PURELY INFINITE CORONA ALGEBRAS AND EXTENSIONS

P. W. NG

Abstract. We classify all essential extensions of the form

$$0 \to \mathcal{B} \to \mathcal{D} \to C(X) \to 0$$

where $\mathcal{B}$ is a nonunital simple separable finite real rank zero $\mathcal{Z}$-stable C*-algebra with continuous scale, and where X is a finite CW complex. In fact, we prove that there is a group isomorphism

$$\mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B}).$$

1. Introduction

Motivated by the problem of classifying essentially normal operators on a separable infinite dimensional Hilbert space, Brown, Douglas and Fillmore (BDF) classified all C*-algebra extensions of the form

$$0 \to \mathcal{K} \to \mathcal{D} \to C(X) \to 0$$

where $\mathcal{K}$ is the C*-algebra of compact operators on a separable infinite dimensional Hilbert space, and X is a compact metric space. This was a starting point for much interesting phenomena in operator theory and has led to the rapid development of extension theory with many effective techniques (especially from KK theory) to compute the Ext-group $Ext(\mathcal{A}, \mathcal{B})$.

However, in general, $Ext(\mathcal{A}, \mathcal{B})$ does not capture all unitary equivalence classes of extensions. Among other things, there can be many nonunitarily equivalent trivial extensions, and also, an extension ϕ with $[\phi] = 0$ in $Ext(C(X), \mathcal{B})$ need not be trivial. (For these and other shortcomings, see, for example, [36], [40], and [41].)

One of the implicit reasons for the success of the original BDF Theory is that $\mathbb{B}(l_2)$ and the Calkin algebra $\mathbb{B}(l_2)/\mathcal{K}$ have particularly nice structure. Among other things, $\mathbb{B}(l_2)$ has strict comparison and real rank zero (it is a von Neumann algebra), and $\mathbb{B}(l_2)/\mathcal{K}$ is simple purely infinite. (For example, the BDF–Voiculescu result that roughly speaking says that all essential extensions are absorbing would not be true without the simplicity of $\mathbb{B}(l_2)/\mathcal{K}$.)

It would be nice to find a class of corona algebras which generalize nice features from $\mathbb{B}(l_2)/\mathcal{K}$, with the goal of developing operator theory and extension theory in an agreeable context, among other things generalizing further the theories developed by BDF, Voiculescu and other workers. These ideas were clearly present[1] in the early literature.

Simple purely infinite corona algebras have been completely characterized. Recall that a simple C*-algebra has *continuous scale* if, roughly speaking, it has a sequential approximate identity which is like a "Cauchy sequence". More precisely:

[1]as a proper subset

Definition 1.1. *Let $\mathcal{B}$ be a nonunital separable simple C*-algebra. Then $\mathcal{B}$ has* continuous scale *if $\mathcal{B}$ has an approximate identity $\{e_n\}_{n=1}^{\infty}$ such that $e_{n+1}e_n = e_n$ for all n, and for every $b \in \mathcal{B}_+ - \{0\}$, there exists an $N \geq 1$ such that for all $m > n \geq N$,*

$$e_m - e_n \preceq b.$$

(See, for example, [34]*.)*

In the above, $\preceq$ is a subequivalence relation for positive elements (generalizing Murray–von Neumann subequivalence for projections) given as follows: for a C*-algebra $\mathcal{D}$, for $a, d \in \mathcal{D}_+$, $a \preceq d$ if there exists a sequence $\{x_n\}$ in $\mathcal{D}$ such that $x_n d x_n^* \to a$.

Theorem 1.2. *Let $\mathcal{B}$ be a nonunital separable simple nonelementary C*-algebra. Then the following statements are equivalent:*

(1) *$\mathcal{B}$ has continuous scale.*
(2) *$\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple.*
(3) *$\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple and purely infinite.*

([34], [46]*; see also* [10], [60]*)*

We note that purely infinite simple C*-algebras have real rank zero ([62]). We further note that for a general nonunital separable simple C*-algebra $\mathcal{D}$, an extension of $\mathcal{D}$ by $C(X)$ can often be decomposed in a way where one piece sits inside the minimal ideal of $\mathcal{M}(\mathcal{D})/\mathcal{D}$, and this piece is essentially an extension of a simple continuous scale algebra (e.g., [41]; see also [27]). Thus, simple purely infinite corona algebras are not just a very nice context, but are part of the general picture.[2]

Nonetheless, difficulties still arise that are not present in the case of $\mathbb{B}(l_2)/\mathcal{K}$. For example, for simple continuous scale $\mathcal{B}$, the K-theory of $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}(\mathcal{B})/\mathcal{B}$ can be much more complicated than that of $\mathbb{B}(l_2)$ and $\mathbb{B}(l_2)/\mathcal{K}$. Moreover, in the case where $\mathcal{B}$ is nonstable, we do not have infinite repeats and the powerful tools of the classical theory of absorbing extensions (e.g., [2], [7], [13], [30], [31], [45], [58]) are no longer completely available.

In effect, one needs to develop a type of nonstable absorption theory, where one takes into account the fine structure of the K-theory. Such a theory has previously been considered with definite results (e.g., [36], [40], [41]). The author of the aforementioned results studied the case where the ideal was a simple nonunital continuous scale algebra with real rank zero, stable rank one, strict comparison and unique tracial state. In the present paper, one of the results removes the unique tracial state condition, but with the addition of the highly restrictive condition of Jiang–Su-stability.

As part of the program, we also have results characterizing (not necessarily simple) purely infinite corona algebras. Under mild regularity conditions on a simple C*-algebra $\mathcal{B}$, we have the equivalences: $\mathcal{B}$ has quasicontinuous scale $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})$ has strict comparison $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})$ has finitely many ideals $\Leftrightarrow$ $\mathcal{I}_{min} = \mathcal{I}_{cont}$ $\Leftrightarrow$ $V(\mathcal{M}(\mathcal{B}))$ has finitely many order ideals. We believe that this category is suitable to the development of a definitive and elegant extension theory, and should be the first case before the construction of an even

[2]And thus, also, the nonstable case is important even for understanding the stable case. We further note that nonstabilization has been a key part of some of the most interesting and difficult results in the field. See, for example, [4], [38], [43], [45].

more general theory. Furthermore, all such corona algebras have real rank zero,[3] and many other related and fundamental results have been investigated. (E.g., [26], [27], [28], [33], [50], [53].)

1.1. **Notation.** We end this section with some brief remarks on notation. In the last part, we also spell out some necessary prerequisites for reading this paper.

For a C*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ denotes the multiplier algebra of $\mathcal{B}$. Thus, $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is the corresponding corona algebra.

For each extension

$$0 \to \mathcal{B} \to \mathcal{D} \to \mathcal{C} \to 0$$

(of $\mathcal{B}$ by $\mathcal{C}$)[4], we will work with the corresponding *Busby invariant* which is a *-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$. We will always work with *essential extensions* which is equivalent to requiring that the corresponding Busby invariant be injective; hence, throughout the paper, when we write "extension", we mean essential extension. An extension is unital if the corresponding Busby invariant is a unital map.

Say that $\phi, \psi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two extensions. We say that ϕ and ψ are *unitarily equivalent* (and write $\phi \sim \psi$) if there exists a unitary $u \in \mathcal{M}(\mathcal{B})$ such that

$$\phi(c) = \pi(u)\psi(c)\pi(u)^*$$

for all $c \in \mathcal{C}$. Here, $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map.

$\mathbf{Ext}(\mathcal{C}, \mathcal{B})$ denotes the set of unitary equivalence classes of nonunital extensions of $\mathcal{B}$ by $\mathcal{C}$. If, in addition, $\mathcal{C}$ is unital, $\mathbf{Ext}_u(\mathcal{C}, \mathcal{B})$ is the set of unitary equivalence classes of unital extensions.

For a unital simple C*-algebra $\mathcal{C}$, $T(\mathcal{C})$ denotes the tracial state space of $\mathcal{C}$. If $\mathcal{C}$ is a nonunital simple C*-algebra, $T(\mathcal{C})$ will denote the class of (norm-) lower semicontinuous, densely defined traces which are normalized at a fixed element $e \in \mathcal{C}_+ - \{0\}$, where e is in the Pedersen ideal of $\mathcal{C}$ (of course, for statements in this paper involving $T(\mathcal{C})$, where $\mathcal{C}$ is nonunital, the choice of e will not be relevant). For $\tau \in T(\mathcal{C})$ (where $\mathcal{C}$ is unital or nonunital), for $c \in \mathcal{C}_+$, $d_\tau(c) =_{df} \lim_{n\to\infty} \tau(c^{1/n})$. (Good references are [26] and [27].)

For a C*-algebra $\mathcal{D}$ and for $a, b \in \mathcal{D}_+$, $a \preceq b$ means that there exists a sequence $\{x_n\}$ in $\mathcal{D}$ such that $x_n b x_n \to a$. (This subsequivalence generalizes Murray–von Neumann subsequence for projections.) For $a \in \mathcal{D}_+$, we let $her_{\mathcal{D}}(a) =_{df} \overline{a\mathcal{D}a}$, the hereditary C*-subalgebra of $\mathcal{D}$ generated by a. Sometimes, for simplicity, we write $her(a)$ in place of $her_{\mathcal{D}}(a)$. Similarly, for a C*-subalgebra $\mathcal{C} \subseteq \mathcal{D}$, we let $her_{\mathcal{D}}(\mathcal{C})$ or $her(\mathcal{C})$ denote $\overline{\mathcal{C}\mathcal{D}\mathcal{C}}$, the hereditary C*-subalgebra of $\mathcal{D}$ generated by $\mathcal{C}$. Finally, for a subset $S \subseteq \mathcal{D}$, we let $Ideal_{\mathcal{D}}(S)$ denote the ideal of $\mathcal{D}$ which is generated by S. Again, we often write $Ideal(S)$ in place of $Ideal_{\mathcal{D}}(S)$.

In this paper, any simple separable stably finite C*-algebra is assumed to have the property that every quasitrace is a trace.

[3] We note that real rank zero was a reoccurring, though implicit, theme in the proof of the original BDF index theorem. Moreover, the Kasparov technical lemma, which is a foundation for the construction of the Kasparov product and the important properties of KK, implies that the corona algebra of a σ-unital algebra is an SAW*-algebra, a property with formal similarities to real rank zero.

[4] In the literature, the terminology is sometimes reversed and this is sometimes called an "extension of $\mathcal{C}$ by $\mathcal{B}$". Following Arveson, BDF, Voiculescu and others, we prefer "$\mathcal{B}$ by $\mathcal{C}$".

Throughout this paper, $\mathcal{Z}$ denotes the Jiang–Su algebra ([25]). A C*-algebra $\mathcal{C}$ is said to be *$\mathcal{Z}$-stable* if $\mathcal{C} \otimes \mathcal{Z} \cong \mathcal{C}$.

Let $\mathcal{A}, \mathcal{C}$ be C*-algebras. Throughout this paper, we will write that a map $\phi : \mathcal{A} \to \mathcal{C}$ is *c.p.c.* if it is linear and completely positive contractive. Let $\mathcal{F} \subset \mathcal{A}$ be a finite subset and let $\delta > 0$. A c.p.c. map $\psi : \mathcal{A} \to \mathcal{C}$ is said to be *$\mathcal{F}$-δ-multiplicative* if $\|\psi(fg) - \psi(f)\psi(g)\| < \delta$ for all $f, g \in \mathcal{F}$.

We will be using, without definition or explanation, many notations and results from KK theory and other theories. The reader will be required to be familiar with the references listed below.

Good references for basic multiplier algebra theory, extension theory, K theory, and KK theory are [5], [29], [44], and [59]. See also [26], [27] and [28] for much of the advanced multiplier algebra machinery. We emphasize that we will be extensively using, without definition or explanation, notation and results from [5], [29] and [44].

For the notation and basic KK-theoretic tools (which, again, we will freely use without definition or explanation), we refer the reader to [15], [16], [23], [36], [40], [45], [43], [44], [48], [52], [54], and the references therein. We emphasize, once more, the nonstable aspects of the theory which can be found in, say, [15] as well as other references mentioned above.

References for simple continuous scale algebras are [34] and [46]. Section 1 of [47] contains computations of the K theory for the multiplier and corona algebras of simple separable continuous scale C*-algebras with real rank zero, stable rank one and strict comparison (see also [50] Propositions 4.2, 4.4 and Corollary 4.6; and also [12]). Other good sources are [26] and [27]. We note that simple continuous scale algebras play a key role in recent outstanding breakthroughs (see, for example, [11]).

The reader should also be familiar with [1], [2], [3], [4], [7], [8], [21], [22], [37], [38], [58], [61], [62].

2. Some results in nonstable absorption

This section is a brief exposition of some results from [54]. Precursors to the results in this section are [2], [7], [13], [30], [32], [36], [41], [58]. This section has the flavour of operator theory, especially Halmos' proof of the Weyl–von Neumann–Berg theorem. (See also the historical remark before Definition 2.2.) We note that this is true also for later parts of the paper (e.g., see Proposition 4.2).

Recall, from the end of the first section, that all our extensions are assumed to be essential.

The following definition/lemma is [54] Remark 1 (after Proposition 2.5).

Definition 2.1. ***(And also Lemma.)***

Let $\mathcal{B}$ be a nonunital separable simple continuous scale C-algebra such that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$, and let X be a compact metric space.*

*Then there is an addition on the class of unital extensions of $\mathcal{B}$ by $C(X)$. More precisely, say that $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two *-monomorphisms. Then the* BDF sum *of ϕ and ψ is given by*

$$S\phi(.)S^* + T\psi(.)T^*$$

where $S, T \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ are isometries such that $SS^* + TT^* = 1$. We denote the above sum by $\phi \oplus \psi$.

The above sum is well-defined up to unitary equivalence. Thus, the above sum induces an addition and hence a semigroup structure on $\mathbf{Ext}_u(C(X), \mathcal{B})$ *(and also on* $\mathbf{Ext}(C(X), \mathcal{B})$*).*

The concepts of *null* and *totally trivial* extensions (see 2.2 and 2.5) are due to Lin (e.g., see [36] and [41]), though we have modified the definitions. Early versions of these concepts were already present in [7].

Recall that in the original BDF case, when X is a compact subset of the plane, uniqueness of the trivial element of $Ext(C(X), \mathcal{K})$ essentially follows from the Weyl–von Neumann–Berg Theorem. Recall also that for a simple separable real rank zero C*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ has the classical Weyl–von Neumann Theorem if and only if $\mathcal{M}(\mathcal{B})$ has real rank zero (e.g., [61], [62]; see also [37]). This is perhaps one clue for the reasons for the assumption that $\mathcal{M}(\mathcal{B})$ has real rank zero in some early papers (see, for example, [36], [41]). All this also indicates the operator-theoretic nature of the present study.

Recall, from the end of the first section, that for extensions ϕ and ψ, $\phi \sim \psi$ means that ϕ and ψ are unitarily equivalent. The next definition is for both the unital and nonunital cases:

Definition 2.2. *Let* $\mathcal{B}$ *be a simple nonunital separable continuous scale C*-algebra. Let* X *be a compact metric space and let* $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *be an essential extension.*

(1) ϕ *is said to be* null *if there exists a commutative AF-subalgebra* $\mathcal{C} \subset \mathcal{M}(\mathcal{B})/\mathcal{B}$ *such that* $Ran(\phi) \subseteq \mathcal{C}$ *and* $[p] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$ *for every projection* $p \in \mathcal{C}$.
(2) *If, in addition,* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$*, then* ϕ *is said to be* self-absorbing *if* $\phi \oplus \phi \sim \phi$.

Proposition 2.3. *Let* $\mathcal{B}$ *be a nonunital simple separable C*-algebra with continuous scale and let* X *be a compact metric space. Then we have the following:*

(1) *There exists a null extension* $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$*. Moreover, we can require* ϕ *to be nonunital or unital (if, additionally,* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$*).*

 Suppose, in addition, that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$*. Then we have the following:*

(2) *Every null extension* $C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *is self-absorbing.*
(3) *Any two unital self-absorbing extensions* $C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *are unitarily equivalent. The same holds for any two nonunital self-absorbing extensions.*
(4) *Every self-absorbing extension must be null.*

Proof. This is [54] Theorem 3.4. □

Theorem 2.4. *Let* $\mathcal{B}$ *be a nonunital separable simple continuous scale C*-algebra such that* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$*.*

Let X *be a compact metric space.*

Then $\mathbf{Ext}_u(C(X), \mathcal{B})$ *is a group where the zero element is the class of a null extension. The same holds for* $\mathbf{Ext}(C(X), \mathcal{B})$*.*

Proof. This is [54] Theorem 3.5. □

Definition 2.5. *Let $\mathcal{B}$ be a nonunital separable C*-algebra, and let X be a compact metric space.*

An extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is totally trivial *if there exist a strictly converging properly increasing sequence $\{e_n\}_{n=1}^{\infty}$ of projections in $\mathcal{B}$, and a dense sequence $\{x_n\}_{n=1}^{\infty}$ in X, with each term repeating infinitely many times, such that*

$$\phi = \pi \circ \psi$$

*where $\psi : C(X) \to \mathcal{M}(\mathcal{B})$ is the *-homomorphism given by $\psi(f) =_{df} \sum_{n=1}^{\infty} f(x_n)(e_n - e_{n-1})$, and where $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map. (Here, $e_0 =_{df} 0$.)*

Sometimes, to save writing, we call a *-homomorphism $\psi : C(X) \to \mathcal{M}(\mathcal{B})$ a *totally trivial extension* if it has the form in Definition 2.5 above.

Theorem 2.6. *Let X be a finite CW complex and let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra with real rank zero, stable rank one and weak unperforation.*

Then an extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is null if and only if ϕ is totally trivial and $K_0(\phi) = 0$.

Proof. This is a part of [54] Theorem 4.10. □

3. Aspects of operator theory: KK Theory

In this section, we gather some relevant results which have their origins in BDF theory and closely related phenomena like Lin's important work on almost commuting self-adjoint matrices. This interesting phenomena have had manifold implications including the important uniqueness and stable uniqueness theorems (e.g., see [15], [16], [17], [43], [45], [48]). We also briefly discuss results concerning the complementary problem of stable existence.

As noted at the end of the first section, we will be freely using, without definition or explanation, notation and basic results from standard references on KK theory, especially with regard to parts of the theory concerning existence, uniqueness, absorbing extensions and nonstable aspects of the theory. A key reference is [44]. Other references are [5], [15], [17], [29], [43], [45], [48], and the other references listed at the end of the first section. The reader is assumed to be familiar with the notation and contents of these references.

Finally, recall that one of our standing hypotheses is that for the unital separable simple stably finite C*-algebras discussed in this paper, we always assume that every quasitrace is a trace.

Let X be a compact metric space and let $\mathcal{A}$ be a C*-algebra. Recall that a *-homomorphism $\phi : C(X) \to \mathcal{A}$ is said to be *finite dimensional* if there exist $x_1, x_2, ..., x_n \in X$ and pairwise orthogonal projections $p_1, p_2, ..., p_n \in \mathcal{A}$ such that $\phi(f) = \sum_{j=1}^{n} f(x_j)p_j$ for all $f \in C(X)$. In this case, the *spectrum* $sp(\phi)$, of ϕ, is defined to be $sp(\phi) =_{df} \{x_1, x_2, ..., x_n\}$.

For all $m \geq 1$, let Y_m be the 2-dimensional CW-complex obtained by attaching a 2-cell to S^1 via the degree m map from S^1 to S^1. Let $C_0(Y_m)$ be the C*-algebra of continuous functions on Y_m which vanish at a fixed point $\infty \in Y_m$. Recall that $K_0(C(Y_m)) = \mathbb{Z} \oplus \mathbb{Z}/m$, $K_0(C_0(Y_m)) = \mathbb{Z}/m$, and $K_1(C(Y_m)) = K_1(C_0(Y_m)) = 0$.

Recall that for any unital C*-algebra $\mathcal{C}$, $\mathbb{P}(\mathcal{C})$ is notation for the collection of projections in $\bigcup_{m=1}^{\infty} \mathbb{M}_{\infty}(C(S^1) \otimes C(Y_m) \otimes \mathcal{C})$. Recall that $\underline{K}(\mathcal{C})_+$, the image of

$\mathbb{P}(\mathcal{C})$ in $\underline{K}(\mathcal{C})_+$, is a positive cone for $\underline{K}(\mathcal{C})$. (See, for example, [40] Section 2.1 or [44].)

To simplify notation, for a subset $\mathcal{P} \subseteq \mathbb{P}(\mathcal{C})$, we also often use $\mathcal{P}$ to denote its image in $\underline{K}(\mathcal{C})_+$.

Definition 3.1. *Let X be a compact metric space, and let $\mathcal{A}$ be a C^*-algebra.*

*Let $\mathcal{P} \subseteq \mathbb{P}(C(X))$. Then $\mathfrak{N}_{\mathcal{P}}$ denotes the set of all maps $\alpha : \mathcal{P} \rightarrow \underline{K}(\mathcal{A})$ such that there exists a finite dimensional *-homomorphism $\phi : C(X) \rightarrow \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi]|_{\mathcal{P}} = \alpha$.*

*Let $\mathfrak{N}$ denote the set of $\alpha \in KL(C(X), \mathcal{A})$ such that there exists a finite dimensional *-homomorphism $\phi : C(X) \rightarrow \mathbb{M}_k \otimes \mathcal{A}$ for which $[\phi] = \alpha$.*

Proposition 3.2. *Let X be a finite CW complex, and let $m =_{df} 2dim(X) + 1$. Let $\epsilon > 0$, a finite subset $\mathcal{F} \subset C(X)$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ be given.*

Then there exist a $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ such that the following statement is true:

For every unital C^*-algebra $\mathcal{A}$ and for every unital c.p.c. $\mathcal{G}$-δ-multiplicative map $\phi : C(X) \rightarrow \mathcal{A}$, there exists a unital c.p.c. $\mathcal{F}$-ϵ-multiplicative map $\psi : C(X) \rightarrow \mathbb{M}_m(\mathcal{A})$ such that

$$[\phi \oplus \psi]|_{\mathcal{P}} \in \mathfrak{N}_{\mathcal{P}}.$$

Proof. This follows from [39] Corollary 1.24. □

Theorem 3.3. *Let X be a finite CW complex, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ a finite subset.*

Then there exists a nonempty finite subset $\mathcal{E} \subset C(X)_+ - \{0\}$ such that for all $\lambda > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following hold:

For all unital separable simple finite real rank zero $\mathcal{Z}$-stable C^*-algebras $\mathcal{A}$, for every unital c.p.c.

$\phi : C(X) \rightarrow \mathcal{A}$,

if $[\phi]|_{\mathcal{P}} : \mathcal{P} \rightarrow \underline{K}(\mathcal{A})$ lies in $\mathfrak{N}_{\mathcal{P}}$, and

$$d_\tau(\phi(g)) > \lambda$$

for all $\tau \in T(\mathcal{A})$ and all $g \in \mathcal{E}$,

*then there exists a unital *-homomorphism $\psi : C(X) \rightarrow \mathcal{A}$ with finite dimensional range such that*

$$\|\phi(f) - \psi(f)\| < \epsilon$$

for all $f \in \mathcal{F}$.

Proof. This is [52] Theorem 2.12. □

We next lemma is a standard exercise.

Lemma 3.4. *For every sequence $\{\delta_n\}_{n=1}^\infty$ in $(0, 1)$, there exists a sequence $\{\delta'_n\}_{n=1}^\infty$ such that the following statements hold:*

Say that X is a compact metric space with metric d and $\mathcal{C}$ is a C^*-algebra. Say that $f \in C(X, \mathcal{C}) = C(X) \otimes \mathcal{C}$ is such that

i. *for all $t \in X$, $\|f(t)^* f(t) - 1\| < \frac{1}{10}$ and $\|f(t)f(t)^* - 1\| < \frac{1}{10}$, and*
ii. *for all $n \geq 1$, for all $s, t \in X$, if $d(s, t) < \delta_n$ then $\|f(s) - f(t)\| < \frac{1}{n}$.*

Let $u \in C(X, \mathcal{C})$ be the unitary from the polar decomposition of f.

Then for all $n \geq 1$, for all $s, t \in X$, if $d(s, t) < \delta'_n$, then $\|u(s) - u(t)\| < \frac{1}{n}$.

The next perturbation argument should certainly be well-known, but we nonetheless provide a sketch of the proof.

Lemma 3.5. *For every $\epsilon > 0$, there exists a $\delta > 0$ such that the following statement is true:*

Let X be a compact metric space, $\mathcal{C}$ be a unital C-algebra, and $\mathcal{D} \subseteq \mathcal{C}$ a unital C*-subalgebra. Let $f \in C(X) \otimes \mathcal{C}$ be a unitary such that for all $s \in X$, $f(s)$ is within δ of an element of $\mathcal{D}$.*

Then there exists a unitary $g \in C(X) \otimes \mathcal{D}$ such that $\|f - g\| < \epsilon$.

Sketch of proof. Since X is compact and f is continuous on X, we can find $d_1, ..., d_n \in \mathcal{D}$ and an appropriate partition of unity $\{h_l\}_{l=1}^{\infty}$ ($h_l : X \to [0, 1]$ continuous for all l) such that for all $t \in X$, $\|f(t) - \sum_{l=1}^{n} h_l(t)d_l\| < 2\delta$. For δ small enough, the function $t \mapsto \sum_{l=1}^{n} h_l(t)d_l$ can be perturbed to a continuous unitary-valued map $g : X \to U(\mathcal{D})$ such that $\|f(t) - g(t)\| < \epsilon$ for all $t \in X$. □

Recall that for a C*-algebra $\mathcal{C}$, and for all $m \geq 2$, $K_1(\mathcal{C}; \mathbb{Z}/m) = K_1(C_0(Y_m)\otimes\mathcal{C})$ and $K_1(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m) = K_1(C(Y_m) \otimes \mathcal{C}) = K_1(\mathcal{C}) \oplus K_1(\mathcal{C}; \mathbb{Z}/m)$. (See, for example, [44].)

Lemma 3.6. *Let $\{\mathcal{A}_N\}_{N=1}^{\infty}$ be a sequence of simple, unital, separable, finite, real rank zero, $\mathcal{Z}$-stable C*-algebras.*

Then for all $m \geq 2$,

$$\prod K_1(\mathcal{A}_N; \mathcal{Z} \oplus \mathbb{Z}/m) = K_1(\prod \mathcal{A}_N; \mathcal{Z} \oplus \mathbb{Z}/m).$$

Proof. Firstly, by [17] Corollary 2.1,

$$K_1(\prod \mathcal{A}_N; \mathcal{Z} \oplus \mathbb{Z}/m) \subseteq \prod K_1(\mathcal{A}_N; \mathcal{Z} \oplus \mathbb{Z}/m).$$

Hence, it suffices to prove the reverse inclusion.

Let $x \in \prod K_1(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$ be given. So x has the form $x = \{x_N\}_{N=1}^{\infty}$ where for all $N \geq 1$, $x_N \in K_1(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$.

By [42], for every $N \geq 1$, there is unital C*-subalgebra $\mathcal{C}_N \subseteq \mathcal{A}_N$ such that $\mathcal{C}_N$ is a simple unital separable TAF-algebra, and the inclusion map $\iota : \mathcal{C}_N \hookrightarrow \mathcal{A}_N$ induces a unital ordered group isomorphism

$$[\iota] : (\underline{K}(\mathcal{C}_N), \underline{K}(\mathcal{C}_N)_+, [1_{\mathcal{C}_N}]) \to (\underline{K}(\mathcal{A}_N), \underline{K}(\mathcal{A}_N)_+, [1_{\mathcal{A}_N}]).$$

Hence, $x_N \in K_1(\mathcal{C}_N; \mathcal{Z} \otimes \mathcal{Z}/m)$ for all $N \geq 1$.

But by [16] Lemma 2.9,

$$\prod K_1(\mathcal{C}_N; \mathbb{Z} \oplus \mathbb{Z}/m) = K_1\left(\prod(\mathcal{C}_N \otimes \mathcal{K}); \mathbb{Z} \oplus \mathbb{Z}/m\right).$$

Hence, let $u \in (C(Y_m) \otimes \prod(\mathcal{C}_N \otimes \mathcal{K}))^{\sim}$ be a unitary such that $x = [u]$ in $K_1(\prod(\mathcal{C}_N \otimes \mathcal{K}); \mathbb{Z} \oplus \mathbb{Z}/m)$.

Let d be a metric for Y_m. Viewing u as a continuous map from Y_m into $U((\prod(\mathcal{C}_N \otimes \mathcal{K}))^{\sim}$, let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1)$ such that for all $n \geq 1$ for all $s, t \in Y_m$, if $d(s, t) < \delta_n$ then $\|u(s) - u(t)\| < \frac{1}{n}$.

Plug $\{\delta_n\}_{n=1}^{\infty}$ into Lemma 3.4 to get a sequence $\{\delta'_n\}_{n=1}^{\infty}$. Then plug $\{\delta'_n\}_{n=1}^{\infty}$ into Lemma 3.4 again to get another sequence $\{\delta''_n\}_{n=1}^{\infty}$.

For all $N \geq 1$, let u_N be the image of u in $(C(Y_m) \otimes \mathcal{C}_N \otimes \mathcal{K})^{\sim}$. Again, we view u_N as a continuous function from Y_m to $U((\mathcal{C}_N \otimes \mathcal{K})^{\sim})$.

For all $N \geq 1$, let $p_N \in 1_{C(Y_m)} \otimes \mathcal{C}_N \otimes \mathcal{K}$ be a projection such that $p_N u_N \approx u_N p_N$, $(1 - p_N) u_N (1 - p_N) \approx \alpha_N (1 - p_N)$ (where $\alpha_N \in S^1$ is a scalar), and if $y_N =_{df} p_N u_N p_N$, then $\|y_N^* y_N - p_N\|, \|y_N y_N^* - p_N\| < \frac{1}{10}$. Let $v_N \in p_N(C(Y_m) \otimes \mathcal{C}_N \otimes \mathcal{K}) p_N$ be the unitary in the polar decomposition of y_N. In fact, we want the above approximations to be so close that u_N is homotopy equivalent to $v_N + \alpha_N(1 - p_N)$ in $U((C(Y_m) \otimes \mathcal{C}_N \otimes \mathcal{K})^{\sim})$.

Now for all $n \geq 1$, for all $s, t \in Y_m$, if $d(s,t) < \delta_n$ then $\|y_N(s) - y_N(t)\| < \frac{1}{n}$. Hence, by Lemma 3.4, for all $n \geq 1$, for all $s, t \in Y_m$, if $d(s,t) < \delta'_n$ then $\|v_N(s) - v_N(t)\| < \frac{1}{n}$.

Note that $x_N = [v_N]$ for all N.

For all $N \geq 1$, since $\mathcal{C}_N$ is TAF and since Y_m is compact, let $q_N \in p_N(\mathcal{C}_N \otimes \mathcal{K}) p_N$ be a projection such that the following statements are true:

(a) $q_N \preceq 1_{\mathcal{C}_N}$ in $\mathcal{C}_N \otimes \mathcal{K}$.
(b) $q_N v_N \approx v_N q_N$.
(c) There exists a finite dimension unital C*-subalgebra $\mathcal{D} \subseteq (p_N - q_N)(\mathcal{C}_N \otimes \mathcal{K})(p_N - q_N)$ such that for all $t \in Y_m$, $(p_N - q_N) v_N(t) (p_N - q_N)$ is close to a unitary in $\mathcal{D}$.
(d) If $z_N =_{df} q_N v_N q_N$, then $\|z_N^* z_N - q_N\| < \frac{1}{10}$ and $\|z_N z_N^* - q_N\| < \frac{1}{10}$.

We can choose the above approximations close enough so that the map $t \mapsto (p_N - q_N) v_N(t)(p_N - q_N)$ can be perturbed to a close unitary valued map $X \to (p_N - q_N)(\mathcal{C}_N \otimes \mathcal{K})(p_N - q_N)$ which, upon application of Lemma 3.5, is close to a unitary valued map $v'_N : X \to \mathcal{D}$. Moreover, by choosing the above approximations close enough, if $w_N \in q_N(\mathcal{C}_N \otimes \mathcal{K}) q_N$ is the unitary in the polar decomposition of z_N, then $v'_N \oplus w_N$ is homotopy-equivalent to v_N in $p_N(C(Y_m) \otimes \mathcal{C}_N \otimes \mathcal{K}) p_N$.

We may choose $\mathcal{D}$ so that each summand has rank greater than 10. Hence, since the stable rank of $C(Y_m)$ is two, and since $K_1(C(Y_m)) = 0$, it follows, by [56] Theorem 10.12, that v'_N is homotopic to $p_N - q_N$ in $C(Y_m) \otimes \mathcal{D}$. Hence, v_N is homotopic to $(p_N - q_N) \oplus w_N$ in $p_N(C(Y_m) \otimes \mathcal{C}_N \otimes \mathcal{K}) p_N$.

Clearly, $x_N = [w_N]$. Also, since $q_N \preceq 1_{\mathcal{C}_N}$, by conjugating by an appropriate partial isometry from $\mathcal{C}_N \otimes \mathcal{K}$ if necessary, we may assume $w_N \in \mathcal{C}_N$.

Finally, for all $n \geq 1$, for all $s, t \in Y_m$, for all N, if $d(s,t) < \delta'_n$ then $\|z_N(s) - z_N(t)\| < \frac{1}{n}$. Hence, by Lemma 3.4, for all $n \geq 1$, for all $s, t \in Y_m$, for all N, if $d(s,t) < \delta''_n$ then $\|w_N(s) - w_N(t)\| < \frac{1}{n}$.

Thus, $w = \{w_N \oplus (1_{\mathcal{C}_N} - q_N)\}_{N=1}^{\infty}$ is a unitary in $C(Y_m) \otimes (\prod \mathcal{C}_N) \subseteq C(Y_m) \otimes (\prod \mathcal{A}_N)$ and $x = [w]$. So $x \in K_1(\prod A_N; \mathbb{Z} \oplus \mathbb{Z}/m)$. ☐

Recall that for all $m \geq 1$, $\mathcal{I}_m$ is notation for the (nonunitized) dimension drop algebra $\mathcal{I}_m = \{f \in C[0,1] \otimes \mathbb{M}_m : f(0) = 0 \text{ and } f(1) \in \mathbb{C}\}$, and $\widetilde{\mathcal{I}_m}$ is the unitization of $\mathcal{I}_m$ (i.e., the unitized dimension drop algebra). Recall that for a C*-algebra $\mathcal{C}$ and for $m \geq 2$, $K_*(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m) = KK(\widetilde{\mathcal{I}_m}, C(S^1) \otimes \mathcal{C})$, $K_0(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m) = KK(\widetilde{\mathcal{I}_m}, \mathcal{C})$ and $K_0(\mathcal{C}; \mathbb{Z}/m) = KK(\mathcal{I}_m, \mathcal{C})$. Under the above identification, recall that $K_0(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m)_{++} =_{df} \{[\phi] : \phi : \widetilde{\mathcal{I}_m} \to \mathcal{C} \otimes \mathcal{K} \text{ is a *-homomorphism}\}$, and $K_*(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m)_{++} =_{df} \{[\phi] : \phi : \widetilde{\mathcal{I}_m} \to C(S^1) \otimes \mathcal{C} \otimes \mathcal{K} \text{ is a *-homomorphism}\}$. $\underline{K}(\mathcal{C})_{++}$ is the subsemigroup of $\underline{K}(\mathcal{C})$ generated by $K_*(\mathcal{C})$ and $K_*(\mathcal{C}; \mathbb{Z} \oplus \mathbb{Z}/m)_{++}$ (for all $m \geq 2$). Recall that $\underline{K}_{++}(\mathcal{C})$ is a cone for $\underline{K}(\mathcal{C})$.

For each $x \in \underline{K}(\mathcal{C})$, we denote $x = \{x(j,m)\}_{0 \leq j \leq 1, 0 \leq m < \infty}$, where $x(j,0) \in K_j(\mathcal{C})$, $x(j,1) = 0$ and for all $m \geq 2$, $x(j,m) \in K_j(\mathcal{C}; \mathbb{Z}/m)$.

Recall that if $\mathcal{C}$ is a unital separable simple finite real rank zero $\mathcal{Z}$-stable C*-algebra, and if X is a connected compact metric space, then $C(X) \otimes \mathcal{C} \otimes \mathcal{K}$ has strict comparison of projections by traces in $T(C(X) \otimes \mathcal{C})$ ([57] Corollary 4.10; the hypothesis of exactness is replaced by our standing assumption that all quasitraces are traces).

In general, for a C*-algebra $\mathcal{C}$, $\underline{K}(\mathcal{C})_+$ and $\underline{K}(\mathcal{C})_{++}$ need not coincide. However, this is so under additional hypotheses. More precisely, suppose that $\mathcal{C}$ is a unital, separable, simple, finite, real rank zero and $\mathcal{Z}$-stable C*-algebra. Recall that, in this case,

$$\underline{K}(\mathcal{C})_+ = \underline{K}(\mathcal{C})_{++} = \{0\} \cup \{x \in \underline{K}(\mathcal{C}) : x(0,0) > 0\}.$$

Let $\{\mathcal{A}_N\}_{N=1}^{\infty}$ be a sequence of unital C*-algebras. Let $m \geq 0$. Define $\prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$ to consist of all those $\{x_N\}_{N=1}^{\infty} \in \prod K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$ such that there exists an $L \geq 1$ where for all $N \geq 1$, there exist projections $P_N, Q_N \in \mathbb{M}_L \otimes C(Y_m) \otimes \mathcal{A}_N$ for which $x_N = [P_N] - [Q_N]$ in $K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$.

Lemma 3.7. *Let $\{\mathcal{A}_N\}_{N=1}^{\infty}$ be a sequence of unital, separable, simple, finite, real rank zero, $\mathcal{Z}$-stable C*-algebras.*

Then for all $m \geq 0$,

$$\prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m) = K_0\left(\prod \mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m\right).$$

Proof. The case $m = 0$ is proven in [17] Corollary 2.1. (And by definition, when $m = 1$, all the groups in the statement of the lemma are zero.)

Hence, let $m \geq 2$ be given.

It is not hard to see that

$$K_0\left(\prod \mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m\right) \subseteq \prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m).$$

Hence, it suffices to prove the reverse inclusion.

Hence, let $x \in \prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$ be given. By [42], for all $N \geq 1$, there exists a unital C*-subalgebra $\mathcal{C}_N \subseteq \mathcal{A}_N$ such that $\mathcal{C}$ is a unital simple AH-algebra with bounded dimension growth and real rank zero, and the inclusion map $\iota : \mathcal{C}_N \to \mathcal{A}_N$ induces a unital ordered group isomorphism

$$[\iota] : (\underline{K}(\mathcal{C}_N), \underline{K}(\mathcal{C}_N)_+, [1_{\mathcal{C}_N}]) \to (\underline{K}(\mathcal{A}_N), \underline{K}(\mathcal{A}_N)_+, [1_{\mathcal{A}_N}]).$$

Note that $C(Y_m) \otimes \mathcal{C}_N \otimes \mathcal{K}$ has strict comparison of projections, for all N. Hence, we may assume that

$$x \in \prod_b K_0(\mathcal{C}_N; \mathbb{Z} \oplus \mathbb{Z}/m).$$

We may further assume that x is positive in each component. More precisely, for all $N \geq 1$, let x_N be the projection of x in $K_0(\mathcal{C}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$. We may assume that there exists $L \geq 1$ such that for all $N \geq 1$, there exists a projection $p_N \in \mathbb{M}_L \otimes C(Y_m) \otimes \mathcal{C}_N$ such that $x_N = [p_N]$.

For simplicity, we may assume that $L = 1$.

Fix $N \geq 1$. $\mathcal{C}_N$ can be realized as a C*-inductive limit

$$\mathcal{C}_N = \lim_k (\mathcal{C}_{N,k}, \phi_{k,k+1})$$

where each building block $\mathcal{C}_{N,k}$ can be decomposed as a finite direct with each summand being either of the form

$$r\mathbb{M}_n(C(Y))r$$

where $r \in Proj(\mathbb{M}_n(C(Y)))$ and where Y is a finite CW complex from the list $\{point, S^1, Y_j : j \geq 1\}$, or of the form

$$\mathbb{M}_n \otimes \widetilde{\mathcal{I}}_l$$

for some l. We may assume that the connecting maps $\phi_{k,k+1}$ are injective, and that the matrix sizes n and projection ranks $rank(r)$ get uniformly arbitrarily large as $k \to \infty$ (i.e., as we move up building blocks).

Let $p_N(0,0)$ be the projection of p_N into $\mathcal{C}_N$. (Recall that $p_N \in C(Y_m) \otimes \mathcal{C}_N$.) So $p_N(0,0)$ is a projection in $\mathcal{C}_N$. Let $x_N(0,m)$ be the projection of x_N into $K_0(\mathcal{C}_N, \mathbb{Z}/m)$. So under the identification $K_0(\mathcal{C}_N; \mathbb{Z} \oplus \mathbb{Z}/m) = K_0(\mathcal{C}_N) \oplus K_0(\mathcal{C}_N, \mathbb{Z}/m)$, $x_N = [p_N(0,0)] + x_N(0,m)$. Note that if $[p_N(0,0)] = 0$ then $x_N(0,m) = 0$. Hence, we may assume that $p_N(0,0)$ is a nonzero.

Throwing away initial building blocks if necessary, we may assume that $p_N(0,0) \in \phi_{1,\infty}(\mathcal{C}_{N,1})$. Let $p'_{N,1}$ denote the preimage of $p_N(0,0)$ in $\mathcal{C}_{N,1}$. Throwing away even more initial building blocks, we may assume that $x_N(0,m) \in K_0(\phi_{1,\infty}(\mathcal{C}_{N,1}); \mathbb{Z}/m)$.

Moreover, the ranks of $\phi_{1,k}(p'_{N,1})$ (in each summand of $\mathcal{C}_{N,k}$) get uniformly arbtrarily large as $k \to \infty$. Hence, it is a standard and well known result in the field that we can find a $K \geq 1$ large enough, and we can find a *-homomorphism $\psi_N : \mathcal{I}_m \to \phi_{K,\infty}(\mathcal{C}_{N,K})$ such that $[\psi_N] = x_N(0,m)$ in $KK(\mathcal{I}_m, \phi_{K,\infty}(\mathcal{C}_{N,K}) = K_0(\phi_{K,\infty}(\mathcal{C}_{N,1}); \mathbb{Z}/m)$. Moreover, by increasing K if necessary (so that the ranks of $\phi_{1,K}(p'_{1,N})$, in each summand, get sufficiently uniformly large), and conjugating $p_N(0,0)$ by a unitary from $\mathcal{C}_N$ if necessary, we may assume that for all $f \in \mathcal{I}_m$, $p_N(0,0)\psi_N(f) = \psi_N(f)$.

Let $\widetilde{\psi}_N : \widetilde{\mathcal{I}_m} \to \mathcal{C}_N$ be given by $\widetilde{\psi}_N|_{\mathcal{I}_m} = \psi_N$ and $\widetilde{\psi}_N(1) = p_N(0,0)$.

Then under the identitfication $KK(\widetilde{\mathcal{I}}_m, \mathcal{C}_N) = K_0(\mathcal{C}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$, $[\widetilde{\psi}_N] = x_N$.

Let $\widetilde{\psi} : \widetilde{\mathcal{I}_m} \to \prod \mathcal{C}_N \subseteq \prod \mathcal{A}_N$ be the *-homomorphism

$$\widetilde{\psi} =_{df} \prod \widetilde{\psi}_N.$$

Then $[\widetilde{\psi}] \in KK(\widetilde{\mathcal{I}}_m, \prod \mathcal{A}_N)$ and for all N, the projection of $[\widetilde{\psi}]$ into $KK(\widetilde{\mathcal{I}}_m, \mathcal{A}_N)$ is equal to x_N.

So $x = [\widetilde{\psi}] \in KK(\widetilde{\mathcal{I}}_m, \prod \mathcal{A}_N) = KK(\prod \mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m)$. □

Let X be a finite CW complex. Recall that there exists a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that for any C*-algebra $\mathcal{D}$, if $\alpha, \beta \in KL(C(X), \mathcal{D})$ satisfy that $\alpha|_{\mathcal{P}} = \beta|_{\mathcal{P}}$ then $\alpha = \beta$ in $KL(C(X), \mathcal{D})$.

Lemma 3.8. *Let X be a finite CW complex, $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given.*

Then there exist an $N \geq 1$ and a finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$ such that the following statement is true:

Suppose that $\mathcal{A}$ is a unital separable simple finite real rank zero $\mathcal{Z}$-stable C-algebra, and say that $\alpha \in KK(C(X), \mathcal{A})$ satisfies that*

$$\alpha([1_{C(X)}]) = [1_{\mathcal{A}}]$$

in $K_0(\mathcal{A})$ *and*

$$\alpha([p]) \geq 0$$

for all $p \in \mathcal{P}$.

Then there exist a unital c.p.c. $\mathcal{F}$-ϵ*-multiplicative map* $\phi : C(X) \to \mathbb{M}_{N+1}(\mathcal{A})$ *and a unital finite dimensional *-homomorphism* $\psi : C(X) \to \mathbb{M}_N(\mathcal{A})$ *such that*

$$[\phi] = \alpha + [\psi]$$

in $KK(C(X), \mathcal{A})$.

Proof. Let X be a finite CW complex. We may assume that X is connected. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$ be given.

Suppose, to the contrary, that the conclusion of Lemma 3.8 is false.

Let $\{\mathcal{P}_N\}_{N=1}^{\infty}$ be an increasing sequence of finite subsets of $\mathbb{P}(C(X))$, $\{\mathcal{A}_N\}_{N=1}^{\infty}$ a sequence of unital separable simple finite real rank zero $\mathcal{Z}$-stable C*-algebras, and $\alpha_N \in KK(C(X), \mathcal{A}_N)$ for all $N \geq 1$ such that

$$\alpha_N([1_{C(X)}]) = [1_{\mathcal{A}_N}]$$

in $K_0(\mathcal{A}_N)$ for all $N \geq 1$,

$$\bigcup_{N=1}^{\infty} [\mathcal{P}_N] = \underline{K}(C(X))_+$$

and there are no unital c.p.c. $\mathcal{F}$-ϵ-multiplicative map $\phi' : C(X) \to \mathbb{M}_{N+1}(\mathcal{A}_N)$ and no unital finite dimensional *-homomorphism $\psi' : C(X) \to \mathbb{M}_N(\mathcal{A}_N)$ for which

$$[\phi'] = \alpha_N + [\psi']$$

in $KK(C(X), \mathcal{A}_N)$.
We denote the above statement by "(*)".

Let $\beta : \underline{K}(C(X)) \to \prod_{N=1}^{\infty} \underline{K}(\mathcal{A}_N)$ be the group homomorphism given by

$$\beta =_{df} \prod_{N=1}^{\infty} \alpha_N.$$

Now let $P \in \mathbb{P}(C(X))$ be arbitrary. For simplicity, let us assume that $m \geq 2$ is such that $[P] \in K_0(C(X); \mathbb{Z} \oplus \mathbb{Z}/m) = K_0(C(Y_m) \otimes C(X))$. Choose $M \geq 1$ so that $[P] \leq M[1_{C(Y_m) \otimes C(X)}] = M[1_{C(X)}]$ in $K_0(C(Y_m) \otimes C(X)) = K_0(C(X)) \oplus K_0(C(X); \mathbb{Z}/m)$.

Since $\{\alpha_N\}$ is asymptotically positive, for sufficiently large N, we must have that $\alpha_N([P]) \geq 0$ and $\alpha_N([P]) \leq M\alpha_N([1_{C(X)}]) = M[1_{\mathcal{A}_N}]$. Hence, since $C(Y_m) \otimes \mathcal{A}_N$ has strict comparison for projections for all N, we must have that

$$\beta([P]) \in \prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m).$$

Since P was arbitrary, we must have that

$$Ran(\beta|_{K_0(C(X);\mathbb{Z}\oplus\mathbb{Z}/m)}) \subseteq \prod_b K_0(\mathcal{A}_N; \mathbb{Z} \oplus \mathbb{Z}/m).$$

Hence, by Lemma 3.6 and Lemma 3.7, we have that

$$Ran(\beta) \subseteq \underline{K}\left(\prod \mathcal{A}_N\right).$$

It is a straightforward (tedious) computation to show that β respects the Bockstein operations. Hence,

$$\beta \in Hom_\Lambda\left(\underline{K}(C(X)), \underline{K}\left(\prod \mathcal{A}_N\right)\right).$$

Hence, by [44] Theorem 6.1.11 (see also [43] Theorem 5.9)[5], there is an integer $L \geq 1$, a unital c.p.c. $\mathcal{F}$-$\epsilon/2$-multiplicative map $\Phi : C(X) \to \mathbb{M}_{L+1} \otimes \prod_{N=1}^\infty \mathcal{A}_N$, and a finite dimensional unital *-homomorphism $\Psi : C(X) \to \mathbb{M}_L \otimes \prod_{N=1}^\infty \mathcal{A}_N$ such that

$$[\Phi] = \beta + [\Psi]$$

in $KK(C(X), \prod_{N=1}^\infty \mathcal{A}_N)$. (Recall that X is a finite CW complex.)

We have decompositions

$$\Phi = \prod_{N=1}^\infty \phi_N$$

and

$$\Psi = \prod_{N=1}^\infty \psi_N$$

where for all N, $\phi_N : C(X) \to \mathbb{M}_{L+1} \otimes \mathcal{A}_N$ is a unital c.p.c. $\mathcal{F}$-$\epsilon/2$-multiplicative map, and $\psi_N : C(X) \to \mathbb{M}_L \otimes \mathcal{A}_N$ is a unital finite dimensional *-homomorphism such that

$$[\phi_N] = \alpha_N + [\psi_N]$$

in $KK(C(X), \mathcal{A}_N)$. This contradicts (*). □

The next result is straightforward, but we nonetheless sketch a proof.

Lemma 3.9. *Let X be a connected finite CW complex. Then there exists a finite subset $\mathcal{P}_X \subset Proj(C(X) \otimes \mathcal{K})$ for which the following is true:*

For every finite subset $\mathcal{P} \subset \mathbb{P}(C(X))$, there exists an integer $N \geq 1$ such that for every unital separable simple finite real rank zero $\mathcal{Z}$-stable C-algebra $\mathcal{A}$, for every $\alpha \in KL(C(X), \mathcal{A})$ with $\alpha([1_{C(X)}] = [1_\mathcal{A}]$ in $K_0(\mathcal{A})$ and*

$$\alpha([p]) \geq 0$$

*for all $p \in \mathcal{P}_X$, and for every unital finite dimensional *-homomorphism $\psi : C(X) \to \mathbb{M}_N \otimes \mathcal{A}$, we have that*

$$(\alpha + [\psi])|_\mathcal{P} \geq 0.$$

Sketch of proof. Since X is a finite CW complex, let $\mathcal{F} \subset Proj(C(X) \otimes \mathcal{K})$ be a finite set whose image, in $K_0(C(X))_+$, generates $K_0(C(X))$. Say that $\mathcal{F} = \{p_1, ..., p_K\}$. For all $1 \leq j \leq K$, let $M_j \geq 1$ be such that $p_j \preceq \bigoplus^{M_j} 1_{C(X)}$.

Define $\mathcal{P}_X =_{df} \{1_{C(X)}, p_j, r_j : 1 \leq j \leq K\}$, where for all j, $r_j \in C(X) \otimes \mathcal{K}$ is a projection such that $[r_j] = (M_j + 1)[1_{C(X)}] - [p_j]$.

Say $\mathcal{P} = \{q_1, q_2, ..., q_L\} \subseteq \mathbb{P}(C(X))$. For each $1 \leq j \leq L$, let $q_j(0,0)$ be a projection of q_j into $Proj(C(X) \otimes \mathcal{K})$. (Recall that for all j, q_j is a projection in $C(Y_m) \otimes C(S^1) \otimes C(X) \otimes \mathcal{K}$ for some m dependent on j. By taking a point evaluation, with point in $Y_m \times S^1$, we get a projection in $C(X) \otimes \mathcal{K}$.) Then $[q_j(0,0)]$ is the K_0 piece of $[q_j]$ in $K_*(C(X); \mathbb{Z} \oplus \mathbb{Z}/m) = K_0(C(X)) \oplus K_1(C(X)) \oplus K_*(C(X); \mathbb{Z}/m)$.

[5] The hypothesis of separability of the codomain algebra, in Lin's existence result, can be easily removed.

For all l, let $m_{l,j}$ be integers such that

$$[q_l(0,0)] = \sum_{j=1}^{K} m_{l,j}[p_j].$$

Let $N = \sum_{l=1}^{L} \sum_{j=1}^{K} (M_j + 10)|m_{l,j}| + 1$. We would be done since an element of $\underline{K}(\mathcal{A})$ is positive if and only if it is either zero or its K_0 component is strictly positive. $\square$

Corollary 3.10. *Let X be a finite CW complex. Then there is a finite subset $\mathcal{P}_X \subset Proj(C(X) \otimes \mathcal{K})$ for which the following is true:*

For every $\epsilon > 0$ and finite subset $\mathcal{F} \subset C(X)$, there exists $N \geq 1$ such that for every unital separable simple finite real rank zero $\mathcal{Z}$-stable C-algebra $\mathcal{A}$, and for every $\alpha \in KK(C(X), \mathcal{A})$ such that*

$$\alpha([1_{C(X)}]) = [1_{\mathcal{A}}]$$

in $K_0(\mathcal{A})$ and

$$\alpha([p]) \geq 0$$

*for all $p \in \mathcal{P}_X$, there exists a unital c.p.c. $\mathcal{F}$-ϵ-multiplicative map $\phi : C(X) \to \mathbb{M}_{N+1}(\mathcal{A})$ and a unital finite dimensional *-homomorphism $\psi : C(X) \to \mathbb{M}_N(\mathcal{A})$ such that*

$$[\phi] = \alpha + [\psi]$$

in $KK(C(X), \mathcal{A})$.

Recall that an extension of C*-algebras

$$0 \to \mathcal{B} \to \mathcal{D} \to \mathcal{A} \to 0$$

is said to be *quasidiagonal* if there exists an approximate unit $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{B}$, consisting of an increasing sequence of projections, such that for all $x \in \mathcal{D}$,

$$\|xe_n - e_n x\| \to 0$$

as $n \to \infty$.

Proposition 3.11. *Let X be a compact metric space, and let $\mathcal{B}$ be a nonunital σ-unital simple C*-algebra with real rank zero and continuous scale.*

Suppose that $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is an essential extension such that

$$[\phi] = 0$$

in $KL(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$.

Then ϕ is quasidiagonal.

Proof. This follows from [40] Theorem 1.5. (See also [47] Theorems 7.10 and 7.11.) $\square$

4. A nonstable Brown–Douglas–Fillmore Theorem

We move towards the technical, operator-theoretic argument of Proposition 4.2. This result and its proof has many precursors, including the Weyl–von Neumann theorem and its many generalizations over the years. (The reader is expected to be comfortable with the references in the last paragraph of Section 1.) We expect the proof of Proposition 4.2 to take a few days to read, even for a very well-prepared reader.

Recall, from previous sections, that we will be using much notation and results from KK theory without definition or explanation. (See the references from previous sections, especially those from the end of Section 1.) Recall, also, our standing assumption that for separable simple unital stably finite C*-algebras, we always assume that every quasitrace is a trace.

We would additionally like to remind the reader of the references (which the reader should be familiar with) [26], [27], [34], [46] and [50]. Recall that for a compact convex set K, $Aff(K)$ is the collection of all real-valued affine continuous functions on K. Recall that with the uniform norm and the natural strict order (i.e., the order where f is below g if $f(s) < g(s)$ for all $s \in K$), $Aff(K)$ is an ordered Banach space. We let $LAff(K)$ denote the class of affine lower semicontinuous functions from K to $(-\infty, \infty]$.

Recall also that $Aff(K)_{++}$ ($LAff(K)_{++}$) denotes the functions in $Aff(K)$ (resp. $LAff(K)$) which are strictly positive at every point in K.

Let $\mathcal{B}$ be a nonunital separable stably finite simple C*-algebra. Here, we follow previously mentioned and universally well-established convention by fixing a nonzero positive element of the Pedersen ideal of $\mathcal{B}$ (if $\mathcal{B}$ has real rank zero, a nonzero projection would do) and defining $T(\mathcal{B})$ to be the set of all densely defined, (norm-) lower semicontinuous traces on $\mathcal{B}$ that are normalized at that fixed positive element. In what follows, when $\mathcal{B}$ is nonunital, the choice of that nonzero positive Pedersen ideal element will not be relevant. (When $\mathcal{B}$ is unital, we always take the Pedersen ideal element to be the unit.) It is well-known that $T(\mathcal{B})$, with the topology of pointwise convergence on $Ped(\mathcal{B})$, is a compact convex set. (See [14].)

Suppose that $\mathcal{B}$, as in the previous paragraph (and nonunital), also has real rank zero. Fix an approximate unit $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{B}$, consisting of an increasing sequence of projections. Recall that for every nonzero $A \in \mathcal{M}(\mathcal{B})_+$, A induces an element $\widehat{A} \in LAff(T(\mathcal{B}))_{++}$ which is defined by

$$\widehat{A}(\tau) =_{df} lim_{n \to \infty} \tau(e_n A e_n).$$

(Note that when $\mathcal{B}$ has continuous scale, $\widehat{A} \in Aff(T(\mathcal{B}))_{++}$, i.e., $\widehat{A}$ is continuous.) The above then extends naturally to a map

$$\widehat{.} : (\mathbb{M}_n \otimes \mathcal{M}(\mathcal{B}))_+ \to LAff(T(\mathcal{B}))_{++} \cup \{0\}$$

for all n.

Recall that there is an ordered group homomorphism

$$\chi : K_0(\mathcal{B}) \to Aff(T(\mathcal{B}))$$

which is given by

$$\chi([p]) =_{df} \widehat{[p]} =_{df} \widehat{p},$$

for all $[p] \in K_0(\mathcal{B})_+$.

Finally, for all C*-algebras $\mathcal{C}, \mathcal{D}$, for any linear map $\sigma : \mathcal{C} \to \mathcal{D}$, we denote again by σ the natural induced linear map $\mathbb{M}_n \otimes \mathcal{C} \to \mathbb{M}_n \otimes \mathcal{D}$, for all n. And for a nonunital C*-algebra $\mathcal{B}$, recall that $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map.

We remind the reader of the following result, which was essentially proven by Lin in 1991:

Theorem 4.1. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra with real rank zero, stable rank one, and weakly unperforated K_0 group. Then we have the following:*

(1) $(K_0(\mathcal{M}(\mathcal{B})), K_0(\mathcal{M}(\mathcal{B}))_+) = (Aff(T(\mathcal{B})), Aff(T(\mathcal{B}))_{++})$.
(2) *For any two projections* $P, Q \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$, $P \sim Q$ *if and only if* $\tau(P) = \tau(Q)$ *for all* $\tau \in T(\mathcal{B})$.
(3) *For any* $f \in Aff(T(\mathcal{B}))_{++}$, *there exists* $k \geq 1$ *and a projection* $P \in \mathbb{M}_k \otimes \mathcal{M}(\mathcal{B}) - \mathbb{M}_k \otimes \mathcal{B}$ *such that* $\widehat{P} = f$. *Moreover, if* $f(\tau) < \tau(1_{\mathcal{M}(\mathcal{B})})$ *for all* $\tau \in T(\mathcal{B})$, *then we can choose* $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$.
(4) *The six-term exact sequence (for the ideal* $\mathcal{B} \subset \mathcal{M}(\mathcal{B})$*) induces a short exact sequence*
$$0 \to Aff(T(\mathcal{B}))/\chi(K_0(\mathcal{B})) \to K_0(\mathcal{M}(\mathcal{B})/\mathcal{B}) \to K_1(\mathcal{B}) \to 0.$$

Proof. The first three statements were proven in [35]. A more widely available version is [47] Theorem 1.4. (See also [12] and [50].)

The last statement can be found in [47] Corollary 1.5. □

Proposition 4.2. *Let* $\mathcal{B}$ *be a nonunital separable simple finite real rank zero* $\mathcal{Z}$*-stable C*-algebra with continuous scale such that* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$.

Let X *be a finite CW-complex.*

Suppose that
$$\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$$
*is a *-monomorphism such that* $KL(\phi) = 0$.

Then ϕ *is a null extension.*

Proof. Since $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$, there exist projections $P, Q \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ with $P \perp Q$, $P + Q = 1$, and
$$[\pi(P)] = [\pi(Q)] = [1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$$
in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$.

By Theorem 2.4, up to unitary equivalence (with unitary from $\mathcal{M}(\mathcal{B})$) we may decompose ϕ into
$$\phi \cong \sigma \oplus \psi$$
with ψ expressible as
$$\psi = \pi \circ \psi'$$
such that the following is true:

(a) $\sigma : C(X) \to P(\mathcal{M}(\mathcal{B})/\mathcal{B})P$ and $\psi' : C(X) \to Q\mathcal{M}(\mathcal{B})Q$ are *-monomorphisms.
(b) There exist finite dimensional *-homomorphisms $\psi_n : C(X) \to Q\mathcal{B}Q$, with pairwise orthogonal ranges, such that for all $f \in C(X)$,
$$\psi'(f) = \sum_{n=1}^{\infty} \psi_n(f)$$
where the sum converges strictly.
(c) $sp(\psi_n) \subset sp(\psi_{n+1})$ for all n.
(d) $X = \overline{\bigcup_{n=1}^{\infty} sp(\psi_n)}$.
(e) $KL(\psi) = 0$.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of strictly positive real numbers such that
$$\epsilon_n \to 0.$$

Let $\mathcal{F} \subset C(X)$ be a finite subset which generates $C(X)$ as a C*-algebra. We may assume that for any C*-algebra $\mathcal{C}$, any c.p.c. $\mathcal{F}$-ϵ_1-multiplicative map $C(X) \to \mathcal{C}$

induces an element of $KL(C(X), \mathcal{C})$ which is positive on the set $[\mathcal{P}_X]$ with $\mathcal{P}_X$ from Corollary 3.10.

Note also that, since X is a finite CW complex, we may assume that the set $[\mathcal{P}_X]$ generates $K_0(C(X))$. We may also assume that $1_{C(X)} \in \mathcal{P}_X$.

Replacing each ψ_n by blocks (of the form $\sum_{j=n'}^{m'} \psi_j$) if necessary, we may additionally assume that for all n, $sp(\psi_n)$ is ϵ_n-dense in X.

For simplicity, let us do some relabelling. For all $n \geq 1$, let

$$\psi'_n =_{df} \psi_{2n}$$

and

$$\psi''_n =_{df} \psi_{2n+1}.$$

Note that for all n, each of $sp(\psi'_n)$ and $sp(\psi''_n)$ is ϵ_{2n}-dense in X. Note also that

$$\psi_1(1) \perp \{\psi'_n(1), \psi''_n(1) : n \geq 1\}.$$

For all n, let $p_n \in \psi''_n(1)\mathcal{B}\psi''_n(1)$ be a nonzero projection such that

$$p_n \preceq p \tag{4.3}$$

for every projection $p \in Ran(\psi''_n)$.

For all n, plug X, ϵ_n and $\mathcal{F}$ into Theorem 3.3 to get a finite subset $\mathcal{E}_n \subset C(X)_+ - \{0\}$. Replacing the ψ'_ns by (pairwise orthogonal) blocks (i.e., finite sums of the form $\sum_{n=n'}^{m'} \psi'_n$) if necessary, we may assume that for all n,

$$d_\tau(\psi'_n(g)) > 0$$

for all $g \in \mathcal{E}_n$ and for all $\tau \in T(\mathcal{B})$. Also, for all n, let $q_n \leq p_n$ be a nonzero proper subprojection such that

$$1000 q_n \preceq p \tag{4.4}$$

for every nonzero projection $p \in Ran(\psi'_n) \cup Ran(\psi'_{n+1})$, and

$$1000 q_n \preceq p_n. \tag{4.5}$$

Let $\psi'''_n : C(X) \to q_n \mathcal{B} q_n$ be a finite dimensional *-homomorphism such that

$$d_\tau(\psi'''_n(g)) > 0$$

for all $g \in \mathcal{E}_n$ and for all $\tau \in T(\mathcal{B})$.

For all n, let $p'_n \in \mathcal{B}$ be a nonzero projection such that

$$p'_n \preceq p \tag{4.6}$$

for every nonzero projection $p \in Ran(\psi'''_n)$. Let $q'_n \leq p_n - q_n$ be a nonzero proper subprojection such that

$$1000 q'_n \preceq p'_n \text{ and } 1000 q'_n \preceq p'_{n+1}. \tag{4.7}$$

Choose $\lambda_n > 0$ so that

$$d_\tau(\psi'_n(g)) > 100\lambda_n \tag{4.8}$$

for all $g \in \mathcal{E}_n$ and for all $\tau \in T(\psi'_n(1)\mathcal{B}\psi'_n(1))$, and

$$d_\tau(\psi'''_n(g)) > 100\lambda_n \tag{4.9}$$

for all $g \in \mathcal{E}_n$ and for all $\tau \in T(\psi'''_n(1)\mathcal{B}\psi'''_n(1))$.

For all n, plug X, ϵ_n, $\mathcal{F}$ and $\mathcal{E}_n$ and λ_n into Theorem 3.3 to get a $\delta_n > 0$ and a finite subset $\mathcal{P}_n \subset \mathbb{P}(C(X))$. (The finite subset of $C(X)$ in the conclusion of

Theorem 3.3 can be taken to be $\mathcal{F}$ again, by making δ_n small enough.) We may assume that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $\epsilon_n > \delta_n > \delta_{n+1}$ for all n.

Let m be the topological dimension of X. For all n, plug X, m, δ_n, $\mathcal{F}$ and $\mathcal{P}_n$ into Proposition 3.2 to get $\delta'_n > 0$. (Again, the finite subset of $C(X)$ in the conclusion of Proposition 3.2 can be taken to be $\mathcal{F}$ again, by making δ'_n small enough.) We may assume that for all n, $\delta_n > \delta'_n > \delta'_{n+1}$.

For all n, plug X, $\mathcal{F}$ and δ'_n into Corollary 3.10 to get an integer $N_n \geq 1$. (Recall our assumption on ϵ_1 concerning $\mathcal{F}$ and $\mathcal{P}_X$.) We may assume that the sequence $\{N_n\}_{n=1}^\infty$ is increasing.

Since $KL(\phi) = KL(\psi) = 0$, it follows that $KL(\sigma) = 0$. Hence, by Proposition 3.11, σ is quasidiagonal. Hence, there exist c.p.c. maps $\sigma_n : C(X) \rightarrow P\mathcal{B}P$, with $\sigma_n(1) \in P\mathcal{B}P$ being projections, and with pairwise orthgonal ranges such that

i. $\sigma_n(fg) - \sigma_n(f)\sigma_n(g) \rightarrow 0$ as $n \rightarrow \infty$, for all $f, g \in C(X)$,
ii. $\sum_{n=1}^\infty \sigma_n(f)$ converges strictly, for all $f \in C(X)$, and
iii. $\sigma(f) = \pi\left(\sum_{n=1}^\infty \sigma_n(f)\right)$, for all $f \in C(X)$.

We now construct σ_0, $\sigma_{0,0}$ and sequences $\{n_k\}_{k=1}^\infty$, $\{\sigma'_k\}_{k=1}^\infty$, $\{\sigma''_k\}_{k=1}^\infty$, $\{\theta_{1,k}\}_{k=1}^\infty$, $\{\theta_{2,k}\}_{k=1}^\infty$, $\{\theta_{3,k}\}_{k=1}^\infty$, $\{\alpha_k\}_{k=0}^\infty$, and $\{u_k\}_{k=1}^\infty$ such that the following statements are true:

(1) $\{n_k\}_{k=1}^\infty$ is a subsequence of the positive integers.
(2) For every $l \geq n_k + 1$, σ_l is $\mathcal{F}$-δ'_{k+1}-multiplicative.
(3)
$$\sum_{l=n_k+1}^{\infty} 100\tau(\sigma_l(1)) < \tau(p)$$
for every nonzero projection $p \in Ran(\psi'_{k+1})$ and for every $\tau \in T(\mathcal{B})$.
(4) $\sigma_0, \sigma_{0,0} : C(X) \rightarrow \psi_1(1)\mathcal{B}\psi_1(1)$ are c.p.c. almost multiplicative maps.
(5) $\sigma'_k, \sigma''_k : C(X) \rightarrow \psi''_k(1)\mathcal{B}\psi''_k(1)$ are c.p.c. maps such that $\sigma'_k(1) \perp \sigma''_k(1) \perp u_k\psi'''_k(1)u_k^* \perp \sigma'_k(1)$, where $u_k \in \mathcal{M}(\mathcal{B})$ is a unitary such that $u_k\psi''_k(1) = \psi''_k(1)u_k$.
(6) $\sigma'_k(1) \oplus \sigma''_k(1) \preceq q'_k$, σ'_k is $\mathcal{F}$-δ'_{k+1}-multiplicative and σ''_k is $\mathcal{F}$-δ_{k+1}-multiplicative.
(7) $[\sigma'_k + \sigma''_k]|_{\mathcal{P}_{k+1}} \in \mathfrak{N}_{\mathcal{P}_{k+1}}$.
(8) $\theta_{1,k}, \theta_{2,k}, \theta_{3,k} : C(X) \rightarrow \mathcal{B}$ are finite dimensional *-homomorphisms.
(9)
$$\theta_{1,1}(1) = \sigma_{0,0}(1) + \sum_{l=1}^{n_1} \sigma_l(1) + \sigma'_1(1) + \psi'_1(1)$$
and
$$\|\theta_{1,1}(f) - (\sigma_{0,0}(f) + \sum_{l=1}^{n_1} \sigma_l(f) + \sigma'_1(f) + \psi'_1(f))\| < \epsilon_1$$
for all $f \in \mathcal{F}$.[6]
(10) For all $k \geq 2$,
$$\theta_{1,k}(1) = \sigma''_{k-1}(1) + \sum_{l=n_{k-1}+1}^{n_k} \sigma_l(1) + \sigma'_k(1) + \psi'_k(1)$$

[6] The role of ψ'_1 in item (9) is to ensure that the "injectivity" condition of Theorem 3.3 is satisfied, and thus allowing for finite dimensional approximation. Similar for ψ'_k in item (10) and $u_k\psi'''_k(.)u_k^*$ in item (11).

and

$$\|\theta_{1,k}(f) - (\sigma''_{k-1}(f) + \sum_{l=n_{k-1}+1}^{n_k} \sigma_l(f) + \sigma'_k(f) + \psi'_k(f))\| < \epsilon_k$$

for all $f \in \mathcal{F}$.[7]

(11) For all $k \geq 1$,

$$\theta_{2,k}(1) = \sigma'_k(1) + \sigma''_k(1) + u_k\psi'''_k(1)u_k^*.$$

(12) For all $k \geq 1$,

$$\|\theta_{2,k}(f) - (\sigma'_k(f) + \sigma''_k(f) + u_k\psi'''_k(f)u_k^*)\| < \epsilon_k$$

for all $f \in \mathcal{F}$.

(13) For all $k \geq 1$, $\theta_{2,k}(1) \perp \theta_{3,k}(1)$ and $\psi''_k(1) = \theta_{2,k}(1) + \theta_{3,k}(1)$.

(14) For all $k \geq 1$, $\|\psi''_k - (\theta_{2,k} + \theta_{3,k})\| < \epsilon_k$.

(15) For all $k \geq 0$, $\alpha_k \in KK(C(X), \mathcal{B})$ and

$$\alpha_k = \alpha_0 - \sum_{l=1}^{n_k} [\sigma_l].$$

Moreover, for all $p \in \mathcal{P}_X$,

$$\widehat{\alpha_k([p])} = \sum_{l=n_k+1}^{\infty} \widehat{\sigma_l([p])}.$$

(Here, we define $n_0 =_{df} 0$.)

(16) $[\sigma_0] - \alpha_0$ is the KK class of a finite dimensional *-homomorphism.

(17) For all $k \geq 1$, $[\sigma'_k] - \alpha_k$ is the KK class of a finite dimensional *-homomorphism.

(18) α_k is asymptotically positive, i.e., for all $p \in \mathbb{P}(C(X))$, $\alpha_k([p]) \geq 0$ for sufficiently large k.[8]

We denote the above statements by (*).

The construction will be by induction on k.

Basis Steps $k = 0, 1$

Since $\{\sigma_k\}_{k=1}^{\infty}$ is asymptotically multiplicative and since X is a finite CW complex, throwing away finitely many initial σ_ks if necessary, we may assume that for all n, σ_n and $\sum_{k=n}^{\infty} \sigma_k$ induce elements of $KL(C(X), \mathcal{B})$ and $KL(C(X), \mathcal{M}(\mathcal{B}))$ respectively. (See for example, [40] Section 2.1.) Moroever, as $n \to \infty$, the induced elements (in KL), $[\sigma_n]$ and $[\sum_{k=n}^{\infty} \sigma_k]$, are asymptotically positive on $(\underline{K}(C(X)), \underline{K}(C(X))_+)$.

[7]This is a key place where the asymptotic multiplicativity of $\{\sigma_l\}$ is used: By Theorem 3.3, the quantities $\sigma''_{k-1} + \sum_{l=n_{k-1}+1}^{n_k} \sigma_l + \sigma'_k + \psi'_k$ have better/closer finite dimensional approximations $(\theta_{1,k})$ as $k \to \infty$, because said quantities are increasingly multiplicative and satisfying other conditions.

[8]Asymptotic positivity is actually stronger than what we need. We actually just need that $\alpha_k([p]) \geq 0$ for all $p \in \mathcal{P}_X$ and for all sufficiently large k. Also, since $\tau(\alpha_k([1_{C(X)}])) \to 0$ uniformly on $T(\mathcal{B})$ ($\tau \in T(\mathcal{B})$), this guarantees, by Corollary 3.10, that we have increasingly multiplicative maps σ'_k (as $k \to \infty$) where we can control the size of $[\sigma'_k(1_{C(X)})]$. This is a key point of the proof.

Note that since $\mathcal{B}$ has continuous scale, the sequence $\sum_{l=n}^{\infty} \widehat{\sigma_l(1)}$ converges to zero uniformly on $T(\mathcal{B})$ as $n \to \infty$. Recall also that $\psi_1(1) \perp \{\psi'_l(1), \psi''_l(1) : l \geq 1\}$. Hence, again throwing away finitely many initial σ_ks if necessary, we may assume that for all l, σ_l is $\mathcal{F}$-δ_1-multiplicative, and that there exists a projection $r \in \mathcal{B}$ with

$$r \perp \{\sigma_k(1), \psi_{k+1}(1) : k \geq 1\}$$

such that

$$100r \preceq p \tag{4.10}$$

for every nonzero projection $p \in Ran(\psi_1) \cup Ran(\psi'_1)$. We may assume that $r \leq \psi_1(1)$.

Throwing away (finitely) more initial terms σ_l if necessary, we may also assume that

$$\sum_{l=1}^{\infty} 100(N_1+1)^2(m+1)\tau(\sigma_l(1)) < \tau(r) \tag{4.11}$$

for all $\tau \in T(\mathcal{B})$. (Use Dini's Theorem.)

Now, let $\alpha_0 \in KL(C(X), \mathcal{B})$ be an element such that $\alpha_0([1]) < [r]$ and for all $p \in \mathcal{P}_X$,

$$\widehat{\alpha_0}([p]) = \sum_{l=1}^{\infty} \widehat{\sigma_l([p])}.$$

(Sketch of straightforward argument: Recall that since $\mathcal{B}$ has continuous scale and since $K_0(\sigma) = 0$, for every $p \in \mathcal{P}_X$,

$$\sum_{l=1}^{\infty} \widehat{\sigma_l([p])} \in \chi(K_0(\mathcal{B})) \subseteq Aff(T(\mathcal{B})).$$

Since $[\mathcal{P}_X]$ generates $K_0(C(X))$, this induces a map from $K_0(C(X))$ to $\chi(K_0(\mathcal{B})) \subseteq Aff(T(\mathcal{B}))$. Since $K_0(C(X))$ is a finitely generated abelian group, there is a decomposition $K_0(C(X)) = \mathbb{F} \oplus \mathbb{T}$, where $\mathbb{F}$ is a free abelian group and $\mathbb{T}$ is a torsion abelian (and hence finite) group. The map from $K_0(C(X))$ to $\chi(K_0(\mathcal{B}))$ has $\mathbb{T}$ in its kernel, and hence, actually is a map $\beta : \mathbb{F} \to \chi(K_0(\mathcal{B}))$. Let $e_1, ..., e_n$ be a basis for $\mathbb{F}$. For each $1 \leq j \leq n$, let $e'_j \in K_0(\mathcal{B})$ be an element such that $\widehat{e'_j} = \beta(e_j)$. We then get a group homomorphism $\beta' : K_0(C(X)) \to K_0(\mathcal{B})$ where $\beta'|_{\mathbb{T}} = 0$ and $\beta'(e_j) = e'_j$ for all j. Using the Universal coefficient theorem, lift β' to $\alpha_0 \in KK(C(X), \mathcal{B})$.)

Note that since $\sum_{l=n}^{\infty} \sigma_l$ is asymptotically multiplicative (as $n \to \infty$), $[\sum_{l=n}^{\infty} \sigma_l]$ is asymptotically positive on $(\underline{K}(C(X)), \underline{K}(C(X))_+)$. Now recall that an element of $\underline{K}(\mathcal{A})$ is positive if and only if it is either zero or has strictly positive K_0 component. Hence, $\alpha_0 - \sum_{l=1}^{n}[\sigma_l]$ is asymptotically positive on $(\underline{K}(C(X)), \underline{K}(C(X))_+)$. (I.e., for all $x \in \underline{K}(C(X))_+$, $\alpha_0(x) - \sum_{l=1}^{n}[\sigma_l](x) \geq 0$ for all sufficiently large n.)

By the definitions of $\mathcal{P}_1$, δ'_1, δ_1, N_1 and r, and by Corollary 3.10 and Proposition 3.2, there exist c.p.c. maps

$$\sigma_0, \sigma_{0,0} : C(X) \to r\mathcal{B}r$$

such that $\sigma_0(1)$ and $\sigma_{0,0}(1)$ are projections, $\sigma_0(1) \perp \sigma_{0,0}(1)$, σ_0 is $\mathcal{F}$-δ'_1-multiplicative, $\sigma_{0,0}$ is $\mathcal{F}$-δ_1-multiplicative, $[\sigma_0]-\alpha_0$ is the class of a finite dimensional *-homomorphism, and $[\sigma_0 \oplus \sigma_{0,0}]|_{\mathcal{P}_1} \in \mathfrak{N}_{\mathcal{P}_1}$.

Now choose $n_1 \geq 1$ so that for all $l \geq n_1 + 1$, σ_l is $\mathcal{F}$-δ_2'-multiplicative,

$$\sum_{l=n_1+1}^{\infty} 100\tau(\sigma_l(1)) < \tau(p)$$

for every nonzero projection $p \in Ran(\psi_2')$ and for all $\tau \in T(\mathcal{B})$, and

$$\sum_{l=n_1+1}^{\infty} 100(N_2+1)^2(m+1)\tau(\sigma_l(1)) < \tau(q_1')$$

for all $\tau \in T(\mathcal{B})$.

Let $\alpha_1 \in KK(C(X), \mathcal{B})$ be given by

$$\alpha_1 =_{df} \alpha_0 - \sum_{l=1}^{n_1} [\sigma_l].$$

Note that for all $p \in \mathcal{P}_X$,

$$\widehat{\alpha_1([p])} = \sum_{l=n_1+1}^{\infty} \widehat{\sigma_l([p])}.$$

In particular,

$$100(N_2+1)^2(m+1)\tau(\alpha_1([1_{C(X)}])) < \tau(q_1')$$

for all $\tau \in T(\mathcal{B})$.

Hence, by the definitions of $\mathcal{P}_2$, δ_2' and δ_2 and by Corollary 3.10 and Proposition 3.2, there exist c.p.c. maps $\sigma_{0,1}', \sigma_{0,1}'' : C(X) \to q_1'\mathcal{B}q_1'$ such that $\sigma_{0,1}'(1) \perp \sigma_{0,1}''(1)$, $\sigma_{0,1}'$ is $\mathcal{F}$-δ_2'-multiplicative, $\sigma_{0,1}''$ is $\mathcal{F}$-δ_2-multiplicative, $[\sigma_{0,1}'] - \alpha_1$ is the class of a finite dimensional $*$-homomorphism, and $[\sigma_{0,1}' \oplus \sigma_{0,1}'']|_{\mathcal{P}_2} \in \mathfrak{N}_{\mathcal{P}_2}$. By our choice of λ_1, q_1' and ψ_1''', it must be the case that

$$d_\tau(\sigma_{0,1}'(g) + \sigma_{0,1}''(g) + \psi_1'''(g)) > \lambda_1$$

for all $g \in \mathcal{E}_1$ and for all $\tau \in T((\sigma_{0,1}'(1) + \sigma_{0,1}''(1) + \psi_1'''(1))\mathcal{B}(\sigma_{0,1}'(1) + \sigma_{0,1}''(1) + \psi_1'''(1)))$. (Note that by (4.6) and (4.7), $1000q_1' \preceq p$ for every nonzero projection p in the range of ψ_1'''. Also, recall equation (4.9).)

Hence, by the definitions of δ_2 and $\mathcal{P}_2$, and by Theorem 3.3, we can find a finite dimensional *-homomorphism $\theta_{0,2,1,} : C(X) \to (q_1 + q_1')\mathcal{B}(q_1 + q_1')$ such that

$$\|\theta_{0,2,1}(f) - (\sigma_{0,1}'(f) + \sigma_{0,1}''(f) + \psi_1'''(f))\| < \epsilon_1$$

for all $f \in \mathcal{F}$.

But since (by equations (4.5) and (4.7)) $1000q_1 \preceq p_1$ and $1000q_1' \preceq p_1' \preceq p_1$, by (4.3), and since $q_1 + q_1' \leq p_1 \leq \psi_1''(1)$, we can find a unitary $u_1 \in \mathcal{M}(\mathcal{B})$ with $u_1\psi_1''(1) = \psi_1''(1)u_1$ and we can find finite-dimensional *-homomorphisms $\theta_{3,1}$: $C(X) \to \psi_1''(1)\mathcal{B}\psi_1''(1)$ such that $u_1\theta_{0,2,1}(1)u_1^* \perp \theta_{3,1}(1)$ and

$$\|\psi_1'' - (u_1\theta_{0,2,1}u_1^* + \theta_{3,1})\| < \epsilon_2.$$

(Recall also that $sp(\psi_1'')$ is ϵ_2-dense in X.)

If we define

$$\sigma_1' =_{df} u_1\sigma_{0,1}'u_1^*,$$
$$\sigma_1'' =_{df} u_1\sigma_{0,1}''u_1^*,$$

and

$$\theta_{2,1} =_{df} u_1\theta_{0,2,1}u_1^*,$$

then

$$\|\theta_{2,1}(f) - (\sigma_1'(f) + \sigma_1''(f) + u_1\psi_1'''(f)u_1^*)\| < \epsilon_1$$

for all $f \in \mathcal{F}$, and

$$\|\psi_1'' - (\theta_{2,1} + \theta_{3,1})\| < \epsilon_2 < \epsilon_1.$$

Now since i. $[\sigma_0] - \alpha_0$ and $[\sigma_1'] - \alpha_1$ are the classes of finite dimensional *-homomorphisms, ii. $[\sigma_0 + \sigma_{0,0}]|_{\mathcal{P}_1} \in \mathfrak{N}_{\mathcal{P}_1}$, and iii. $\alpha_1 = \alpha_0 - \sum_{l=1}^{n_1}[\sigma_l]$, we must have that

$$\left([\sigma_{0,0}] + \sum_{l=1}^{n_1}[\sigma_l] + [\sigma_1']\right)|_{\mathcal{P}_1} \in \mathfrak{N}_{\mathcal{P}_1}.$$

Also, note that by (4.7), (4.6) and (4.4),

$$\sigma_1'(1) \preceq q_1' \preceq 1000q_1' \preceq p_1' \preceq \psi_1'''(1) \leq q_1 \preceq 1000q_1 \preceq \psi_1'(1).$$

Hence, since $\sigma_{0,0}(1) \leq r$, and by equations (4.10), (4.11) and (4.8), we must have that

$$d_\tau\left(\sigma_{0,0}(g) + \sum_{l=1}^{n_1}\sigma_l(g) + \sigma_1'(g) + \psi_1'(g)\right) > \lambda_1$$

for all $g \in \mathcal{E}_1$, for all $\tau \in T\,(her\,(\sigma_{0,0}(1) + \sum_{l=1}^{n_1}\sigma_l(1) + \sigma_1'(1) + \psi_1'(1)))$.

Note that all the above maps are $\mathcal{F}$-δ_1-multiplicative.

Hence, by the definitions of δ_1, λ_1, $\mathcal{E}_1$ and $\mathcal{P}_1$, and by Theorem 3.3, let $\theta_{1,1} : C(X) \to \mathcal{B}$ be a finite dimensional *-homomorphism with

$$\theta_{1,1}(1) = \sigma_{0,0}(1) + \sum_{l=1}^{n_1}\sigma_l(1) + \sigma_1'(1) + \psi_1'(1)$$

such that

$$\left\|\theta_{1,1}(f) - \left(\sigma_{0,0}(f) + \sum_{l=1}^{n_1}\sigma_l(f) + \sigma_1'(f) + \psi_1'(f)\right)\right\| < \epsilon_1$$

for all $f \in \mathcal{F}$.

Induction Step:

The argument of the Induction Step is very similar to that of the Basis Steps. We provide the argument for the convenience of the reader.

Suppose that n_k, σ_k', σ_k'', $\theta_{1,k}$, $\theta_{2,k}$, $\theta_{3,k}$, α_k and u_k have been constructed.

We now construct n_{k+1}, σ_{k+1}', σ_{k+1}'', $\theta_{1,k+1}$, $\theta_{2,k+1}$, $\theta_{3,k+1}$, α_{k+1} and u_{k+1}.

Choose $n_{k+1} \geq n_k + 1$ so that for all $l \geq n_{k+1} + 1$, σ_l is $\mathcal{F}$-δ_{k+2}'-multiplicative,

$$\sum_{l=n_{k+1}+1}^{\infty} 100\tau(\sigma_l(1)) < \tau(p) \tag{4.12}$$

for every nonzero projection $p \in Ran(\psi_{k+2}')$ and for all $\tau \in T(\mathcal{B})$, and

$$\sum_{l=n_{k+1}+1}^{\infty} 100(N_{k+2}+1)^2(m+1)\tau(\sigma_l(1)) < \tau(q_{k+1}')$$

for every $\tau \in T(\mathcal{B})$.

Let $\alpha_{k+1} \in KK(C(X), \mathcal{B})$ be given by

$$\alpha_{k+1} =_{df} \alpha_k - \sum_{l=n_k+1}^{n_{k+1}} [\sigma_l] = \alpha_0 - \sum_{l=1}^{n_{k+1}} [\sigma_l].$$

Note that for all $p \in \mathcal{P}_X$,

$$\widehat{\alpha_{k+1}([p])} = \sum_{l=n_{k+1}+1}^{\infty} \widehat{\sigma_l([p])}.$$

In particular,

$$100(N_{k+2}+1)^2(m+1)\tau(\alpha_{k+1}([1_{C(X)}])) < \tau(q'_{k+1})$$

for all $\tau \in T(\mathcal{B})$.

Hence, by the definitions of $\mathcal{P}_{k+2}$, δ'_{k+2} and δ_{k+2} and by Corollary 3.10 and by Proposition 3.2, there exist c.p.c. maps $\sigma'_{0,k+1}, \sigma''_{0,k+1} : C(X) \to q'_{k+1}\mathcal{B}q'_{k+1}$ such that $\sigma'_{0,k+1}(1) \perp \sigma''_{0,k+1}(1)$, $\sigma'_{0,k+1}$ is $\mathcal{F}$-δ'_{k+2}-multiplicative, $\sigma''_{0,k+1}$ is $\mathcal{F}$-δ_{k+2}-multiplicative, $[\sigma'_{0,k+1}] - \alpha_{k+1}$ is the class of a finite dimensional *-homomorphism, and $[\sigma'_{0,k+1} \oplus \sigma''_{0,k+1}]|_{\mathcal{P}_{k+2}} \in \mathfrak{N}_{\mathcal{P}_{k+2}}$. By our choice of λ_{k+1}, q'_{k+1} and ψ'''_{k+1}, it must be the case that

$$d_\tau(\sigma'_{0,k+1}(g) + \sigma''_{0,k+1}(g) + \psi'''_{k+1}(g)) > \lambda_{k+1}$$

for all $g \in \mathcal{E}_{k+1}$ and for all $\tau \in T(her(\sigma'_{0,k+1}(1) + \sigma''_{0,k+1}(1) + \psi'''_{k+1}(1)))$. (Note that by equations (4.7) and (4.6), $1000q'_{k+1} \preceq p'_{k+1} \preceq \psi'''_{k+1}(1)$. Also recall equation (4.9).)

Hence, by the definition of δ_{k+2} and by Theorem 3.3, we can find a finite dimensional *-homomorphism $\theta_{0,2,k+1} : C(X) \to (q_{k+1} + q'_{k+1})\mathcal{B}(q_{k+1} + q'_{k+1})$ such that

$$\|\theta_{0,2,k+1}(f) - (\sigma'_{0,k+1}(f) + \sigma''_{0,k+1}(f) + \psi'''_{k+1}(f))\| < \epsilon_{k+1}$$

for all $f \in \mathcal{F}$.

But by equations (4.5) and (4.7), $1000q_{k+1} \preceq p_{k+1}$ and $1000q'_{k+1} \preceq p'_{k+1} \preceq p_{k+1}$. Also, by (4.3), $q_{k+1}, q'_{k+1}, p_{k+1} \leq \psi''_{k+1}(1)$ and $p_{k+1} \preceq p$ for every nonzero projection p in the range of ψ''_{k+1}. Hence, we can find a unitary $u_{k+1} \in \mathcal{M}(\mathcal{B})$ with $u_{k+1}\psi''_{k+1}(1) = \psi''_{k+1}(1)u_{k+1}$ and we can find a finite dimensional *-homomorphism $\theta_{3,k+1} : C(X) \to \psi''_{k+1}(1)\mathcal{B}\psi''_{k+1}(1)$ such that $u_{k+1}\theta_{0,2,k+1}(1)u^*_{k+1} \perp \theta_{3,k+1}(1)$ and

$$\|\psi''_{k+1} - (u_{k+1}\theta_{0,2,k+1}u^*_{k+1} + \theta_{3,k+1})\| < \epsilon_{2k+2} < \epsilon_{k+1}.$$

(Recall that $sp(\psi''_{k+1})$ is at least ϵ_{2k+2}-dense in X.)

If we define

$$\sigma'_{k+1} =_{df} u_{k+1}\sigma'_{0,k+1}u^*_{k+1},$$
$$\sigma''_{k+1} =_{df} u_{k+1}\sigma''_{0,k+1}u^*_{k+1},$$

and

$$\theta_{2,k+1} =_{df} u_{k+1}\theta_{0,2,k+1}u^*_{k+1}$$

then

$$\|\theta_{2,k+1}(f) - (\sigma'_{k+1}(f) + \sigma''_{k+1}(f) + u_{k+1}\psi'''_{k+1}(f)u^*_{k+1})\| < \epsilon_{k+1}$$

for all $f \in \mathcal{F}$, and

$$\|\psi''_{k+1} - (\theta_{2,k+1} + \theta_{3,k+1})\| < \epsilon_{k+1}.$$

Now since i. $[\sigma'_k] - \alpha_k$ and $[\sigma'_{k+1}] - \alpha_{k+1}$ are both classes of finite dimensional *-homomorphisms, ii. $[\sigma'_k + \sigma''_k]|_{\mathcal{P}_{k+1}} \in \mathfrak{N}_{\mathcal{P}_{k+1}}$ (by the induction hypothesis), and iii. $\alpha_{k+1} = \alpha_k - \sum_{l=n_k+1}^{n_{k+1}} [\sigma_l]$, we must have that

$$\left([\sigma''_k] + \sum_{l=n_k+1}^{n_{k+1}} [\sigma_l] + [\sigma'_{k+1}] \right) |_{\mathcal{P}_{k+1}} \in \mathfrak{N}_{\mathcal{P}_{k+1}}.$$

Now, by the induction hypothesis and equations (4.7), (4.6) and (4.4),

$$1000\sigma''_k(1) \preceq 1000q'_k \preceq p'_{k+1} \preceq \psi'''_{k+1}(1) \preceq q_{k+1} \preceq 1000q_{k+1} \preceq p$$

for every nonzero projection p in $Ran(\psi'_{k+1})$.

Also, by a similar argument,

$$1000\sigma'_{k+1}(1) \preceq 1000q'_{k+1} \preceq p'_{k+1} \preceq p$$

for every nonzero projection $p \in Ran(\psi'_{k+1})$.

Also, by the induction hypothesis,

$$\sum_{l=n_k+1}^{\infty} 100\tau(\sigma_l(1)) < \tau(p)$$

for every nonzero projection $p \in Ran(\psi'_{k+1})$ and for every $\tau \in T(\mathcal{B})$.

Hence, from the above and from (4.8), we must have that

$$d_\tau \left(\sigma''_k(g) + \sum_{l=n_k+1}^{n_{k+1}} \sigma_l(g) + \sigma'_{k+1}(g) + \psi'_{k+1}(g) \right) > \lambda_{k+1}$$

for every $g \in \mathcal{E}_{k+1}$ and $\tau \in T(her(\sigma''_k(1) + \sum_{l=n_k+1}^{n_{k+1}} \sigma_l(1) + \sigma'_{k+1}(1) + \psi'_{k+1}(1)))$.

Note that all the above maps are $\mathcal{F}$-δ_{k+1}-multiplicative.

Hence, by the definitions of δ_{k+1}, $\lambda_{k+1}, \mathcal{E}_{k+1}$ and $\mathcal{P}_{k+1}$, and by Theorem 3.3, let $\theta_{1,k+1} : C(X) \to \mathcal{B}$ be a finite dimensional *-homomorphism with

$$\theta_{1,k+1}(1) = \sigma''_k(1) + \sum_{l=n_k+1}^{n_{k+1}} \sigma_l(1) + \sigma'_{k+1}(1) + \psi'_{k+1}(1)$$

such that

$$\left\| \theta_{1,k+1}(f) - (\sigma''_k(f) + \sum_{l=n_k+1}^{n_{k+1}} \sigma_l(f) + \sigma'_{k+1}(f) + \psi'_{k+1}(f)) \right\| < \epsilon_{k+1}$$

for all $f \in \mathcal{F}$.

This completes the inductive construction.

Let

$$\Phi : C(X) \to \mathcal{M}(\mathcal{B})$$

be the *-homomorphism given by

$$\Phi(f) =_{df} \sum_{k=1}^{\infty} (\theta_{1,k}(f) + u_k \psi'''_k(f) u_k^* + \theta_{3,k}(f))$$

for every $f \in C(X)$.

Then

$$\pi \circ \Phi = \phi.$$

Moreover, we can vary the spectrum of Φ to get Ψ so that $\pi \circ \Phi = \pi \circ \Psi$ and where each point in $sp(\Psi)$ repeats infinitely many times.

Hence, ϕ is totally trivial. Since $K_0(\phi) = 0$, it follows, from Theorem 2.6, that ϕ is a null extension. $\square$

We note that there is a gap in the proof of [40] Theorem 2.8. In particular, on page 1280 lines 14–15, the statement "... we may assume that $\rho''_n = e_n \rho e_n$ is also δ_n-$\mathcal{G}_n$-multiplicative..." is not correct. However, [40] Theorem 2.8 can be proven by following the argument of the above proposition.

Let X be a compact metric space and $\mathcal{C}$ be a unital C*-algebra. Let $\gamma : KK(C(X), \mathcal{C}) \to Hom(K_*(C(X)), K_*(\mathcal{C}))$ be the surjective map from the Universal coefficient theorem.

We let $KK_u(C(X), \mathcal{C})$ be the set of elements $x \in KK(C(X), \mathcal{C})$ such that $\gamma(x)$ maps $[1_{C(X)}]$ (in $K_0(C(X))$) to $[1_{\mathcal{C}}]$ (in $K_0(\mathcal{C})$).

Finally, denote by $\mathfrak{J}$ the natural map $\mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B}) : [\phi]_{\mathbf{Ext}} \mapsto [\phi]_{KK}$. Similar for the (natural) analogous map $\mathbf{Ext}_u(C(X), \mathcal{B}) \to KK_u(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$.

Theorem 4.13. *Let X be a finite CW complex, and let $\mathcal{B}$ be a nonunital simple finite real rank zero $\mathcal{Z}$-stable C*-algebra with continuous scale.*

Then the map

$$\mathfrak{J} : \mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

is a bijection.

Proof. The argument is a minor variation on the argument of [40] Theorem 2.10. We provide the proof for the convenience of the reader.

Firstly, surjectivity of the map $\mathfrak{J}$ follows from [39] Theorem 1.17. Hence, it suffices to prove that $\mathfrak{J}$ is injective.

To prove injectivity of $\mathfrak{J}$, we reduce to the unital case. We claim that it suffices to prove that

$$\mathfrak{J} : \mathbf{Ext}_u(C(X), \mathcal{B}) \to KK_u(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

is injective. We denote the above claim by "(*)".

Let $Y =_{df} \{y\} \sqcup X$, where the union is disjoint.

Suppose that $\phi_1 : C(X) \to e(\mathcal{M}(\mathcal{B})/\mathcal{B})e$ and $\phi_2 : C(X) \to e'(\mathcal{M}(\mathcal{B})/\mathcal{B})e'$ are two unital *-monomorphisms with $\mathfrak{J}(\phi_1) = \mathfrak{J}(\phi_2)$, where $e, e' \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ are proper subprojections of the unit. Thus $[e] = [e']$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$.

Let $\phi'_1, \phi'_2 : C(Y) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be unital *-monomorphism given by $\phi'_1(f) =_{df} f(y)(1-e) + \phi_1(f|_X)$ and $\phi'_2(f) =_{df} f(y)(1-e') + \phi_2(f|_X)$ for all $f \in C(Y)$. Then if ϕ_1 and ϕ_2 are unitarily equivalent then so are ϕ'_1 and ϕ'_2. (See [54] Propositions 2.4 and 2.5.) And if $\mathfrak{J}(\phi_1) = \mathfrak{J}(\phi_2)$ then $\mathfrak{J}(\phi'_1) = \mathfrak{J}(\phi'_2)$.

Conversely, suppose that $\phi'_1, \phi'_2 : C(Y) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two unital *-monomorphisms. Then if ϕ'_1 and ϕ'_2 are unitarily equivalent then $\phi'_1|_{C(X)}$ and $\phi'_2|_{C(X)}$ are unitarily equivalent. And if $\mathfrak{J}(\phi'_1) = \mathfrak{J}(\phi'_2)$ then $\mathfrak{J}(\phi'_1|_{C(X)}) = \mathfrak{J}(\phi'_2|_{C(X)})$.

This completes to proof of the claim in (*).

From the claim in (*), we thus proceed to proving injectivity in the unital case. To finish the proof, we need to diverge into two further cases.

Case 1: Assume that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{B})$.

By Theorem 2.4, $Ext_u(C(X), \mathcal{B})$ is a group. Since $\mathfrak{J}$ (on $Ext_u(C(X), \mathcal{B})$) is a group homomorphism, it suffices to prove that for all $[\phi] \in Ext_u(C(X), \mathcal{B})$, if $\mathfrak{J}(\phi) = 0$ then ϕ is a null extension. But this has been proven in Proposition 4.2.

This completes the proof of Case 1.

Case 2: General case.

Let $p \in \mathcal{M}(\mathcal{B})$ be a projection such that $[\pi(p)] + [1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$.

Let $\mathcal{D} =_{df} (1 \oplus p)\mathbb{M}_2(\mathcal{B})(1 \oplus p)$. Then $\mathcal{D}$ is a nonunital separable simple real rank zero finite $\mathcal{Z}$-stable C*-algebra with continuous scale such that $[1_{\mathcal{M}(\mathcal{D})/\mathcal{D}}] = 0$ in $K_0(\mathcal{M}(\mathcal{D})/\mathcal{D})$.

Supose that $\phi_1, \phi_2 : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two unital *-monomorphisms such that $\mathfrak{J}(\phi_1) = \mathfrak{J}(\phi_2)$.

Let $Y =_{df} \{y\} \sqcup X$, where the union is disjoint. Let $\phi'_1, \phi'_2 : C(Y) \to \mathcal{M}(\mathcal{D})/\mathcal{D}$ be two unital *-monomorphisms given by $\phi'_1(f) =_{df} f(y)\pi(p) \oplus \phi_1(f|_X)$ and $\phi'_2(f) =_{df} f(y)\pi(p) \oplus \phi_2(f|_X)$ for all $f \in C(Y)$. Then $\mathfrak{J}(\phi'_1) = \mathfrak{J}(\phi'_2)$.

Hence, by Case 1, ϕ'_1 and ϕ'_2 are unitarily equivalent. Hence, ϕ_1 and ϕ_2 are unitarily equivalent.

This completes Case 2 and the whole proof. □

Remark 4.14. *The proof of Theorem 4.13 actually shows that, under the hypotheses of Theorem 4.13, when $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$, the map*

$$\mathfrak{J} : \mathbf{Ext}_u(C(X), \mathcal{B}) \to KK_u(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

is a group isomorphism.

Similar for when $\mathfrak{J}$ maps $\mathbf{Ext}(C(X), \mathcal{B})$ to $KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$.

Remark 4.15. *As pointed out in* [54] *Subsection 3.1, one can give a group structure to $\mathbf{Ext}(C(X), \mathcal{B})$ even without the hypothesis that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$. The idea is due to* [12] *and involves a natural generalization of the BDF sum. More precisely, suppose that $\mathcal{B}$ is a nonunital separable simple continuous scale C*-algebra and suppose that X is a compact metric space. Let $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be two nonunital extensions. Then the* (generalized) BDF sum *of ϕ and ψ is defined to be*

$$S\phi(.)S^* + T\psi(.)T^*$$

where $S, T \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two isometries such that $SS^* + TT^* \leq 1$.

The above sum is, up to unitary equivalence, independent of the choices of S and T. Thus, we get a well defined addition and then group structure on $\mathbf{Ext}(C(X), \mathcal{B})$. (See [54] *Subsection 3.1.) And this is so even if $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] \neq 0$.*

With the aforementioned group structure on $\mathbf{Ext}(C(X), \mathcal{B})$, the map $\mathfrak{J}$ is a group homomorphism, and thus, the bijection in Theorem 4.13 is a group isomorphism. We state this result.

Theorem 4.16. *Let X be a finite CW complex and let $\mathcal{B}$ be a nonunital simple separable finite real rank zero $\mathcal{Z}$-stable C*-algebra with continuous scale.*

Then the map

$$\mathfrak{J} : \mathbf{Ext}(C(X), \mathcal{B}) \to KK(C(X), \mathcal{M}(\mathcal{B})/\mathcal{B})$$

is a group isomorphism.

References

[1] C. Apostol, L. A. Fialkow, D. A. Herrero and D. Voiculescu, Approximation of Hilbert space operators. II, Research Notes in Math. **102** (London–Boston–Melbourne: Pitman Books, Ltd., 1984)

[2] W. Arveson, *Notes on extensions of C*-algebras.* Duke Math. J. **44** (1977), no. 2, 329–355.

[3] I. D. Berg, *An extension of the Weyl–von Neumann theorem to normal operators,* Trans. AMS **160** (1971), 365–371.

[4] I. D. Berg and K. Davidson, *Almost commuting matrices and a quantitative version of the Brown–Douglas–Fillmore theorem,* Acta Math. **166** (1991), 121–161.

[5] B. Blackadar, K-theory for operator algebras. Second edition. Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.

[6] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras.* J. Funct. Anal. **45** (1982), no. 3, 297–340.

[7] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras.* Proceedings of a Conference on Operator Theorem (Dalhouse Univ., Halifax, N.S., 1973), pp58–128. Lecture Notes in Math., Vol 345, Springer, Berlin, 1973.

[8] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of C*-algebras and K-homology,* Ann. Math. **105** (1977), 265–324.

[9] G. Corach and A. R. Larotonda, *A stabilization theorem for Banach algebras.* Journal of Algebra **101** (1986), 433–449.

[10] G. Elliott, *Derivations of matroid C*-algebras. II.* Annals of Mathematics **100** (1974), no. 2, 407–422.

[11] G. Elliott, G. Gong, H. Lin and Z. Niu, *The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension.* Preprint. A copy can be found at https://arxiv.org/abs/1712.09463.

[12] G. Elliott and D. Handelman, *Addition of C*-algebra extensions.* Pacific Journal of Mathematics **137** (1989), no. 1, 87–121.

[13] G. Elliott and D. Kucerovsky, *An abstract Voiculescu–Brown–Douglas–Fillmore absorption theorem.* Pacific J. Math. **198** (2001), no. 2, 385–409.

[14] G. Elliott, L. Robert and L. Santiago, *The cone of lower semicontinuous traces on a C*-algebra.* Amer. J. Math. **133** (2011), no. 4, 969–1005.

[15] G. Gong and H. Lin, *Classification of homomorphisms from $C(X)$ to simple C*-algebras of real rank zero.* Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 2, 181–206.

[16] G. Gong and H. Lin, *Almost multiplicative morphisms and almost commuting matrices.* J. Operator Theory **40** (1998), 217–275.

[17] G. Gong and H. Lin, *Almost multiplicative morphisms and K-theory.* Internat. J. Math. **11** (2000), 983–1000.

[18] K. R. Goodearl, *Riesz decomposition in inductive limit C*-algebras.* Rocky Mountain Journal of Mathematics **24** (1994), no. 4, 1405–1430.

[19] P. S. Guinand and L. Marcoux, *Between the unitary and similarity orbits of normal operators.* Pacific J. Math. **159** (1993), no. 2, 299–355.

[20] D. Hadwin, *Nonseparable approximate unitary equivalence.* Trans. Amer. Math. Soc. **266** (1981), no. 1, 203–231.

[21] P. R. Halmos, *Ten problems in Hilbert space.* Bull. AMS **76** (1970), 887–933.

[22] D. A. Herrero, Approximation of Hilbert space operators. I, Research Notes in Math. **72** (London–Boston–Melbourne: Pitman Books, Ltd., 1982)

[23] S. Hu, H. Lin and Y. Xue, *Limits of homomorphisms with finite dimensional range.* Internat. J. Math. **16** (2005), no. 7, 807–821.

[24] D. Husemoeller, Fibre Bundles, Third editon, 1994, Springer–Verlag, New York.

[25] X. Jiang and H. Su, *On a simple unital projectionless C*-algebra.* Amer. J. Math. **121** (1991), no. 2, 359–413.

[26] V. Kaftal, P. W. Ng and S. Zhang, *Strict comparison of positive elements in multiplier algebras.* Canadian Journal of Mathematics **69** (2017), no. 2, 373–407.

[27] V. Kaftal, P. W. Ng and S. Zhang, *The minimal ideal in a multiplier algebra.* Journal of Operator Theory **79** (2018), no. 2, 419–462.

[28] V. Kaftal, P. W. Ng S. Zhang, *Purely infinite corona algebras.* Preprint (2014). A copy is available at arxiv.org/abs/1804.02618.

[29] G. G. Kasparov, *The operator K-functor and extensions of C*-algebras.* Izv. Akad. Nauk SSSR Ser. Math. **44** (1980), no. 3, 571–636.

[30] G. G. Kasparov, *Hilbert C*-modules: theorems of Stinespring and Voiculescu.* J. Operator Theory **4** (1980), no. 1, 133–150.

[31] D. Kucerovsky, *Extensions contained in ideals.* Trans. Amer. Math. Soc. **356** (2004), no. 3, 1025–1043.

[32] D. Kucerovsky and P. W. Ng, *Completely positive maps into corona algebras.* Mathematical Proceedings of the Cambridge Philosophical Society. **147** (2009), no. 2, 323–337.

[33] D. Kucerovsky, P. W. Ng and F. Perera, *Purely infinite corona algebras of simple C*-algebras.* Math. Ann. **346** (2010), no. 1, 23–40.

[34] H. Lin, *Simple C*-algebras with continuous scales and simple corona algebras.* Proceedings of the AMS. **112** (1991), no. 3, 871–880.

[35] H. Lin, *Notes on K-theory of multiplier algebras.* (1992), University of Victoria Library Technical Reports. (Mathematics and Statistics) A copy is available at https://dspace.library.uvic.ca/handle/1828/2827
filename: DMS-603-IR.

[36] H. Lin, *Extensions by C*-algebras of real rank zero. II.* Proc. London Math. Soc. (3) **71** (1995), no. 3, 641–674.

[37] H. Lin, *Generalized Weyl–von Neumann theorems. II.* Math. Scand. **77** (1995), no. 1, 129–147.

[38] H. Lin, *Almost commuting self-adjoint matrices and applications.* Fields Inst. Comm. **13**, Amer. Math. Soc. (1997), 193–233.

[39] H. Lin, *Almost multiplicative morphisms and some applications.* J. Operator Theory **37** (1997), no. 1, 121–154.

[40] H. Lin, *Extensions of $C(X)$ by simple C*-algebras of real rank zero.* Amer. J. Math. **119** (1997), no. 6, 1263–1289.

[41] H. Lin, *Extensions by C*-algebras of real rank zero. III.* Proc. London Math. Soc. (3) **76** (1998), no. 3, 634–666.

[42] H. Lin, *Embedding an AH algebra into a simple C*-algebra with prescribed KK data.* K-theory **24** (2001), no. 2, 135–156.

[43] H. Lin, *Classification of simple tracially AF C*-algebras.* Canad. J. Math. **53** (2001), no. 1, 161–194.

[44] H. Lin, An introduction to the classification of amenable C*-algebras, World Scientific Publishing Company, Inc., River Edge, NJ, 2001.

[45] H. Lin, *Stable approximate unitary equivalence of homomorphisms.* J. Operator Theory **47** (2002), no. 2, 343–378.

[46] H. Lin, *Simple corona algebras.* Proceedings of the American Mathematical Society **132** (2004), no. 11, 3215–3224.

[47] H. Lin, *Extensions by simple C*-algebras: quasidiagonal extensions.* Canad. J. Math. **57** (2005), no. 2, 351–399.

[48] H. Lin, *Classification of homomorphisms and dynamical systems.* Trans. Amer. Math. Soc. **359** (2007), no. 2, 859–895.

[49] H. Lin. Personal communication, (2014).

[50] H. Lin and P. W. Ng, *The corona algebra of stabilized Jiang–Su algebra.* J. Funct. Anal. **270** (2016), no. 3, 1220–1267.

[51] P. W. Ng, *A double commutant theorem for purely large C*-subalgebras of real rank zero corona algebras.* Studia Mathematica **190** (2009), no. 2, 135–145.

[52] P. W. Ng, *Approximate uniqueness for maps from $C(X)$ into simple real rank zero C*-algebras.* Banach Journal of Mathematical Analysis **12** (2016), no. 1, 126–143. A copy is available at http://www.ucs.louisiana.edu/∼pwn1677/references.html.

[53] P. W. Ng, *Real rank zero for purely infinite corona algebras.* Preprint (2015).

[54] P. W. Ng, *Nonstable absorption.* Accepted for publication at the Houston Journal of Mathematics. A copy is available at http://www.ucs.louisiana.edu/∼pwn1677/references.html.

[55] N. C. Phillips, *Approximation by unitaries with finite spectrum in purely infinite C*-algebras.* J. Funct. Anal. **120** (1994), no. 1, 98–106.

[56] M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras.* Proc. London Math. Soc. (3), **46** (1983), no. 2, 301–333.

[57] M. Rordam, *The stable and real rank of $\mathcal{Z}$-absorbing C*-algebras.* Internat. J. Math. **15** (2004), no. 10, 1065–1084.

[58] D. Voiculescu, *A noncommutative Weyl–von Neumann theorem.* Rev. Roumaine Math. Pures Appl. **21** (1976), 97–113.

[59] N. E. Wegge-Olsen, K-theory and C*-algebras. A friendly approach. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[60] S. Zhang, *A Riesz decomposition property and ideal structure of multiplier algebras.* J. Operator Theory **24** (1990), no. 2, 209–225.

[61] S. Zhang, *K_1-groups, quasidiagonality, and interpolation by multiplier projections.* Trans. Amer. Math. Soc. **325** (1991), no. 2, 793–818.

[62] S. Zhang, *Certain C*-algebras with real rank zero and their corona and multiplier algebras. Part I.* Pacific Journal of Mathematics **155** (1992), no. 1, 169–197.

Mathematics Department, University of Louisiana at Lafayette, P. O. Box 43568, Lafayette, Louisiana, 70504–3568, USA

E-mail address: png@louisiana.edu