite CW complexes), our K theoretic condition is the same as that of [6] and a special case of that of [16]. (See, for example, the discussion in [9] 1.2.) We also note that since X is a finite CW complex, $KK(C(X), \mathcal{A}) = KL(C(X), \mathcal{A})$.

Theorem 2.13. *Let X be a finite CW complex and let $\mathcal{A}$ be a unital simple separable finite $\mathcal{Z}$-stable C*-algebra with real rank zero. Let $\phi : C(X) \to \mathcal{A}$ be a unital *-monomorphism.*

*Then ϕ is the pointwise-norm limit of finite dimensional *-homomorphisms if and only if $[\phi] \in \mathfrak{N}$.*

Proof. This follows immediately from Theorem 2.12. □

References

[1] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of C*-algebras and K-homology.* Ann. of Math. **105** (1977), 265–324.

[2] G. A. Elliott and T. A. Loring, *AF embeddings of $C(\mathbb{T}^2)$ with prescribed K-theory.* J. Funct. Anal. (1995), no. 103, 1–25.

[3] G. Gong, X. Jiang and H. Su, *Obstructions to $\mathcal{Z}$-stability for unital simple C*-algebras.* Canad. Math. Bull. **43** (2000), no. 4, 418–426.

[4] G. Gong and H. Lin, *Classification of homomorphisms from $C(X)$ to simple C*-algebras of real rank zero.* Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 2, 181–206.

[5] G. Gong and H. Lin, *Almost multiplicative morphisms and K-theory.* Internat. J. Math. **11** (2000), 983–1000.

[6] S. Hu, H. Lin and Y. Xue, *Limits of homomorphisms with finite dimensional range.* Internat. J. Math. **16** (2005), no. 7, 807–821.

[7] X. Jiang and H. Su, *On a simple unital projectionless C*-algebras.* Amer. J. Math. **121** (1999), no. 2, 359–413.

[8] H. Lin, *Approximation by normal elements with finite spectra in C*-algebras of real rank zero.* Pacific Journal of Mathematics **173** (1996), no. 2, 443–489.

[9] H. Lin, *Extensions of $C(X)$ by simple C*-algebras of real rank zero.* American Journal of Mathematics **119** (1997), no. 6, 1263–1289.
[10] H. Lin, *Almost commuting self-adjoint matrices and applications,* Fields Inst. Comm. **13**, Amer. Math. Soc., (1997), 193–233.
[11] H. Lin, *Homomorphisms from $C(X)$ into C*-algebras.* Can. J. Math. **49**(5) (1997), 963–1009.
[12] H. Lin, *Embedding and AH algebra into a simple C*-algebra with prescribed KK data.* K-theory **24** (2001), no. 2, 135–156.
[13] H. Lin, An introduction to the classification of amenable C*-algebras, World Scientific Publishing Company, Inc., River Edge, NJ, 2001.
[14] H. Lin, *Classification of homomorphisms and dynamical systems.* Trans. Amer. Math. Soc. **359** (2007), no. 2, 859–895.
[15] H. Lin and Z. Niu, *Homomorphisms into simple $\mathcal{Z}$-stable C*-algebras.* J. Operator Theory **71** (2014), no. 2, 517–569.
[16] J. Liu and Y. Zhang, *Approximated by finite-dimensional homomorphisms into simple C*-algebras with tracial rank one.* Proc. Amer. Math. Soc. **142** (2014), 3941–3952.
[17] P. W. Ng, *Simple real rank zero algebras with locally Hausdorff spectrum.* Proc. Amer. Math. Soc. **134** (2006), no. 8, 2223–2228.
[18] P. W. Ng and W. *Commutative C*-subalgebras of simple stably finite C*-algebras with real rank zero.* Indiana Univ. Math. J. **57** (2008), no. 7, 3209–3239.

Department of Mathematics, University of Louisiana at Lafayette, P. O. Box 43568, Lafayette, Louisiana, 70504–3568, USA
E-mail address: png@louisiana.edu