NONSTABLE ABSORPTION

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ABSTRACT. Let X be a finite CW complex and $\mathcal{B}$ be a nonunital separable simple C*-algebra with continuous scale. We show that $\mathbf{Ext}(C(X), \mathcal{B})$ is a group, and we also characterize the neutral element of $\mathbf{Ext}(C(X), \mathcal{B})$, with further information when $\mathcal{B}$ has additional regularity properties. We have similar results for the unital version $\mathbf{Ext}_u(C(X), \mathcal{B})$.

In the process, we prove some results involving nonstable absorption, some of which work for general (not necessarily simple) purely infinite corona algebras.

1. Introduction

Motivated by the problem of classifying essential normal operators on a separable infinite dimensional Hilbert space, Brown, Douglas and Fillmore (BDF) classified all C*-algebra extensions of the form

$$0 \to \mathcal{K} \to \mathcal{D} \to C(X) \to 0$$

where $\mathcal{K}$ is the C*-algebra of compact operators on a separable infinite dimensional Hilbert space, and X is a compact metric space. This was a starting point for much interesting phenomena in operator theory (including the important stable uniqueness theorems of Classification Theory; [6], [33]), and has led to the rapid development of extension theory with many effective techniques (especially from KK theory) to compute the Ext-group $Ext(\mathcal{A}, \mathcal{B})$.

However, in general, $Ext(\mathcal{A}, \mathcal{B})$ does not capture all unitary equivalence classes of extensions. Among other things, there can be many nonunitarily equivalent trivial extensions, and also, an extension ϕ with $[\phi] = 0$ in $Ext(C(X), \mathcal{B})$ need not be trivial. (For these and other shortcomings, see, for example, [26], [29], and [30].)

One of the implicit reasons for the success of the original BDF Theory is that $\mathbb{B}(l_2)$ and the Calkin algebra $\mathbb{B}(l_2)/\mathcal{K}$ have particularly nice structure. Among other things, $\mathbb{B}(l_2)$ has strict comparison and real rank zero (it is a von Neumann algebra), and $\mathbb{B}(l_2)/\mathcal{K}$ is simple purely infinite. (For example, the BDF–Voiculescu result that all essential extensions are absorbing would not be true without the simplicity of $\mathbb{B}(l_2)/\mathcal{K}$.)

One would like to single out a class of corona algebras which generalize nice features from $\mathbb{B}(l_2)/\mathcal{K}$, with the goal of developing operator theory and extension theory in an agreeable context, among other things generalizing further the theories developed by BDF, Voiculescu and other workers. Such thoughts were clearly present[1] in the early literature.

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[1]as a proper subset

Simple purely infinite corona algebras have been completely characterized. Recall that a simple C*-algebra has *continuous scale* if, roughly speaking, it has a sequential approximate identity which is a "Cauchy sequence". More precisely:

Definition 1. *Let $\mathcal{B}$ be a nonunital separable simple C*-algebra. Then $\mathcal{B}$ has* continuous scale *if $\mathcal{B}$ has an approximate identity $\{e_n\}_{n=1}^{\infty}$ such that $e_{n+1}e_n = e_n$ for all n, and for every $b \in \mathcal{B}_+ - \{0\}$, there exists an $N \geq 1$ such that for all $m > n \geq N$,*

$$e_m - e_n \preceq b.$$

(See, for example, [24]*.)*

In the above, $\preceq$ is a subequivalence relation for positive elements (generalizing Murray–von Neumann subequivalence for projections) given as follows: for a C*-algebra $\mathcal{D}$, for $a, d \in \mathcal{D}_+$, $a \preceq d$ if there exists a sequence $\{x_n\}$ in $\mathcal{D}$ such that $x_n d x_n^* \to a$.

Theorem 1.1. *Let $\mathcal{B}$ be a nonunital separable simple nonelementary C*-algebra. Then the following statements are equivalent:*

(1) *$\mathcal{B}$ has continuous scale.*
(2) *$\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple.*
(3) *$\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple and purely infinite.*

([24], [34]*; see also* [7], [45]*)*

We note that purely infinite simple C*-algebras have real rank zero ([47]). We further note that for a general nonunital separable simple C*-algebra $\mathcal{D}$, an extension of $\mathcal{D}$ by $C(X)$ can often be decomposed in a way where one piece sits inside the minimal ideal of $\mathcal{M}(\mathcal{D})/\mathcal{D}$, and this piece is essentially an extension of a simple continuous scale algebra (e.g., [30]; see also [18]). Thus, simple purely infinite corona algebras are not just a very nice context, but are part of the general picture.

Nonetheless, difficulties still arise that are not present in the case of $\mathbb{B}(l_2)/\mathcal{K}$. For example, for simple continuous scale $\mathcal{B}$, the K-theory of $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}(\mathcal{B})/\mathcal{B}$ can be much more complicated than that of $\mathbb{B}(l_2)$ and $\mathbb{B}(l_2)/\mathcal{K}$. Moreover, in the case where $\mathcal{B}$ is nonstable, we do not have infinite repeats and the powerful tools of the classical theory of absorbing extensions (e.g., [1], [4], [10], [20], [21], [33], [43]) are no longer completely available.

In effect, one needs to develop a type of nonstable absorption theory, where one takes into account the fine structure of the K-theory. Such a theory has previously been considered with definite results (e.g., [26], [29], [30]). The author of the aforementioned results studied the case where the ideal was a simple nonunital continuous scale algebra with real rank zero, stable rank one, strict comparison and unique tracial state. In the present paper, we develop analogous absorption results in a more general context. Some results even work for the case where the corona algebra is nonsimple (purely infinite).

The theory makes use of the techniques from the Elliott Classification Program, which is interesting since one source of the important stable uniqueness theorems of the Classification Program is BDF Theory itself [2]. Also, in addition to their importance in extension theory, we note the prominence of continuous scale algebras in the striking recent classification work of Elliott–Gong–Lin–Niu ([8]).

[2] Another source is Elliott's work.

As part of the program, we also have results characterizing (not necessarily simple) purely infinite corona algebras. Under mild regularity conditions on a simple C*-algebra $\mathcal{B}$, we have the equivalences: $\mathcal{B}$ has quasicontinuous scale $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})$ has strict comparison $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite $\Leftrightarrow$ $\mathcal{M}(\mathcal{B})$ has finitely many ideals $\Leftrightarrow$ $\mathcal{I}_{min} = \mathcal{I}_{cont}$. Furthermore, all such corona algebras have real rank zero, and many related fundamental results have been proven. (E.g., [17], [18], [19], [23], [37], [40].)

In [41], we will use the results in this paper to classify all extensions of the form

$$0 \to \mathcal{B} \to \mathcal{D} \to C(X) \to 0$$

where $\mathcal{B}$ is a simple real rank zero finite Jiang–Su-stable continuous scale C*-algebra and where X is a finite CW complex.

1.1. **Notation.** We end this section with some brief remarks on notation.

For a C*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ denotes the multiplier algebra of $\mathcal{B}$. Thus, $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is the corresponding corona algebra.

For each extension

$$0 \to \mathcal{B} \to \mathcal{D} \to \mathcal{C} \to 0$$

(of $\mathcal{B}$ by $\mathcal{C}$[3]) we will work with the corresponding *Busby invariant* which is a *-homomorphism $\phi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$. We will always work with *essential extensions* which is equivalent to requiring that the corresponding Busby invariant to be injective; hence, throughout the paper, when we write "extension", we mean essential extension. An extension is unital if the corresponding Busby invariant is a unital map.

Say that $\phi, \psi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two extensions. We say that ϕ and ψ are *unitarily equivalent* (and write $\phi \sim \psi$) if there exists a unitary $u \in \mathcal{M}(\mathcal{B})$ such that

$$\phi(c) = \pi(u)\psi(c)\pi(u)^*$$

for all $c \in \mathcal{C}$. Here, $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map.

$\mathbf{Ext}(\mathcal{C}, \mathcal{B})$ denotes the set of unitary equivalence classes of nonunital essential extensions of $\mathcal{B}$ by $\mathcal{C}$. If, in addition, $\mathcal{C}$ is unital, $\mathbf{Ext}_u(\mathcal{C}, \mathcal{B})$ is the set of unitary equivalence classes of unital essential extensions.

For a unital simple C*-algebra $\mathcal{C}$, $T(\mathcal{C})$ denotes the tracial state space of $\mathcal{C}$. If $\mathcal{C}$ is a nonunital simple C*-algebra, $T(\mathcal{C})$ will denote that class of lower semicontinous, densely defined traces which are normalized at a fixed element $e \in \mathcal{C}_+ - \{0\}$, where e is in the Pedersen ideal of $\mathcal{C}$ (of course, for statements involving $T(\mathcal{C})$, where $\mathcal{C}$ is nonunital, the choice of e is not relevant). For $\tau \in T(\mathcal{C})$ (where $\mathcal{C}$ is unital or nonunital), for $a \in \mathcal{C}_+$, $d_\tau(c) =_{df} \lim_{n\to\infty} \tau(c^{1/n})$. (Good references are [17] and [18].)

For a C*-algebra $\mathcal{D}$, and for $a \in \mathcal{D}_+$, we let $her_{\mathcal{D}}(a) =_{df} \overline{a\mathcal{D}a}$, the hereditary C*-subalgebra of $\mathcal{D}$ generated by a. Sometimes, for simplicity, we write $her(a)$ in place of $her_{\mathcal{D}}(a)$. Similarly, for a C*-subalgebra $\mathcal{C} \subseteq \mathcal{D}$, we let $her_{\mathcal{D}}(\mathcal{C})$ or $her(\mathcal{C})$ denote $\overline{\mathcal{C}\mathcal{D}\mathcal{C}}$, the hereditary C*-subalgebra of $\mathcal{D}$ generated by $\mathcal{C}$. Finally, for a subset $S \subseteq \mathcal{D}$, we let $Ideal_{\mathcal{D}}(S)$ denote the ideal of $\mathcal{D}$ which is generated by S. Again, we often write $Ideal(S)$ in place of $Ideal_{\mathcal{D}}(S)$.

[3]We note that, in the literature, such extensions are often called "extensions of $\mathcal{C}$ by $\mathcal{B}$". However, like some other authors ([1], [4], [5]), we prefer "extensions of $\mathcal{B}$ by $\mathcal{C}$", and will use this in the present paper.

In this paper, any simple separable stably finite C*-algebra is assumed to have the property that every quasitrace is a trace.

Let $\mathcal{A}, \mathcal{C}$ be C*-algebras. Throughout this paper, we will write that a map $\phi : \mathcal{A} \to \mathcal{C}$ is *c.p.c.* if it is linear and completely positive contractive. Let $\mathcal{F} \subset \mathcal{A}$ be a finite subset and let $\delta > 0$. A c.p.c. map $\psi : \mathcal{A} \to \mathcal{C}$ is said to be *$\mathcal{F}$-δ-multiplicative* if $\|\psi(fg) - \psi(f)\psi(g)\| < \delta$ for all $f, g \in \mathcal{F}$.

Good references for basic multiplier algebra theory, extension theory, K theory, and KK theory are [2], [32], and [44]. See also [17], [18] and [19] for much of the advanced multiplier algebra machinery.

For the notation and basic KK-theoretic tools from Classification Theory used in this paper, we refer the reader to [11], [15], [26], [29], [32], [36], and the references therein.

References for simple continuous scale algebras are [24] and [34]. Section 1 of [35] contains computations of the K theory for the multiplier and corona algebras of simple separable continuous scale C*-algebras with real rank zero, stable rank one and strict comparison (see also [37] Propositions 4.2, 4.4 and Corollary 4.6; and also [9]). Other good sources are [17] and [18].

2. Some nonstable decomposition theorems for purely infinite corona algebras

In this section, we provide some Voiculescu-type decomposition theorems for purely infinite corona algebras. We do not have the strongest possible technical results, but enough for the main goal of this paper. Precursors to the results in this section are [1], [10], [20], [22], [43].

The first result partly generalizes [22] Theorem 3.4.

Theorem 2.1. *Let $\mathcal{B}$ be a simple nonunital separable C*-algebra such that $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite and has finitely many ideals.*[4]

*Let X be a compact metric space and $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a *-monomorphism.*

*Say that $\psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a *-homomorphism such that for all $f \in C(X)$, $\psi(f) \in Ideal(\phi(f))$.*

Then there exists a $V \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that

$$\psi(f) = V\phi(f)V^*$$

for all $f \in C(X)$.

Proof. Since ϕ is injective, we may assume that $C(X)$ is a C*-subalgebra of $\mathcal{M}(\mathcal{B})/\mathcal{B}$ and that ϕ is the inclusion map.

Let $\{\mathcal{F}_n\}_{n=1}^\infty$ be a increasing sequence of finite subsets of $C(X)_+$ such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is norm dense in the closed unit ball of $C(X)_+$. We may assume that $1_{C(X)} \in \mathcal{F}_1$.

Let $\{\epsilon_n\}_{n=1}^\infty$ be a strictly decreasing sequence of numbers in $(0,1)$ such that $\sum_{n=1}^\infty \epsilon_n < \infty$. Let $\{\delta_n\}_{n=1}^\infty$ be another decreasing sequence of numbers in $(0,1)$ such that $\sum_{n=1}^\infty \delta_n < \infty$, and such that for all n, for all $f \in \mathcal{F}_n$, for all $x, y \in X$, if $d(x,y) < 2\delta_n$ then $|f(x) - f(y)| < \epsilon_n/10$.

For all n, let $x_{n,1}, x_{n,2}, ..., x_{n,k_n} \in X$ and $O_{n,1}, O_{n,2}, ..., O_{n,k_n}$ be open balls in X such that the following statements are true:

[4] Actually, under mild regularity conditions on the canonical ideal, purely infinite corona algebras will have finitely many ideals ([19]).

(1) $x_{n,k} \in O_{n,k}$ for all n, k.
(2) $diam(O_{n,k}) < \delta_n/4$ for all n, k.
(3) $X = \bigcup_{k=1}^{k_n} O_{n,k}$ for all n, k.
(4) For all $f \in C_0(O_{n,k})_+$ such that $f(x_{n,k}) \neq 0$, $Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(f) = Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(C_0(O_{n,k}))$.

For all n, let $\{g_{n,k}\}_{k=1}^{k_n}$ be a partition of unity for X subordinate to $\{O_{n,k}\}_{k=1}^{k_n}$, i.e.,

(1) $g_{n,k} \in C(X)_+$ for all n, k,
(2) $\overline{supp(g_{n,k})} \subseteq O_{n,k}$ for all n, k, and
(3) $\sum_{k=1}^{k_n} g_{n,k}(x) = 1$ for all $x \in X$, for all n.

Let $\widetilde{\psi} : C(X) \to \mathcal{M}(\mathcal{B})$ be a c.p.c. map such that $\psi = \pi \circ \widetilde{\psi}$.

For all n, let $\widetilde{\psi}_n : C(X) \to \mathcal{M}(\mathcal{B})$ be the c.p.c. map given by $\widetilde{\psi}_n(f) =_{df} \sum_{k=1}^{k_n} f(x_{n,k})\widetilde{\psi}(g_{n,k})$ for all $f \in C(X)$. Let $\psi_n =_{df} \pi \circ \widetilde{\psi}_n$. Hence, for all $f \in C(X)$, $\psi_n(f) = \sum_{k=1}^{k_n} f(x_{n,k})\psi(g_{n,k}) = \psi(\sum_{k=1}^{k_n} f(x_{n,k})g_{n,k})$. As a consequence, for all $f \in \mathcal{F}_n$, $\|\psi(f) - \psi_n(f)\| < \epsilon_n/10$.

Claim: There exists a sequence $\{d_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{B})/\mathcal{B}$ such that the following statements hold:

i. $\|d_n\| < 2$ for all n.
ii. $\psi(f) \approx_{\epsilon_n} d_n f d_n^*$ for all $f \in \mathcal{F}_n$, for all n.
iii. $\|d_n f d_{n+1}^*\| < \epsilon_n$ for all $f \in C(X)$, for all n.

Proof of Claim: Since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite and has finitely many ideals, let $\{h_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{k_n}$ be a collection of elements in $(\mathcal{M}(\mathcal{B})/\mathcal{B})_+$ such that

(a) $\|h_{n,k}\| = 1$ for all n, k,
(b) $h_{n,k} \in her_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(C_0(O_{n,k}))$ for all n, k,
(c) $Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}((h_{n,k}^2 - \frac{9}{10})_+) = Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(C_0(O_{n,k}))$ for all n, k,
(d) $h_{n,k} f h_{n,k} \approx_{\frac{\epsilon_n}{10(k_n+1)}} f(x_{n,k}) h_{n,k}^2$ for all $f \in \mathcal{F}_n$, for all n, k,
(e) $h_{n,k} h_{n,l} = 0$, for all n, for all $k \neq l$, and
(f) $\|h_{n,k} f h_{n,l}\| < \frac{\epsilon_n}{10(k_n^2+1)}$ for all n, for all $k \neq l$.
(g) $\|h_{n+1,k} f h_{n,l}\| < \frac{\epsilon_{n+1}}{10(\|k_n\|^2 + \|k_{n+1}\|^2 + 1)}$ for all n, for all k, l.

By hypothesis, for all n, k,

$$\psi(g_{n,k}) \in Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(g_{n,k}) \subseteq Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(C_0(O_{n,k})) = Ideal_{\mathcal{M}(\mathcal{B})/\mathcal{B}}(h_{n,k}).$$

Since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite, for all n, k,

$$\psi(g_{n,k}) \preceq \left(h_{n,k}^2 - \frac{9}{10}\right)_+.$$

Hence, since $\|\psi(g_{n,k})\| \leq 1$ and $\|h_{n,k}\| \leq 1$, let $d'_{n,k} \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ with $\|d'_{n,k}\| \leq 3/2$ be such that

$$d'_{n,k} h_{n,k}^2 (d'_{n,k})^* \approx_{\frac{\epsilon_n}{10(k_n+1)}} \psi(g_{n,k})$$

for all n, k.

Hence, for all n, for all $f \in \mathcal{F}_n$,

$$\begin{array}{rcl} \psi(f) & \approx_{\epsilon_n/10} & \psi_n(f) \\ & = & \sum_{k=1}^{k_n} f(x_{n,k})\psi(g_{n,k}) \\ & \approx_{\epsilon_n/10} & \sum_{k=1}^{k_n} f(x_{n,k}) d'_{n,k} h_{n,k}^2 (d'_{n,k})^* \\ & = & \sum_{k=1}^{k_n} d'_{n,k} f(x_{n,k}) h_{n,k}^2 (d'_{n,k})^* \\ & \approx_{\epsilon_n/4} & \sum_{k=1}^{k_n} d'_{n,k}(h_{n,k} f h_{n,k})(d'_{n,k})^* \\ & \approx_{\epsilon_n/10} & d_n f d_n^* \end{array}$$

where $d_n =_{df} \sum_{k=1}^{k_n} d'_{n,k} h_{n,k} \in \mathcal{M}(\mathcal{B})/\mathcal{B}$.

Hence, for all $f \in \mathcal{F}_n$,

$$\psi(f) \approx_{\epsilon_n} d_n f d_n^*.$$

Also, $1_{C(X)} \in \mathcal{F}_1 \subseteq \mathcal{F}_n$. Therefore, $\psi(1) \approx_{\epsilon_n} d_n 1 d_n^* = d_n d_n^*$. So $\|d_n\| < 2$, for all n.

Also, since $\|h_{n,k} f h_{n+1,l}\| < \frac{\epsilon_{n+1}}{10(\|k_n\|^2 + \|k_{n+1}\|^2 + 1)}$ for all n, k, l and for all $f \in \mathcal{F}_n$, we have that for all n and for all $f \in \mathcal{F}_n$,

$$\begin{array}{rcl} \|d_n f d_{n+1}^*\| & \leq & \sum_{k=1}^{k_n} \sum_{l=1}^{k_{n+1}} \|d'_{n,k} h_{n,k} f h_{n+1,l} (d'_{n+1,l})^*\| \\ & < & \epsilon_n/4. \end{array}$$

We have thus constructed a sequence $\{d_n\}_{n=1}^{\infty}$ as in the Claim. This completes the proof of the Claim.

For all n, let $\widetilde{d_n} \in \mathcal{M}(\mathcal{B})$ be such that $\|\widetilde{d_n}\| < 2$ and $\pi(\widetilde{d_n}) = d_n$.

For all $f \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$, let $\widetilde{f} \in \mathcal{M}(\mathcal{B})$ with $\|\widetilde{f}\| \leq 2\|f\| \leq 2$ be such that $\pi(\widetilde{f}) = f$. For all n, let $\widetilde{\mathcal{F}_n} =_{df} \{\widetilde{f} : f \in \mathcal{F}_n\}$. Let $\{e_m\}_{m=1}^{\infty}$ be an approximate unit for $\mathcal{B}$ such that $e_{m+1}e_m = e_m$ for all m, and such that $\{e_m\}_{m=1}^{\infty}$ quasicentralizes $\bigcup_{n=1}^{\infty}(\widetilde{\mathcal{F}_n} \cup \widetilde{\psi}(\mathcal{F}_n) \cup \{\widetilde{d_n}\})$.

Note that for all n, since $\psi(f) \approx_{\epsilon_n} d_n f d_n^*$ and $d_n f d_{n+1}^* \approx_{\epsilon_n} 0$ for all $f \in \mathcal{F}_n$, we must have that $\widetilde{\psi}(f)(1 - e_m) \approx_{\epsilon_n} \widetilde{d_n}\widetilde{f}\widetilde{d_n^*}(1 - e_m)$ and $\widetilde{d_n}\widetilde{f}\widetilde{d_{n+1}^*}(1 - e_m) \approx_{\epsilon_n} 0$ for all sufficiently large m and for all $f \in \mathcal{F}_n$.

Hence, let $\{m(n)\}_{n=1}^{\infty}$ be a subsequence of the positive integers such that the following statements hold:

Take $m(0) =_{df} 1$.

For all $n \geq 1$, let $m(n) \geq m(n-1) + 1$ be such that the following statements are true:

i. For all $f \in \mathcal{F}_{n+1}$, for all $m \geq m(n)$,

$$\begin{aligned} & \widetilde{\psi}(f)(e_m - e_{m(n)}) \\ \approx_{\epsilon_{n+1}} \quad & \widetilde{d}_{n+1}\widetilde{f}\widetilde{d}^*_{n+1}(e_m - e_{m(n)}) \\ \approx_{\epsilon_{n+1}/10} \quad & (e_m - e_{m(n)})^{1/4}\widetilde{d}_{n+1}(e_m - e_{m(n)})^{1/4}\widetilde{f}(e_m - e_{m(n)})^{1/4}\widetilde{d}^*_{n+1}(e_m - e_{m(n)})^{1/4}. \end{aligned}$$

ii. For all $f \in \mathcal{F}_{n+1}$, for all $m \geq m(n)$, for all $m' \geq m(n-1)$,

$$(e_{m'} - e_{m(n-1)})^{1/4}\widetilde{d}_n(e_{m'} - e_{m(n-1)})^{1/4} f (e_m - e_{m(n)})^{1/4}\widetilde{d}^*_{n+1}(e_m - e_{m(n)})^{1/4} \approx_{\epsilon_{n+1}} 0.$$

iii. For all $f \in \mathcal{F}_{n+1}$, for all $m \geq m(n)$,

$$(e_m - e_{m(n)})^{1/4} f \approx_{\epsilon_{n+1}/10} f(e_m - e_{m(n)})^{1/4}.$$

Let $\widetilde{V} =_{df} \sum_{n=1}^{\infty}(e_{m(n+1)} - e_{m(n)})^{1/4}\widetilde{d}_{n+1}(e_{m(n+1)} - e_{m(n)})^{1/4} \in \mathcal{M}(\mathcal{B})$, where the series converges strictly.

Let $V =_{df} \pi(\widetilde{V}) \in \mathcal{M}(\mathcal{B})/\mathcal{B}$.

Then $VfV^* = \psi(f)$, for all $f \in C(X)$. □

In addition to the references cited at the beginning of this section, we also note that an early (operator theoretic) analogue of the next result was already present implicitly in [4].

Corollary 2.2. *Let $\mathcal{B}$ be a simple nonunital separable C^*-algebra such that $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite and has finitely many ideals.*

*Let X be a compact metric space and $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a *-monomorphism.*

*Say that $\psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a *-homomorphism such that for all $f \in C(X)$, $\psi(f) \in Ideal(\phi(f))$.*

*Then there exists a partial isometry $W \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ with $W^*W = \psi(1)$ and *-homomorphism*

$$\sigma : C(X) \to (1 - WW^*)(\mathcal{M}(\mathcal{B})/\mathcal{B})(1 - WW^*)$$

such that

$$\phi(.) = W\psi(.)W^* + \sigma(.).$$

Proof. By Theorem 2.1, there exists $W \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that

$$\psi(f) = W^*\phi(f)W$$

for all $f \in C(X)$.

Replacing W with $\phi(1)W$ if necessary, we have that W is a partial isometry such that $W^*W = \psi(1)$.

Hence, $WW^* \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a projection. Since ψ is a *-homomorphism, for all $f \in C(X)$ with $f \geq 0$,

$$W^*\phi(f^2)W = W^*\phi(f)WW^*\phi(f)W$$

and hence,

$$W^*\phi(f)(1 - WW^*)\phi(f)W = 0,$$

which implies that

$$WW^*\phi(f)(1 - WW^*) = 0.$$

Hence, for all $f \in C(X)$, WW^* commutes with $\phi(f)$. Hence,

$$C(X) \to (1 - WW^*)(\mathcal{M}(\mathcal{B})/\mathcal{B})(1 - WW^*) : f \mapsto (1 - WW^*)\phi(f)$$

is a *-homomorphism. Also, note that for all $f \in C(X)$,

$$WW^*\phi(f) = WW^*\phi(f)WW^* = W\psi(f)W^*$$

From the above, for all $f \in C(X)$,

$$\phi(f) = (1 - WW^*)\phi(f) \oplus W\psi(f)W^*.$$

We can therefore take

$$\sigma(f) =_{df} (1 - WW^*)\phi(f)$$

for all $f \in C(X)$. □

The above result immediately implies the following unital version:

Corollary 2.3. *Let $\mathcal{B}$ be a simple nonunital separable C*-algebra such that $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite and has finitely many ideals.*

*Let X be a compact metric space and $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a unital *-monomorphism.*

*Say that $\psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a unital *-homomorphism such that for all $f \in C(X)$, $\psi(f) \in Ideal(\phi(f))$.*

*Then there exists an isometry $W \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ and a unital *-homomorphism*

$$\sigma : C(X) \to (1 - WW^*)(\mathcal{M}(\mathcal{B})/\mathcal{B})(1 - WW^*)$$

such that

$$\phi(.) = W\psi(.)W^* + \sigma(.).$$

Recall that for a nonunital C*-algebra $\mathcal{B}$, for a C*-algebra $\mathcal{C}$ and for extensions $\phi, \psi : \mathcal{C} \to \mathcal{M}(\mathcal{B})/\mathcal{B}$, ϕ and ψ are said to be *weakly unitarily equivalent* if there exists a unitary $U \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that

$$\phi(.) = U\psi(.)U^*.$$

Recall also that Lin proved that for every separable simple nonunital and nonelementary C*-algebra $\mathcal{B}$, $\mathcal{B}$ has continuous scale if and only if $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite (see Theorem 1.1). In particular, such a $\mathcal{B}$ satisfies the more technical assumption on ideals in the previous results of this section (Th. 2.1, Cor. 2.2 and Cor. 2.3).

Proposition 2.4. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra, and let X be a compact metric space.*

Then any two unital essential extensions of $\mathcal{B}$ by $C(X)$ that are weakly unitarily equivalent are unitarily equivalent (i.e., with unitary coming from $\mathcal{M}(\mathcal{B})$).

Proof. Say that $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are unital *-monomorphisms that are weakly unitarily equivalent. Hence, let $U \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a unitary such that

$$U\phi(.)U^* = \psi(.).$$

Pick a point $x_0 \in X$, and let $\rho : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be the unital *-homomorphism given by

$$\rho(f) =_{df} f(x_0)1$$

for all $f \in C(X)$.

By Collorary 2.3, let $W \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ be an isometry and let $\sigma : C(X) \rightarrow (1 - WW^*)(\mathcal{M}(\mathcal{B})/\mathcal{B})(1 - WW^*)$ be a unital *-homomorphism such that

$$\phi(.) = W\rho(.)W^* + \sigma(.).$$

Since $WW^* \sim 1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}$ and since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite, let $V_0 \in WW^*(\mathcal{M}(\mathcal{B})/\mathcal{B})WW^*$ be a unitary such that

$$V =_{df} U(V_0 + (1 - WW^*))$$

is a unitary in the path-connected component of the identity in the unitary group of $\mathcal{M}(\mathcal{B})/\mathcal{B}$.

Note that since ρ is centre-valued,

$$V\phi(.)V^* = U\phi(.)U^* = \psi(.)$$

and V can be lifted to a unitary in $\mathcal{M}(\mathcal{B})$. □

A modification of the above argument, replacing Corollary 2.3 with Corollary 2.2, gives us the nonunital case. However, we will give a different proof (though similar in spirit) where $C(X)$ is replaced with a general separable unital C*-algebra $\mathcal{A}$.

Proposition 2.5. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra, and let $\mathcal{A}$ be a separable unital C*-algebra.*

Then any two nonunital essential extensions of $\mathcal{B}$ by $\mathcal{A}$ that are weakly unitarily equivalent are unitarily equivalent (i.e., with unitary coming from $\mathcal{M}(\mathcal{B})$).

Proof. Say that $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B}$ are nonunital *-monomorphisms which are weakly unitarily equivalent. Hence, let $U \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a unitary such that

$$U\phi(.)U^* = \psi(.).$$

Since ϕ is nonunital, $1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} - \phi(1)$ is nonzero. Since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite, let $V_0 \in (1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} - \phi(1))(\mathcal{M}(\mathcal{B})/\mathcal{B})(1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} - \phi(1))$ be a unitary such that

$$V =_{df} U(V_0 + (1_{\mathcal{M}(\mathcal{B})/\mathcal{B}} - \phi(1))$$

is a unitary in the path-connected component of the identity in the unitary group of $\mathcal{M}(\mathcal{B})/\mathcal{B}$. Hence,

$$V\phi(.)V^* = U\phi(.)U^* = \psi(.)$$

and V can be lifted to a unitary in $\mathcal{M}(\mathcal{B})$. □

Remark 1. *Recall that all our extensions are assumed to be essential. Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra such that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$, and let X be a compact metric space.*

*Then there is addition on the class of unital extensions of $\mathcal{B}$ by $C(X)$. More precisely, say that $\phi, \psi : C(X) \rightarrow \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two *-monomorphisms. Then the* BDF sum *of ϕ and ψ is given by*

$$S\phi(.)S^* + T\psi(.)T^*$$

where $S, T \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ are isometries such that $SS^* + TT^* = 1$.

Note that, by Proposition 2.4 and 2.5, the above sum is well-defined up to unitary equivalence. Thus, the above sum induces an addition and hence a semigroup structure on $\mathbf{Ext}_u(C(X), \mathcal{B})$ (and also on $\mathbf{Ext}(C(X), \mathcal{B})$).

Henceforth, in the rest of this paper (except for Subsection 3.1), whenever $\mathcal{B}$ is a simple continuous scale C*-algebra with $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$ and whenever $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two homomorphisms with at least one injective, if we write

$$\phi \oplus \psi$$

then we mean that we are taking a BDF-sum of ϕ *and* ψ, which, by Remark 1, is well defined up to unitary equivalence.

Theorem 2.6. *Let* $\mathcal{B}$ *be a nonunital separable simple continuous scale C*-algebra such that* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$*. Let* $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *be unital *-homomorphisms with* ϕ *injective.*

*Then there exists a unital *-homomorphism* $\sigma : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *such that*

$$\phi \sim \psi \oplus \sigma.$$

Moreover, we can require σ *to be injective.*

Proof. Apply Corollary 2.3 to ϕ and $\psi \oplus \psi$ to get that either

$$\phi \sim \psi \oplus \psi$$

or there exists a unital *-homomorphism σ' such that

$$\phi \sim \psi \oplus \psi \oplus \sigma'.$$

In the former case, we can take $\sigma =_{df} \psi$, and in the latter case, we take $\sigma =_{df} \psi \oplus \sigma'$. Note that since ϕ is injective, σ is injective. □

Again, by a similar argument, we get the nonunital case.

Theorem 2.7. *Let* $\mathcal{B}$ *be a nonunital separable simple continuous scale C*-algebra such that* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *in* $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$*. Let* $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *be *-homomorphisms with* ϕ *injective and nonunital.*

*Then there exists a *-homomorphism* $\sigma : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *such that*

$$\phi = \psi \oplus \sigma.$$

Moreover, we can require σ *to be injective.*

3. When $\mathbf{Ext}_u$ is a group

For a unital C*-algebra $\mathcal{C}$, recall that $U(\mathcal{C})$ denotes the unitary group of $\mathcal{C}$, and $U(\mathcal{C})_0$ denotes the path-connected component of the identity of $U(\mathcal{C})$.

Lemma 3.1. *Let* $\mathcal{C}$ *be a unital simple purely infinite C*-algebra, and let* $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in \mathcal{C}$ *be nonzero projections such that*

(1) $p_j \sim q_j$ *for all* $1 \leq j \leq n$*, and*
(2) *either*

$$\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1_{\mathcal{C}}$$

or

$$\sum_{j=1}^n p_j \neq 1_{\mathcal{C}} \neq \sum_{j=1}^n q_j.$$

Then there exists $u \in U(\mathcal{C})_0$ *such that*

$$up_j u^* = q_j$$

for all $1 \leq j \leq n$.

Proof. Since $\mathcal{C}$ is simple purely infinite, if $\sum_{j=1}^n p_j \neq 1_{\mathcal{C}} \neq \sum_{j=1}^n q_j$ then $1_{\mathcal{C}} - \sum_{j=1}^n p_j \sim 1_{\mathcal{C}} - \sum_{j=1}^n q_j$. Hence, we may assume that

$$\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1_{\mathcal{C}}.$$

The case $n = 1$ is clear. So let us assume that $n \geq 2$.

Now suppose that $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in \mathcal{C}$ are nonzero projections for which $p_j \sim q_j$ for all j and $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$.

Hence, let $v \in U(\mathcal{C})$ be such that $vp_j v^* = q_j$ for all j.

Since $\mathcal{C}$ is simple purely infinite, let $w \in U(p_1 \mathcal{C} p_1)$ be such that v is homotopy equivalent to $w + (1 - p_1)$ in $U(\mathcal{C})$. Hence, $u =_{df} v(w^* + (1 - p_1)) \in U(\mathcal{C})_0$ and for all j, $up_j u^* = vp_j v^* = q_j$. □

We thank Professor Huaxin Lin for pointing out the next result and its proof to us.

Theorem 3.2. *Let* $\mathcal{B}$ *be a nonunital separable simple continuous scale C*-algebra, and let* X *be a zero dimensional compact metric space.*

Suppose that $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ *are unital *-monomorphisms such that*

$$K_0(\phi) = K_0(\psi).$$

Then there exists a unitary $U \in \mathcal{M}(\mathcal{B})$ *such that*

$$\phi(f) = \pi(U)\psi(f)\pi(U)^*$$

for all $f \in C(X)$.

Proof. Let $b \in \mathcal{B}_+$ be a strictly positive element with $\|b\| = 1$.

Let $\{n_j\}_{j=1}^\infty$ be a sequence of positive integers and $\{E_{i_1,i_2,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k, 1 \leq k < \infty}$ a collection of clopen subsets of X such that the following statements are true:

(1) For all k, $E_{i_1,i_2,...,i_k} \cap E_{i'_1,i'_2,...,i'_k} = \emptyset$ if $i_j \neq i'_j$ for some j.
(2) $X = \bigcup_{i=1}^{n_1} E_i$.
(3) For all k, $E_{i_1,i_2,...,i_k} = \bigcup_{i=1}^{n_{k+1}} E_{i_1,i_2,...,i_k,i}$.
(4) For all k, $diam(E_{i_1,...,i_k}) < 1/100^k$.

For all $i_1, i_2, ..., i_k$, let $e_{i_1,i_2,...,i_k} \in C(X)$ be the projection given by $e_{i_1,...,i_k} =_{df} \chi_{E_{i_1,i_2,...,i_k}}$, where

$$\chi_{E_{i_1,i_2,...,i_k}} : X \to [0, 1]$$

is the characteristic function of $E_{i_1,i_2,...,i_k}$. Let $p_{i_1,...,i_k}, q_{i_1,...,i_k} \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ be projections that are given by $p_{i_1,...,i_k} =_{df} \phi(e_{i_1,...,i_k})$ and $q_{i_1,...,i_k} =_{df} \psi(e_{i_1,...,i_k})$.

We claim that there exists a sequence $\{u_k\}_{k=1}^\infty$ of unitaries in $\mathcal{M}(\mathcal{B})$ and collections $\{A_{i_1,i_2,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k, 1 \leq k < \infty}$, $\{A'_{i_1,i_2,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k, 1 \leq k < \infty}$, $\{B_{i_1,i_2,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k, 1 \leq k < \infty}$ and $\{B'_{i_1,i_2,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k, 1 \leq k < \infty}$ in $\mathcal{M}(\mathcal{B})_+$ such that the following statements hold:

(1) $\|A_{i_1,i_2,...,i_k}\| = \|A'_{i_1,i_2,...,i_k}\| = \|B_{i_1,i_2,...,i_k}\| = \|B'_{i_1,i_2,...,i_k}\| \leq 1$ for all $i_1, i_2, ..., i_k$.

(2) $A_{i_1,i_2,...,i_k} \in her((A'_{i_1,i_2,...,i_k} - 1/2)_+)$ and
$B_{i_1,i_2,...,i_k} \in her((B'_{i_1,i_2,...,i_k} - 1/2)_+)$ for all $i_1, i_2, ..., i_k$.
(3) $A'_{i_1,i_2,...,i_k} \perp A'_{i'_1,i'_2,...,i'_k}$ and $B'_{i_1,i_2,...,i_k} \perp B'_{i'_1,i'_2,...,i'_k}$ if $i_j \neq i'_j$ for some j.
(4) $A'_{i_1,i_2,...,i_k,i} \in her(A_{i_1,i_2,...,i_k})$ and $B'_{i_1,i_2,...,i_k,i} \in her(B_{i_1,i_2,...,i_k})$ for all $i_1, i_2, ..., i_k, i$.
(5) $\pi(A_{i_1,...,i_k}) = \pi(A'_{i_1,...,i_k}) = p_{i_1,...,i_k}$ and
$\pi(B_{i_1,...,i_k}) = \pi(B'_{i_1,...,i_k}) = q_{i_1,...,i_k}$ for all $i_1, ..., i_k$.
(6) For all k, let $v_k =_{df} u_k u_{k-1}...u_1$. Then $v_k A_{i_1,...,i_l} v_k^* \in her(B_{i_1,i_2,...,i_l})$ and $v_k^* B_{i_1,...,i_l} v_k \in her(A_{i_1,i_2,...,i_l})$ for all $i_1, ..., i_l$ and for all $l \leq k$.
(7) $\pi(v_k) p_{i_1,...,i_l} \pi(v_k)^* = q_{i_1,...,i_l}$ for all $i_1, i_2, ..., i_l$ and for all $l \leq k$.
(8) $\|v_k b - v_{k-1} b\|, \|b v_k - b v_{k-1}\| < 1/100^k$ for all k.

The construction will be by induction on k.

Basis step:

Since $K_0(\phi) = K_0(\psi)$, $p_i \sim q_i$ for $1 \leq i \leq n$. Hence, let $u'_1 \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a unitary such that $u'_1 p_i (u'_1)^* = q_i$ for all i. Since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite and by Lemma 3.1, we may choose u'_1 to be in the connected component of the identity of the unitary group of $\mathcal{M}(\mathcal{B})/\mathcal{B}$.

Let $A_i, A'_i \in \mathcal{M}(\mathcal{B})_+$ with $\|A_i\| = \|A'_i\| \leq 1$ be such that $A_i \in her((A'_i - 1/2)_+)$, $\pi(A_i) = \pi(A'_i) = p_i$ for $1 \leq i \leq n$, and $A'_i \perp A'_j$ for $i \neq j$. Since $u'_1 \in U(\mathcal{M}(\mathcal{B})/\mathcal{B})_0$, lift u'_1 to a unitary $u_1 \in \mathcal{M}(\mathcal{B})$.

Let $B_i, B'_i \in \mathcal{M}(\mathcal{B})_+$ be given by $B_i =_{df} u_1 A_i (u_1)^*$ and $B'_i =_{df} u_1 A'_i (u_1)^*$, for $1 \leq i \leq n$.

Then $\|B_i\| = \|B'_i\| \leq 1$ for all i, $B_i \in her((B'_i - 1/2)_+)$, $B'_i \perp B'_j$ for $i \neq j$, and $\pi(B_i) = \pi(B'_i) = q_i$ for all i.

Induction step:

Suppose that $\{u_l\}_{l=1}^k$, $\{A_{i_1,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k}$, $\{A'_{i_1,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k}$, $\{B_{i_1,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k}$ and $\{B'_{i_1,...,i_k}\}_{1 \leq i_j \leq n_j, 1 \leq j \leq k}$ have been constructed.

By the induction hypothesis, $\pi(v_k) p_{i_1,...,i_l} \pi(v_k)^* = q_{i_1,...,i_l}$, $v_k A_{i_1,...,i_l} v_k^* \in her(B_{i_1,...,i_l})$, and $v_k^* B_{i_1,...,i_l} v_k \in her(A_{i_1,...,i_l})$ for all $i_1, i_2, ..., i_l$ and for all $l \leq k$.

So for all $i_1, ..., i_k$, for all i, $\pi(v_k) p_{i_1,...,i_k,i} \pi(v_k)^* \leq q_{i_1,...,i_k}$.

Let $u'_{i_1,...,i_k} \in her(q_{i_1,...,i_k})$ be a unitary such that for all i, $u'_{i_1,i_2,...,i_k} \pi(v_k) p_{i_1,...,i_k,i} \pi(v_k)^* (u'_{i_1,i_2,...,i_k})^* = q_{i_1,...,i_k,i}$. Since $q_{i_1,...,i_k}(\mathcal{M}(\mathcal{B})/\mathcal{B}) q_{i_1,...,i_k}$ is simple purely infinite and by Lemma 3.1, we may assume that $u'_{i_1,i_2,...,i_k}$ is in the connected component of the identity in $U(q_{i_1,...,i_k}(\mathcal{M}(\mathcal{B})/\mathcal{B}) q_{i_1,...,i_k})$.

Fix $\vec{i} =_{df} (i_1, i_2, ..., i_k)$.

Since $u'_{i_1,i_2,...,i_k} \in U(q_{i_1,...,i_k}(\mathcal{M}(\mathcal{B})/\mathcal{B}) q_{i_1,...,i_k})_0$ and since $q_{i_1,...,i_k}(\mathcal{M}(\mathcal{B})/\mathcal{B}) q_{i_1,...,i_k}$ is simple purely infinite, by [42], let $C_{\vec{i}}, D_{\vec{i}} \in her(q_{i_1,...,i_k})_{SA}$ be such that $u'_{i_1,i_2,...,i_k} = e^{iC_{\vec{i}}} e^{iD_{\vec{i}}}$.

Let $C'_{\vec{i}}, D'_{\vec{i}} \in her(B_{i_1,..,i_k})_{SA}$ be such that $\pi(C'_{\vec{i}}) = C_{\vec{i}}$ and $\pi(D'_{\vec{i}}) = D_{\vec{i}}$.

Let $\{e_m\}$ be an approximate unit for $\mathcal{B}$. For all m, let $C'_{\vec{i}}(m) =_{df} (C'_{\vec{i}})_+^{1/2}(1 - e_m)(C'_{\vec{i}})_+^{1/2} - (C'_{\vec{i}})_-^{1/2}(1 - e_m)(C'_{\vec{i}})_-^{1/2}$ and $D'_{\vec{i}}(m) =_{df} (D'_{\vec{i}})_+^{1/2}(1 - e_m)(D'_{\vec{i}})_+^{1/2} - (D'_{\vec{i}})_-^{1/2}(1 - e_m)(D'_{\vec{i}})_-^{1/2}$

Hence, $C'_{\vec{i}}(m), D'_{\vec{i}}(m) \to 0$ strictly as $m \to \infty$.

Hence, $u_{\vec{i}}(m) =_{df} e^{iC'_{\vec{i}}(m)} e^{iD'_{\vec{i}}(m)} \to 1_{\mathcal{M}(\mathcal{B})}$ strictly as $m \to \infty$.

Note that $\pi(u_{\vec{i}}(m)) = u'_{i_1,i_2,...,i_k} + (1 - q_{i_1,i_2,...,i_k})$ and $u_{\vec{i}}(m) \in her(B_{i_1,...,i_k})^{\sim}$ for all m.

Note also that since $B_{i_1,...,i_k} \perp B_{i'_1,...,i'_k}$ if $i_j \neq i'_j$ for some j, we must have that $u_{\vec{i}}(m)u_{\vec{i}'}(m) = u_{\vec{i}'}(m)u_{\vec{i}}(m)$ where $\vec{i} =_{df} (i_1, i_2, ..., i_k)$ and $\vec{i}' =_{df} (i'_1, i'_2, ..., i'_k)$, for all $\vec{i}, \vec{i}'$ and for all m.

Choose $m_1 \geq 1$ large enough such that if $u_{k+1} =_{df} \prod_{\vec{i}} u_{\vec{i}}(m_1)$ and $v_{k+1} =_{df} u_{k+1}v_k$ then $\|bv_{k+1} - bv_k\|, \|v_{k+1}b - v_kb\| < 1/100^{k+1}$.

Now, $u_{k+1} \in \prod_{i_1,...,i_k} U(her(B_{i_1,...,i_k})^{\sim})$

(And recall that since $B_{i_1,...,i_k} \perp B_{i'_1,...,i'_k}$ if $i_j \neq i'_j$ for some j, every element of $her(B_{i_1,...,i_k})^{\sim}$ commutes with every element of $her(B_{i'_1,...,i'_k})^{\sim}$ if $i_j \neq i'_j$ for some j.)

Hence, for all $i_1, ..., i_l$, for all $l \leq k$, $u_{k+1}B_{i_1,...,i_l}u^*_{k+1} \subseteq her(B_{i_1,...,i_l})$ and $u^*_{k+1}B_{i_1,...,i_l}u_{k+1} \subseteq her(B_{i_1,...,i_l})$.

Hence, for all $i_1, ..., i_l$, for all $l \leq k$, $v_{k+1}A_{i_1,...,i_l}v^*_{k+1} \subseteq her(B_{i_1,...,i_l})$ and $v^*_{k+1}B_{i_1,...,i_l}v_{k+1} \subseteq her(A_{i_1,...,i_l})$.

Note that $\pi(v_{k+1})p_{i_1,...,i_l}\pi(v_{k+1})^* = q_{i_1,...,i_l}$ for all $i_1, ..., i_l$ for all $l \leq k$.

For all $i_1, ..., i_k$, let $\{B_{i_1,..,i_k,i}\}_{i=1}^{n_{k+1}}$ and $\{B'_{i_1,...,i_k,i}\}_{i=1}^{n_{k+1}}$ be two collections of pairwise orthogonal contractive elements of $her(B_{i_1,...,i_k})_+$ such that $B_{i_1,...,i_k,i} \in her((B'_{i_1,...,i_k,i} - 1/2)_+)$ and $\pi(B_{i_1,...,i_k,i}) = \pi(B'_{i_1,...,i_k,i}) = q_{i_1,...,i_k,i}$ for all i.

Then define $A_{i_1,...,i_k,i_{k+1}} =_{df} v^*_{k+1}B_{i_1,...,i_k,i_{k+1}}v_{k+1}$ and $A'_{i_1,...,i_k,i_{k+1}} =_{df} v^*_{k+1}B'_{i_1,...,i_k,i_{k+1}}v_{k+1}$ for all $i_1, ..., i_{k+1}$.

This completes the inductive construction.

Let $v =_{df} \lim_{k\to\infty} v_k \in U(\mathcal{M}(\mathcal{B}))$, where (by the Claim) the limit converges strictly in $\mathcal{M}(\mathcal{B})$.

Therefore, by the Claim, $vA_{i_1,...,i_k}v^* \subseteq her(B'_{i_1,...,i_k})$ and $v^*B_{i_1,...,i_k}v \subseteq her(A'_{i_1,...,i_k})$.

Therefore, $\pi(v)p_{i_1,...,i_k}\pi(v)^* = q_{i_1,...,i_k}$ for all $i_1, ..., i_k$.

Therefore, $\pi(v)\phi(f)\pi(v)^* = \psi(f)$ for all $f \in C(X)$. □

A similar proof yields the nonunital case.

Theorem 3.3. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C^*-algebra, and let X be a zero dimensional compact metric space.*

*Suppose that $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are both nonunital *-monomorphisms such that*

$$K_0(\phi) = K_0(\psi).$$

Then there exists a unitary $U \in \mathcal{M}(\mathcal{B})$ such that

$$\phi(f) = \pi(U)\psi(f)\pi(U)^*$$

for all $f \in C(X)$.

Proof. Note that since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite and since both ϕ and ψ are nonunital,

$$1 - \phi(1) \sim 1 - \psi(1).$$

Hence, conjugating with a unitary (and using Proposition 2.5) if necessary, we may assume that $\phi(1) = \psi(1)$.

Now proceed as in the proof of Theorem 3.2. □

The concepts of *null* and *totally trivial* extensions (see 2 and 7) are due to Lin (e.g., see [26] and [30]), though we have modified the definitions. Early versions of these concepts were already present in [4].

Recall that in the original BDF case, when X is a compact subset of the plane, uniqueness of the trivial element of $Ext(C(X), \mathcal{K})$ follows from the Weyl–von Neumann–Berg Theorem. Recall also that for a simple separable real rank zero C*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ has the classical Weyl–von Neumann Theorem if and only if $\mathcal{M}(\mathcal{B})$ has real rank zero (e.g., [46], [47]; see also [27]). This is perhaps one clue for the reasons for the assumption that $\mathcal{M}(\mathcal{B})$ has real rank zero in [26], [30] and other early papers.

The next definition is for both the unital and nonunital cases:

Definition 2. *Let $\mathcal{B}$ be a simple nonunital separable continuous scale C*-algebra.*

Let X be a compact metric space and let $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be an essential extension.

(1) *ϕ is said to be* null *if there exists a commutative AF-subalgebra $\mathcal{C} \subset \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that $Ran(\phi) \subseteq \mathcal{C}$ and $[p] = 0$ for every projection $p \in \mathcal{C}$.*
(2) *Suppose, in addition, that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$. ϕ is said to be* self-absorbing *if $\phi \oplus \phi \sim \phi$.*

Theorem 3.4. *Let $\mathcal{B}$ be a nonunital simple separable C*-algebra with continuous scale and let X be a compact metric space. Then we have the following:*

(1) *There exists a null essential extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$. Moreover, we can require ϕ to be nonunital or unital (if, additionally, $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$).*

   Suppose, in addition, that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$. Then we have the following:

(2) *Every null essential extension $C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is self-absorbing.*
(3) *Any two unital self-absorbing essential extensions $C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ are unitarily equivalent. The same holds for any two nonunital self-absorbing essential extensions.*
(4) *Every self-absorbing essential extension must be null.*

Proof. (1): We firstly construct a nonunital null essential extension. (The construction for the unital case is similar.) Since $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite, choose a nonzero projection $q \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that $[q]_{K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})} = 0$ and q is a proper subprojection of $1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}$. There is a unital embedding $O_2 \hookrightarrow q(\mathcal{M}(\mathcal{B})/\mathcal{B})q$. By a classical result of topology, there also exists a unital embedding $C(X) \hookrightarrow C(K)$, where K is the Cantor space. Since there is also a unital embedding $C(K) \hookrightarrow O_2$, the above maps compose to a unital embedding $C(X) \hookrightarrow q(\mathcal{M}(\mathcal{B})/\mathcal{B})q$ with range contained in a commutative AF-algebra which is zero in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$. We thus get a nonunital null extension of $\mathcal{B}$ by $C(X)$.

For the unital case, note that, by assumption, $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$. Proceed as in the argument for the nonunital case, replacing q with $1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}$.

(2) follows from Theorems 3.2 (in the unital case) and 3.3 (in the nonunital case).

(3): We prove the unital case. The proof of the nonunital case is similar.

Suppose that ϕ and ψ are both unital self-absorbing essential extensions of $\mathcal{B}$ by $C(X)$. By Theorem 2.6, let $\rho : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a unital (essential) extension such that $\phi \sim \rho \oplus \psi$. Then

$$\begin{aligned} & \phi \oplus \psi \\ \sim \quad & \rho \oplus \psi \oplus \psi \\ \sim \quad & \rho \oplus \psi \text{ (since } \psi \text{ is self-absorbing)} \\ \sim \quad & \phi. \end{aligned}$$

By a similar argument, $\phi \oplus \psi \sim \psi$. Hence, $\phi \sim \psi$.

(4) follows from (1), (2) and (3). □

We remind the reader that, throughout this paper, when we write "extension", we mean essential extension (though often we will add "essential").

Theorem 3.5. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra such that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$.*

Let X be a compact metric space.

Then $\mathbf{Ext}_u(C(X), \mathcal{B})$ is a group where the zero element is the class of a null essential extension. The same holds for $\mathbf{Ext}(C(X), \mathcal{B})$.

Proof. Again, we prove the unital case, and the proof of the nonunital case is similar.

By Remark 1, we have that $\mathbf{Ext}_u(C(X), \mathcal{B})$, with the BDF sum, is a commutative semigroup. Hence, it suffices to prove that $\mathbf{Ext}_u(C(X), \mathcal{B})$ has a neutral element and that every element has an inverse.

Note that by Theorem 3.2, all null unital extensions are unitarily equivalent.

Let $\sigma : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be a null unital extension.

Let $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be an arbitrary unital extension. By Theorem 2.6, there exists a unital extension $\phi_0 : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that

$$\phi = \phi_0 \oplus \sigma.$$

By Theorem 3.4, $\sigma \oplus \sigma \sim \sigma$. Hence, $\phi \oplus \sigma \sim \phi_0 \oplus \sigma \oplus \sigma \sim \phi_0 \oplus \sigma \sim \phi$. Hence, $[\sigma]$ gives a zero element for the semigroup $\mathbf{Ext}_u(C(X), \mathcal{B})$.

Again, let $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be an arbitrary unital extension. We will prove that $[\phi]$ has an inverse in $\mathbf{Ext}_u(C(X), \mathcal{B})$.

By Theorem 2.6, there exists a unital extension $\phi_0 : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ such that $\sigma \sim \phi_0 \oplus \phi$. Then $[\phi_0]$ is the inverse of $[\phi]$.

Since ϕ was arbitrary, $\mathbf{Ext}_u(C(X), \mathcal{B})$ is a group.

The argument for the case of $\mathbf{Ext}(C(X), \mathcal{B})$ is exactly the same, with the appropriate modifications. □

3.1. **Ext is a group.** In this short subsection, we briefly diverge from the main exposition and describe how **Ext** can be given group structure even when $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] \neq 0$, by following an idea of [9] which generalizes the BDF sum in a natural way.

Definition 3. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra and let X be a compact metric space.*

Let $\phi, \psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ be two nonunital essential extensions.

Then the (generalized) BDF sum *of ϕ and ψ is given by*

$$S\phi(.)S^* + T\psi(.)T^*$$

where $S, T \in \mathcal{M}(\mathcal{B})/\mathcal{B}$ are two isometries such that $SS^* + TT^* \leq 1$.

Remark 2. *One can show, using Proposition 2.5, that, up to unitary equivalence, the sum defined in Definition 3 is independent of the choices of* S *and* T*. Thus, even without the assumption that* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$*, one can give a semigroup structure to* $\mathbf{Ext}(C(X), \mathcal{B})$*.*

We emphasize that the key to Proposition 2.5 (and thus to the well-definedness, up to unitary equivalence, of this version of the BDF-sum) is that the relevant extensions are nonunital. In fact, this addition will be well-defined even when $C(X)$ *is replaced by a general separable unital C*-algebra (as in Proposition 2.5). However, since this paper does not focus on such a general setting, we will not continue to proceed down this path. We do additionally point out, though, that our proof of well-definededness of the BDF-sum in the unital setting (i.e.,* $\mathbf{Ext}_u(C(X), \mathcal{B})$*; see Remark 1) utilizes that* $C(X)$ *has a one-dimensional *-representation (see the proof of Proposition 2.4) and thus does not immediately generalize to arbitrary unital separable C*-algebras.*

The unital setting can be tricky and must be dealt with carefully. For example, for all $n \geq 3$*, any two nonunital essential extensions of* $\mathcal{K}$ *by* $\mathbb{M}_n$ *are unitarily equivalent (and hence, trivially, any two such which are weakly unitarily equivalent are unitarily equivalent). On the other hand, there are pairs of unital essential extensions of* $\mathcal{K}$ *by* $\mathbb{M}_n$ *which are weakly unitarily equivalent but not unitarily equivalent. (See, for example, the discussion in* [2] *15.4.1(b) and 15.6.6(a).) Since this paper focusses on* $C(X)$*, we will not elaborate further on this.*

Many concepts and results (with the same proofs) work with the new sum in the nonunital setting. Thus, one can remove the hypothesis that $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ *from Theorems 2.6 and 2.7, Definition 2, and Theorems 3.4, 3.5 and 4.10.*

To keep the paper short, we here state only the new version of Theorem 3.5:

Theorem 3.6. *Let* $\mathcal{B}$ *be a nonunital separable simple continuous scale C*-algebra and let* X *be a compact metric space.*

Then, with the addition operation defined as in Definition 3, $\mathbf{Ext}(C(X), \mathcal{B})$ *is a group where the zero element is the class of a null essential extension.*

Remark 3. *Suppose that* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$*. Then the group* $\mathbf{Ext}(C(X), \mathcal{B})$ *defined in Theorem 3.6 (using the generalized BDF-sum in Definition 3) will be isomorphic to the group* $\mathbf{Ext}(C(X), \mathcal{B})$ *defined in Theorem 3.5 (using the sum in Remark 1 and which requires the assumption* $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$*). This is not hard to see.*

To save notation, let us denote the first group by $\mathbf{Ext}^{sum1}$ *and the second group by* $\mathbf{Ext}^{sum2}$*.*

Firstly, note that for any nonunital essential extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$*, the class of* ϕ *in* $\mathbf{Ext}^{sum1}$ *is the same as the class of* ϕ *in* $\mathbf{Ext}^{sum2}$ *(both are unitary equivalence classes), which we denote by* $[\phi]$*.*

Hence, we have a natural map (the identity map)

$$\mathbf{Ext}^{sum2} \to \mathbf{Ext}^{sum1} : [\phi] \mapsto [\phi]. \tag{3.1}$$

This map is definitely bijective.

The sum defined in Remark 1 (for $\mathbf{Ext}^{sum2}$*) is a special case of the sum defined in Definition 3 (for* $\mathbf{Ext}^{sum1}$*), and thus, the map in (3.1) preserves addition.*

It is not hard to see that the neutral element (or the class of an essential self-absorbing extension) gets mapped to the neutral element, and thus the map in (3.1) is a group isomorphism.

4. Null extensions

In the previous sections, we defined the class of null essential extensions which are the self-absorbing essential extensions and which give the zero element in $\mathbf{Ext}_u$ and $\mathbf{Ext}$.

In this section, we show that totally trivial extensions (see Definition 7) with zero K_0 are null extensions.

Recall that for a compact convex set K, $Aff(K)$ is the collection of all real-valued affine continuous functions on K. With the uniform norm and the natural order, $Aff(K)$ is an ordered Banach space.

Recall also that $Aff(K)_{++}$ denotes the functions in $Aff(K)$ which are strictly positive at every point in K.

Next, we remind the reader about our standing assumption that all quasitraces are traces.

Let $\mathcal{C}$ be a unital stably finite simple C*-algebra. Recall that for all n and for all $c \in \mathbb{M}_n(\mathcal{C})_+$, c induces an element

$$\widehat{c} \in Aff(T(\mathcal{C})) \tag{4.1}$$

which is given by

$$\widehat{c}(\tau) =_{df} \tau \otimes Tr_n(c)$$

for all $\tau \in T(\mathcal{C})$, where Tr_n is the nonnormalized trace on $\mathbb{M}_n$. Note that if $c \neq 0$ then $\widehat{c} \in Aff(T(\mathcal{C}))_{++}$.

Also, to simplify notation, we often write "τ" for $\tau \otimes Tr_n$ or $\tau \otimes Tr_{\mathcal{K}}$, where $Tr_{\mathcal{K}}$ is the standard densely defined (norm-) lower semicontinuous trace on $\mathcal{K}_+$. For a nonunital simple σ-unital C*-algebra $\mathcal{B}$ with densely defined (norm-) lower semicontinuous trace τ, we will often write "τ" also for the strictly continuous extension to $\mathcal{M}(\mathcal{B})_+$.

Recall that there is a well-defined ordered group homomorphism

$$\chi : K_0(\mathcal{C}) \to Aff(T(\mathcal{C})) \tag{4.2}$$

which is determined by

$$\chi([p]) =_{df} \widehat{[p]} =_{df} \widehat{p}$$

for every projection $p \in \mathcal{C} \otimes \mathcal{K}$.

(Finally, recall, from the remarks on notation at the beginning, that when $\mathcal{C}$ is nonunital, we take $T(\mathcal{C})$ be the class of densely defined lower semicontinuous traces that are normalized at a fixed nonzero positive element of the Pedersen ideal of $\mathcal{C}$.)

Lemma 4.1. *Let $\mathcal{A}$ be a unital separable simple nonelementary C*-algebra with real rank zero, stable rank one and weakly unperforated K_0 group. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of nonzero projections in $\mathcal{A} \otimes \mathcal{K}$ such that*

$$\sum_{n=1}^{\infty} \widehat{p}_n \in Aff(T(\mathcal{A})),$$

and let S be an arbitrary infinite subset of $\mathbb{Z}_+$.

Then for all $n \in S$, there exists a subprojection $q_n \leq p_n$ such that

$$\sum_{n \notin S} \widehat{p}_n + \sum_{n \in S} \widehat{q}_n \in \chi(K_0(\mathcal{A})_+).$$

(χ is as defined in (4.2).)

Moreover, we can choose $\{q_n\}$ *so that for all* n, q_n *and* $p_n - q_n$ *are nonzero projections.*

Note that as $\sum_{n=1}^{\infty} \widehat{p}_n, \sum_{n \notin S} \widehat{p}_n, \sum_{n \in S} \widehat{q}_n \in Aff(T(\mathcal{A}))$, *these sums all converge uniformly on* $T(\mathcal{A})$.

Proof. Say that S corresponds to the (strictly increasing) subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers.

For all $k \geq 1$, let

$$\alpha_k =_{df} \inf_{\tau \in T(\mathcal{A})} \tau(p_{n_k}) > 0.$$

($\alpha_k > 0$ since $\mathcal{A}$ is simple.)

Now $\chi(K_0(\mathcal{A}))$ is uniformly dense in $Aff(T(\mathcal{A}))$. (See, for example, [2] Theorem 6.9.3, [3] III.) Hence, we can find a function $f \in \chi(K_0(\mathcal{A})_+ - \{0\}) \subseteq Aff(T(\mathcal{A}))_{++}$ such that

$$f(\tau) < \sum_{n=1}^{\infty} \tau(p_n)$$

and

$$|f(\tau) - \sum_{n=1}^{\infty} \tau(p_n)| < \alpha_1/10$$

for all $\tau \in T(\mathcal{A})$.

Again, since $\chi(K_0(\mathcal{A}))$ is uniformly dense in $Aff(T(\mathcal{A}))$, we can find a sequence $\{r_k\}_{k=1}^{\infty}$ of nonzero projections in $\mathcal{A} \otimes \mathcal{K}$ such that

$$\tau(r_k) < \alpha_k$$

and

$$f(\tau) + \sum_{n=1}^{\infty} \tau(r_k) = \sum_{n=1}^{\infty} \tau(p_n)$$

for all $\tau \in T(\mathcal{A})$.

Since $\mathcal{A} \otimes \mathcal{K}$ has strict comparison of projections, $r_k \preceq p_{n_k}$ for all $\tau \in T(\mathcal{A})$ and for all k. Hence, for all k, let $s_{n_k} \leq p_{n_k}$ be a projection such that $r_k \sim s_{n_k}$. And let $q_{n_k} =_{df} p_{n_k} - s_{n_k}$ (which is necessarily a nonzero projection in $p_{n_k}(\mathcal{A} \otimes \mathcal{K})p_{n_k}$).

Then

$$\sum_{n \notin S} \widehat{p}_n + \sum_{n \in S} \widehat{q}_n = \sum_{n=1}^{\infty} \widehat{p}_n - \sum_{k=1}^{\infty} \widehat{r}_k = f \in \chi(K_0(\mathcal{A})_+)$$

as required.

□

For a C*-algebra $\mathcal{D}$, we let $Proj(\mathcal{D})$ denote the projections in $\mathcal{D}$.

Definition 4. *Let* X *be a metric space and let* $\mathcal{B}$ *be a nonunital C*-algebra. A countable subset* $\Lambda \subset X \times Proj(\mathcal{B})$ *is called a* clump *over* $X \times \mathcal{B}$ *if the following statements are true:*

1. *The set* $\{x \in X : \exists p \in Proj(\mathcal{B})$ *s.t.* $(x, p) \in \Lambda\}$ *is dense in* X.
2. *For all* $x \in X$, *if* $(x, p) \in \Lambda$ *for some projection* p, *then* $(x, q) \in \Lambda$ *for infinitely many distinct projections* q.
3. *For all* $(x, p), (y, q) \in \Lambda$, *if either* $x \neq y$ *or* $p \neq q$ *then* $p \perp q$.
4. *Let* $S \subseteq \{p \in Proj(\mathcal{B}) : \exists x \in X$ *s.t.* $(x, p) \in \Lambda\}$. *Then* $\sum_{p \in S} p$ *converges strictly in* $\mathcal{M}(\mathcal{B})$.

Let X be a metric space and $\mathcal{B}$ a nonunital C*-algebra. Suppose that Λ is a clump over $X \times \mathcal{B}$. We denote

$$\Lambda_X =_{df} \{x \in X : \exists p \in \mathcal{B} \text{ s.t. } (x, p) \in \Lambda\}$$

and

$$\Lambda_{\mathcal{B}} =_{df} \{p \in \mathcal{B} : \exists x \in X \text{ s.t. } (x, p) \in \Lambda\}.$$

Let X be a compact metric space, $\mathcal{B}$ a nonunital C*-algebra and $\phi : C(X) \to \mathcal{M}(\mathcal{B})$ a *-homomorphism. Suppose $\{x_n\}_{n=1}^{\infty}$ is a dense sequence in X such that each term repeats infinitely many times, and suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence of nonzero pairwise orthogonal projections in $\mathcal{B}$ such that

$$\phi(f) = \sum_{n=1}^{\infty} f(x_n) p_n$$

for all $f \in C(X)$, where the sum converges strictly in $\mathcal{M}(\mathcal{B})$. Then the set $\Lambda_\phi =_{df} \{(x_n, p_n) : n \geq 1\}$ is a clump on $X \times \mathcal{B}$, and we say that Λ_ϕ is a *clump affiliated with* ϕ. Note that more than one clump can be affiliated with a $*$-homomorphism.

Definition 5. *Let X be a metric space and $\mathcal{B}$ a nonunital C*-algebra. Suppose that Λ and Γ are clumps over $X \times \mathcal{B}$.*

Then Γ is said to be obtained from Λ by a splitting operation *if there exist sequences of points $\{x_k\}$, $\{y_k\}$ and $\{z_k\}$ in X, and sequences of pairwise orthogonal projections $\{p_k\}$, $\{q_k\}$, $\{r_k\}$ in $\mathcal{B}$ such that*

(a) $\Lambda_0 =_{df} \{(x_k, p_k) : k \geq 1\} \subseteq \Lambda$,
(b) $p_k = q_k + r_k$ *for all* k,
(c) $d(x_k, y_k), d(x_k, z_k) \to 0$, *and*
(d) $\Gamma = (\Lambda - \Lambda_0) \cup \{(y_k, q_k), (z_k, r_k) : k \geq 1\}$.

Note that in the above definition, every projection in $\Gamma_{\mathcal{B}}$ is a subprojection of some projection in $\Lambda_{\mathcal{B}}$.

Note also that the above definition implies that

$$\sum_{(x,p) \in \Lambda} p = \sum_{(y,q) \in \Gamma} q,$$

and for every bounded uniformly continuous function f on X,

$$\sum_{(x,p) \in \Lambda} f(x) p - \sum_{(y,q) \in \Gamma} f(y) q \in \mathcal{B},$$

where all sums converge strictly in $\mathcal{M}(\mathcal{B})$. Since this remark and its conclusion are used multiple times, we will make a definition.

Definition 6. *Let X be a metric space and $\mathcal{B}$ a nonunital C*-algebra. Suppose that Λ, Γ are clumps over $X \times \mathcal{B}$.*

We write $\Lambda \sim \Gamma$ if

$$\sum_{(x,p) \in \Lambda} p = \sum_{(y,q) \in \Gamma} q,$$

and for every bounded uniformly continuous function $f : X \to \mathbb{C}$,

$$\sum_{(x,p) \in \Lambda} f(x) p - \sum_{(y,q) \in \Lambda} f(y) q \in \mathcal{B}.$$

Lemma 4.2. *Let X be a metric space and let $\mathcal{B}$ be a nonunital separable simple C^*-algebra with real rank zero, stable rank one and weakly unperforated K_0 group.*

Let $Y, Z \subset X$ be disjoint subsets such that $X = Y \cup Z$ and $dist(Y, Z) = 0$.

Suppose that Λ is a clump over $X \times \mathcal{B}$ such that $\Lambda_X \cap Y$ is dense in Y, $\Lambda_X \cap Z$ is dense in Z, and

$$\sum_{(x,p)\in\Lambda} \widehat{p} \in Aff(T(\mathcal{B})).$$

Then there exists a clump Γ over $X \times \mathcal{B}$ such that the following statements are true:

i. Γ *is obtained from* Λ *by a splitting operation.*
ii. $\sum_{(x,p)\in\Gamma,\ x\in Y} \widehat{p} \in \chi(K_0(\mathcal{B})_+)$.
(See (4.2) for the definition of χ*.)*
iii. $\Lambda_X = \Gamma_X$.
iv. *Every projection in* $\Gamma_{\mathcal{B}}$ *is a subprojection of some projection in* $\Lambda_{\mathcal{B}}$.

Proof. We can express Λ as

$$\Lambda = \{(x_n, p_n) : n \geq 1\} \cup \{(x'_n, p'_n) : n \geq 1\}$$

where for all m, n, $x_m \in Y, x'_m \in Z$, $p_m \perp p'_n$, and if $m \neq n$ then $p_m \perp p_n$ and $p'_m \perp p'_n$.

Recall that, by the definition of clump, each term in $\{x_n\}$ repeats infinitely many times, and each term in $\{x'_n\}$ also repeats infinitely many times.

Since $dist(Y, Z) = 0$ and since $\Lambda_X \cap Y$ and $\Lambda_X \cap Z$ are dense in Y and Z respectively, we can find subsequences $\{x_{n_k}\}$, $\{x'_{m_k}\}$ of $\{x_n\}$, $\{x'_n\}$ respectively such that $d(x_{n_k}, x'_{m_k}) \to 0$.

Also, since $\sum_{(x,p)\in\Lambda} \widehat{p} \in Aff(T(\mathcal{B}))$, it follows that

$$\sum_{n\geq 1} \widehat{p}_n \in Aff(T(\mathcal{B})).$$

By Lemma 4.1, let $\{q_{n_k}\}$ be a sequence of projections in $\mathcal{B}$ such that

(1) $q_{n_k} \leq p_{n_k}$ for all k,
(2) both $q_{n_k} \neq 0$ and $r_{n_k} =_{df} p_{n_k} - q_{n_k} \neq 0$ for all k, and
(3) $\sum_{n\neq n_k \forall k} \widehat{p}_n + \sum_{k\geq 1} \widehat{q}_{n_k} \in \chi(K_0(\mathcal{B})_+)$.

Define

$$\Gamma =_{df} (\Lambda - \{(x_{n_k}, p_{n_k}) : k \geq 1\}) \cup \{(x_{n_k}, q_{n_k}), (x'_{m_k}, r_{n_k}) : k \geq 1\}.$$

□

Lemma 4.3. *Let $\mathcal{B}$ be a separable nonunital C^*-algebra and let X be a metric space. Let $Y_1, Y_2, ..., Y_N \subseteq X$ be subsets such that $\overline{Y_1 \cup Y_2 \cup ... \cup Y_N} = X$. Say that Λ is a clump over $X \times \mathcal{B}$.*

Then there exists a clump Γ over $X \times \mathcal{B}$ such that

(1) $\Lambda \sim \Gamma$,
(2) $\Gamma_X \subseteq Y_1 \cup Y_2 \cup ... \cup Y_N$,
(3) $\Gamma_X \cap Y_n$ *is dense in* Y_n *for all* n, *and*
(4) $\Gamma_{\mathcal{B}} = \Lambda_{\mathcal{B}}$.

Proof. For all $1 \leq n \leq N$, let $\{y_{n,k}\}_{k=1}^{\infty}$ be a dense sequence in Y_n.

We construct an increasing sequence

$$\Gamma_1 \subset \Gamma_2 \subset .... \subset \Gamma_m \subset ...$$

of finite subsets of $(\bigcup_{n=1}^{N} Y_n) \times \mathcal{B}$ and a decreasing sequence

$$\Lambda_1 \supset \Lambda_2 \supset .... \supset \Lambda_m \supset ...$$

of clumps over $X \times \mathcal{B}$.

The construction is by induction on m.

Basis step $m = 1$:

Since Λ is a clump, we can find $(x_1, p_1), (x_2, p_2), ..., (x_N, p_N) \in \Lambda$ such that $(x_k, p_k) \neq (x_l, p_l)$ for all $k \neq l$ and $d(x_j, y_{j,1}) < 1/10$ for all $1 \leq j \leq N$.

Let

$$\Gamma_1 =_{df} \{(y_{j,1}, p_j) : 1 \leq j \leq N\}$$

and

$$\Lambda_1 =_{df} \Lambda - \{(x_j, p_j) : 1 \leq j \leq N\}.$$

Note that Λ_1 is also a clump over $X \times \mathcal{B}$.

Induction step:

Say that Γ_m and Λ_m have already been constructed. We now construct Γ_{m+1} and Λ_{m+1}.

By the induction hypothesis, Λ_m is a clump over $X \times \mathcal{B}$. Hence, we can find $\{(x_{j,l}, p_{j,l}) : 1 \leq j \leq N, 1 \leq l \leq m+1\} \subseteq \Lambda_m$ such that $(x_{j,l}, p_{j,l}) \neq (x_{i,k}, p_{i,k})$ if $(j,l) \neq (i,k)$. and

$$d(y_{j,l}, x_{j,l}) < \frac{1}{10^{m+1}}$$

for all j, l.

Let

$$\Gamma_{m+1} =_{df} \Gamma_m \cup \{(y_{j,l}, p_{j,l}) : 1 \leq j \leq N, 1 \leq l \leq m+1\}$$

and

$$\Lambda_{m+1} =_{df} \Lambda_m - \{(x_{j,l}, p_{j,l}) : 1 \leq j \leq N, 1 \leq l \leq m+1\}.$$

Note that Λ_{m+1} is also a clump over $X \times \mathcal{B}$.

The inductive construction is complete.

Let

$$\Gamma_\infty =_{df} \bigcup_{m=1}^{\infty} \Gamma_m$$

and let

$$\Lambda_\infty =_{df} \bigcap_{m=1}^{\infty} \Lambda_m.$$

Note that Γ_∞ is a clump over $X \times \mathcal{B}$ and $(\Gamma_\infty)_X \cap Y_n$ is dense in Y_n for all n.

If $\Lambda_\infty = \emptyset$, then we can take $\Gamma =_{df} \Gamma_\infty$ and we would be done.

Suppose that $\Lambda_\infty \neq \emptyset$. Let us also assume that Λ_∞ is an infinite set. (For the case where Λ_∞ is finite, the argument is the same as that of the infinite case, but easier.)

Since Λ_∞ is an infinite set, let $\{(x'_j, p'_j) : 1 \leq j < \infty\}$ be an enumeration of Λ_∞. For all j, let $z_j \in \{y_{n,k} : n, k \geq 1\}$ be such that $d(x'_j, z_j) < \frac{1}{10^j}$.

Let

$$\Gamma =_{df} \Gamma_\infty \cup \{(z_j, p'_j) : 1 \leq j < \infty\}.$$

Then Γ is the required clump over $X \times \mathcal{B}$.

□

Recall that by a *closed ball*, we mean the closed unit ball of $\mathbb{R}^n$ for some n. If E is a closed ball, by $Int(E)$ we mean the interior of E (as a subset of the appropriate $\mathbb{R}^n$), i.e., the open unit ball of $\mathbb{R}^n$.

We will consider singleton points to be 0-dimensional closed balls, and (as a simplifying convention for this paper), we will define the interior $Int(\{x_0\})$ of a singleton $\{x_0\}$ to be $\{x_0\}$.

The next lemma follows immediately from the definition of finite CW complex, since we may cut each n-cell into smaller pieces with the desired diameters.

Lemma 4.4. *Let X be a finite CW complex with a metric, and let $\epsilon > 0$ be given.*

Then we can find a collection $\{L_n\}_{n=1}^N$ of pairwise disjoint subsets of X such that

(a) *$diameter(L_n) < \epsilon$, for all n,*
(b) *$X = \bigcup_{n=1}^N \overline{L_n}$, and*
(c) *for all n, there exists a closed ball E_n and a continuous surjection $\rho_n : E_n \rightarrow \overline{L_n}$ such that $\rho_n(Int(E_n)) = L_n$ and $\rho_n|_{Int(E_n)} : Int(E_n) \rightarrow L_n$ is a homeomorphism.*

A compact metric space X is called a *compact cell* if there exists a closed ball E, an open dense subset $X^0 \subseteq X$, and a continuous surjection $\rho : E \rightarrow X$ such that $\rho(Int(E)) = X^0$ and the restriction map $\rho|_{Int(E)} : Int(E) \rightarrow X^0$ is a homeomorphism. Sometimes, we denote X^0 by $Int(X)$.

Lemma 4.5. *Let X be a compact cell with a metric, and let $\epsilon > 0$ be given.*

Then we can find a collection $\{L_n\}_{n=1}^N$ of pairwise disjoint subsets of $Int(X)$ such that

(a) *$diameter(L_n) < \epsilon$, for all n,*
(b) *$X = \bigcup_{n=1}^N \overline{L_n}$, and*
(c) *for all n, there exists a closed ball E_n and a continuous surjection $\rho_n : E_n \rightarrow \overline{L_n}$ such that $\rho_n(Int(E_n)) = L_n$ and $\rho_n|_{Int(E_n)} : Int(E_n) \rightarrow L_n$ is a homeomorphism.*

Lemma 4.6. *Let X be a compact cell with a metric and let $\mathcal{B}$ be a nonunital separable simple C*-algebra with real rank zero, stable rank one and weakly unperforated K_0 group.*

Let $\epsilon > 0$ be given.

Let Λ be a clump over $X \times \mathcal{B}$ such that

$$\sum_{(x,p) \in \Lambda} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

Then there exists a clump Γ over $X \times \mathcal{B}$ and there exists a sequence $\{L_n\}_{n=1}^N$ of pairwise disjoint subsets of $Int(X)$ such that the following statements are true:

i. $X = \overline{\bigcup_{n=1}^N L_n}$.
ii. *$diam(L_n) < \epsilon$, for all n.*
iii. *For all n, there exists a closed ball E and there exists a continuous surjection $\rho : E \to \overline{L_n}$ such that $\rho(Int(E)) = L_n$ and $\rho|_{Int(E)} : Int(E) \to L_n$ is a homeomorphism.*
iv. $\Lambda \sim \Gamma$.
v. *For all n, $\Gamma_X \cap L_n$ is a dense subset of L_n.*
vi. $\Gamma_X \subseteq \bigcup_{n=1}^N L_n$.
vii. *Every projection in $\Gamma_{\mathcal{B}}$ is a subprojection of a projection in $\Lambda_{\mathcal{B}}$.*
viii. *For all n, $\sum_{(x,p)\in\Gamma,\ x\in L_n} \widehat{p} \in \chi(K_0(\mathcal{B}))$.*

Proof. Apply Lemma 4.5 to X and ϵ to get a collection $\{L_n\}_{n=1}^N$ of pairwise disjoint subsets of $Int(X)$, a collection $\{E_n\}_{n=1}^N$ of closed balls, and a collection $\{\rho_n\}_{n=1}^N$ of continuous maps that satisfy conditions (a)-(c) of Lemma 4.5.

By applying Lemma 4.3 to X and $\{L_n\}_{n=1}^N$, we may assume that $\Lambda_X \subseteq \bigcup_{n=1}^N L_n$ and for all n, $\Lambda_X \cap L_n$ is dense in L_n.

Since X is connected and since $X = \bigcup_{n=1}^N \overline{L_n}$, there exists a finite sequence $\{m_k\}_{k=1}^M$ of positive integers such that

(a) $\{m_k : 1 \leq k \leq M\} = \{1, 2, 3, ..., N\}$,
(b) $m_M = N$, and
(c) for all $1 \leq k \leq M-1$, $dist(L_{m_k}, L_{m_{k+1}}) = 0$.

We now repeatedly apply Lemma 4.2. In particular, we will apply Lemma 4.2 $M-1$ times to get a sequence $\Lambda_1, \Lambda_2, ..., \Lambda_M$ of clumps over $X \times \mathcal{B}$.

This is the procedure:

Let $\Lambda_1 =_{df} \Lambda$.

Let $1 \leq k \leq M-1$. Suppose that we have obtained Λ_k. We apply Lemma 4.2 with $Y = L_{m_k}$ and $Z = L_{m_{k+1}}$ to obtain Λ_{k+1}.

Note that by Lemma 4.2, $(\Lambda_{k+1})_X = (\Lambda_k)_X$, and every projection in $(\Lambda_{k+1})_{\mathcal{B}}$ is a subprojection of a projection in $(\Lambda_k)_{\mathcal{B}}$.

Now let

$$\Gamma =_{df} \Lambda_M.$$

Then, by construction, Γ is a clump over $X \times \mathcal{B}$, $\Gamma \sim \Lambda$, $\Gamma_X = \Lambda_X$, and every projection in $\Gamma_{\mathcal{B}}$ is a subprojection of some projection in $\Lambda_{\mathcal{B}}$.

Also, by our construction, for all $1 \leq n \leq N-1$,

$$\sum_{(x,p)\in\Gamma,\ x\in L_n} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

But by hypothesis and since splitting operations result in $\sim$ equivalent clumps, we have that

$$\sum_{n=1}^N \sum_{(x,p)\in\Gamma,\ x\in L_n} \widehat{p} = \sum_{(x,p)\in\Gamma} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

Hence, since $\chi(K_0(\mathcal{B}))$ is a group,

$$\sum_{(x,p)\in\Gamma,\ x\in L_N} \widehat{p} = \sum_{(x,p)\in\Gamma} \widehat{p} - \sum_{n=1}^{N-1} \sum_{(x,p)\in\Gamma,\ x\in L_n} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

□

Lemma 4.7. *Let X be a finite CW complex with a metric and let $\mathcal{B}$ be a nonunital separable simple C*-algebra with real rank zero, stable rank one and weakly unperforated K_0 group.*

Let $\epsilon > 0$ be given.

Let Λ be a clump over $X \times \mathcal{B}$ such that for every connected component $C \subseteq X$,

$$\sum_{(x,p)\in\Lambda,\ x\in C} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

Then there exists a clump Γ over $X \times \mathcal{B}$ and there exists a sequence $\{L_n\}_{n=1}^N$ of pairwise disjoint subsets of X such that the following statements are true:

i. $X = \overline{\bigcup_{n=1}^N L_n}$.
ii. *$diam(L_n) < \epsilon$, for all n.*
iii. *For all n, there exists a closed ball E and there exists a continuous surjection $\rho : E \to \overline{L_n}$ such that $\rho(Int(E)) = L_n$ and $\rho|_{Int(E)} : Int(E) \to L_n$ is a homeomorphism.*
iv. $\Lambda \sim \Gamma$.
v. *For all n, $\Gamma_X \cap L_n$ is a dense subset of L_n.*
vi. $\Gamma_X \subseteq \bigcup_{n=1}^N L_n$.
vii. *Every projection in $\Gamma_{\mathcal{B}}$ is a subprojection of a projection in $\Lambda_{\mathcal{B}}$.*
viii. *For all n, $\sum_{(x,p)\in\Gamma,\ x\in L_n} \widehat{p} \in \chi(K_0(\mathcal{B}))$.*

Proof. The proof is very similar to the proof of Lemma 4.6, where Lemma 4.5 is replaced with Lemma 4.4 and where we work in each connected component of X. (Note that X has finitely many connected components.) □

Definition 7. *Let $\mathcal{B}$ be a nonunital separable C*-algebra, and let X be a compact metric space.*

An essential extension $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is totally trivial *if there exist a strictly converging properly increasing sequence $\{e_n\}_{n=1}^\infty$ of projections in $\mathcal{B}$, and a dense sequence $\{x_n\}_{n=1}^\infty$ in X, with each term repeating infinitely many times, such that*

$$\phi = \pi \circ \psi$$

*where $\psi : C(X) \to \mathcal{M}(\mathcal{B})$ is the *-homomorphism given by $\psi(f) =_{df} \sum_{n=1}^\infty f(x_n)(e_n - e_{n-1})$, and where $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the quotient map. (Here, $e_0 =_{df} 0$.)*

Sometimes, to save writing, we call a *-homomorphism $\psi : C(X) \to \mathcal{M}(\mathcal{B})$ a *totally trivial extension* if it has the form in Definition 7 above. Note that such a ψ has an affiliated clump over $X \times \mathcal{B}$. (See the second paragraph after Definition 4.)

We require the following result, parts of which were first proven by Lin in 1991:

Theorem 4.8. *Let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra with real rank zero, stable rank one, and weakly unperforated K_0 group. Then we have the following:*

(1) $(K_0(\mathcal{M}(\mathcal{B})), K_0(\mathcal{M}(\mathcal{B}))_+) = (Aff(T(\mathcal{B})), Aff(T(\mathcal{B}))_{++})$.
(2) *For any two projections $P, Q \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$, $P \sim Q$ if and only if $\tau(P) = \tau(Q)$ for all $\tau \in T(\mathcal{B})$.*

(3) *For any* $f \in Aff(T(\mathcal{B}))_{++}$*, there exists* $k \geq 1$ *and a projection* $P \in \mathbb{M}_k \otimes \mathcal{M}(\mathcal{B}) - \mathbb{M}_k \otimes \mathcal{B}$ *such that* $\widehat{P} = f$*. Moreover, if* $f(\tau) < \tau(1_{\mathcal{M}(\mathcal{B})})$ *for all* $\tau \in T(\mathcal{B})$*, then we can choose* $P \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$*.*

(4) *The six-term exact sequence (for the ideal* $\mathcal{B} \subset \mathcal{M}(\mathcal{B})$*) induces a short exact sequence*

$$0 \to Aff(T(\mathcal{B}))/\chi(K_0(\mathcal{B})) \to K_0(\mathcal{M}(\mathcal{B})/\mathcal{B}) \to K_1(\mathcal{B}) \to 0.$$

Proof. The first three statements were proven in [25]. A more widely available version is [35] Theorem 1.4. (See also [9] and [37].)

The last statement is proven in [35] Corollary 1.5. □

Recall that for a nonunital C*-algebra $\mathcal{B}$, $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is the standard quotient map.

The next lemma, once more, works for both the nonunital and unital cases.

Lemma 4.9. *Let* X *be a finite CW complex and let* $\mathcal{B}$ *be a nonunital separable simple continuous scale C*-algebra with real rank zero, stable rank one and weakly unperforated* K_0 *group.*

Suppose that $\phi : C(X) \to \mathcal{M}(\mathcal{B})$ *is a totally trivial essential extension with*

$$K_0(\pi \circ \phi) = 0.$$

Then there exists a commutative AF subalgebra $\mathcal{D} \subseteq \mathcal{M}(\mathcal{B})$ *with* $1_{\mathcal{D}} = \phi(1_{C(X)})$ *and* $\mathcal{D} \cap \mathcal{B} = \{0\}$ *such that*

i. *every element of* $Ran(\phi)$ *commutes with every element of* $\mathcal{D}$*,*
ii. $Ran(\pi \circ \phi) \subseteq \pi(\mathcal{D})$*, and*
iii. $\widehat{p} \in \chi(K_0(\mathcal{B}))$*, for every projection* $p \in \mathcal{D}$*.*

In particular, $\pi \circ \phi$ *is a null essential extension.*

Proof. Since X is metrizable, let us assume that there is a metric on X.

Replacing $\mathcal{B}$ with $\phi(1)\mathcal{B}\phi(1)$ if necessary, we may assume that and $\phi(1) = 1_{\mathcal{M}(\mathcal{B})}$.

Let Λ_ϕ be a clump affiliated with ϕ. Note that by Theorem 4.8, the hypothesis that $K_0(\pi \circ \phi) = 0$ implies that for every connected component $C \subseteq X$, $\sum_{(x,p) \in \Lambda_\phi, x \in C} \widehat{p} \in \chi(K_0(\mathcal{B}))$.

We will construct an increasing sequence

$$\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_3 \subseteq ....$$

of finite dimensional commutative unital C*-subalgebras of $\mathcal{M}(\mathcal{B})$ for which the following statements are true:

(1) For every $n \geq 1$, there exist a collection $\{X_{n,k}\}_{k=1}^{N_n}$ of pairwise disjoint subsets of X such that $X = \overline{\bigcup_{k=1}^{N_n} X_{n,k}}$.
(2) For all n, k, $diam(\overline{X_{n,k}}) < \frac{1}{n}$.
(3) For all n, k, there exists an l such that $X_{n+1,k} \subseteq X_{n,l}$.
(4) For all n, k, there exists a closed ball $E_{n,k}$ and a continuous surjection $\rho_{n,k} : E_{n,k} \to \overline{X_{n,k}}$ such that $\rho_{n,k}(Int(E_{n,k})) = X_{n,k}$ and the map $\rho_{n,k}|_{Int(E_{n,k})} : Int(E_{n,k}) \to X_{n,k}$ is a homeomorphism.
(5) For all n, there exist projections $P_{n,1}, P_{n,2}, ..., P_{n,N_n} \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ which are the minimal projections of $\mathcal{D}_n$. For all n, k, $P_{n,k}$ commutes with every element of $Ran(\phi)$, and $\widehat{P}_{n,k} \in \chi(K_0(\mathcal{B}))$.

(6) For every n, there exists a totally trivial unital extension

$$\phi_n : C(X) \to \mathcal{M}(\mathcal{B})$$

with an affiliated clump Λ_{ϕ_n} such that $\phi_n(f) - \phi(f) \in \mathcal{B}$ for all $f \in C(X)$, and every projection in $(\Lambda_{\phi_n})_{\mathcal{B}}$ is a subprojection of a projection in $(\Lambda_{\phi})_{\mathcal{B}}$. Moreover, for all n, k, $P_{n,k}$ commutes with every element of $Ran(\phi_n)$.

(7) For every n, k, the kernel of the *-homomorphism

$$C(X) \to P_{n,k}\mathcal{M}(\mathcal{B})P_{n,k} : f \mapsto P_{n,k}\phi_n(f)P_{n,k} = \phi_n(f)P_{n,k}$$

is $C_0(X - \overline{X_{n,k}})$. In other words, the above *-homomorphism induces a unital *-monomorphism

$$\phi_{n,k} : C(\overline{X_{n,k}}) \to P_{n,k}\mathcal{M}(\mathcal{B})P_{n,k}$$

where for all $f \in C(X)$,

$$\phi_{n,k}(f|_{\overline{X_{n,k}}}) = \phi_n(f)P_{n,k}.$$

$\phi_{n,k}$ will be a unital totally trivial extension of $P_{n,k}\mathcal{B}P_{n,k}$ by $C(\overline{X_{n,k}})$ such that $K_0(\pi \circ \phi_{n,k}) = 0$.

(8) For every n, k, there exists a totally trivial unital extension

$$\psi_{n,k} : C(E_{n,k}) \to P_{n,k}\mathcal{M}(\mathcal{B})P_{n,k}$$

with $K_0(\pi \circ \psi_{n,k}) = 0$ such that for all $f \in C(X)$,

$$\pi \circ \psi_{n,k}(f \circ \rho_{n,k}) = \pi \circ \phi_{n,k}(f|_{\overline{X_{n,k}}}).$$

(9) For all n, k, there is a clump $\Lambda_{\psi_{n,k}}$ on $E_{n,k} \times P_{n,k}\mathcal{B}P_{n,k}$ affiliated with $\psi_{n,k}$ such that $(\Lambda_{\psi_{n,k}})_{\mathcal{B}} \subseteq (\Lambda_{\phi_n})_{\mathcal{B}}$, and thus, every projection in $(\Lambda_{\psi_{n,k}})_{\mathcal{B}}$ is a subprojection of some projection in $(\Lambda_{\phi})_{\mathcal{B}}$.

We denote the above statements by "(*)".

The construction (and proof of the statements in (*)) will be by induction on n.

Basis step $n = 1$.

Since X is a finite CW complex, X can be partitioned $X = \bigsqcup_{j=1}^{J} K_j$ where for all j, K_j is clopen and connected. By hypothesis, $\widehat{\phi(\chi_{K_j})} \in \chi(K_0(\mathcal{B}))$ for all j. (Here, χ_{K_j} is the characteristic function on K_j.) Hence, replacing $\mathcal{B}$ with $\phi(\chi_{K_j})\mathcal{B}\phi(\chi_{K_j})$ $(1 \leq j \leq J)$ if necessary, we may assume that X is connected. (Note that for all j, $\mathcal{M}(\phi(\chi_{K_j})\mathcal{B}\phi(\chi_{K_j})) = \phi(\chi_{K_j})\mathcal{M}(\mathcal{B})\phi(\chi_{K_j})$.)

The rest of the Basis step essentially follows immediately from Lemma 4.7. More precisely, we proceed as follows:

We have that Λ_ϕ is a clump over $X \times \mathcal{B}$ and, by hypothesis, $\sum_{(x,p) \in \Lambda_\phi} \widehat{p} \in \chi(K_0(\mathcal{B}))$. Hence, by Lemma 4.7, let Λ' be a clump over $X \times \mathcal{B}$ and let $\{X_{1,k}\}_{k=1}^{N_1}$ be a collection of pairwise disjoint subsets of X such that the following statements are true:

i. $X = \overline{\bigcup_{k=1}^{N_1} X_{1,k}}$.
ii. $diam(\overline{X_{1,k}}) < \frac{1}{2}$ for all $1 \leq k \leq N_1$.
iii. For all $1 \leq k \leq N_1$, there exist a closed ball $E_{1,k}$ and a continuous surjection $\rho_{1,k} : E_{1,k} \to \overline{X_{1,k}}$ such that $\rho_{1,k}(Int(E_{1,k})) = X_{1,k}$ and the map $\rho_{1,k}|_{Int(E_{1,k})} : Int(E_{1,k}) \to X_{1,k}$ is a homeomorphism.

iv. $\Lambda' \sim \Lambda_\phi$.
v. For all k, $\Lambda'_X \cap X_{1,k}$ is dense in $X_{1,k}$.
vi. $\Lambda'_X \subseteq \bigcup_{k=1}^{N_1} X_{1,k}$.
vii. Every projection in $\Lambda'_{\mathcal{B}}$ is a subprojection of a projection in Λ_ϕ.
viii. For all k, $\sum_{(x,p)\in\Lambda',\ x\in X_{1,k}} \widehat{p} \in \chi(K_0(\mathcal{B}))$.

Let $\phi_1 : C(X) \to \mathcal{M}(\mathcal{B})$ be the totally trivial unital extension given by

$$\phi_1(f) =_{df} \sum_{(x,p)\in\Lambda'} f(x)p$$

for all $f \in C(X)$.

Since Λ' is a clump affiliated with ϕ_1, we define

$$\Lambda_{\phi_1} =_{df} \Lambda'.$$

Since $\Lambda \sim \Lambda'$,

$$\pi \circ \phi_1 = \pi \circ \phi,$$

and since $\Lambda_{\phi_1} = \Lambda'$, every projection in $(\Lambda_{\phi_1})_{\mathcal{B}}$ is a subprojection of a projection in Λ_ϕ.

For all $1 \leq k \leq N_1$, let

$$P_{1,k} =_{df} \sum_{(x,p)\in\Lambda',\ x\in X_{1,k}} p \in \mathcal{M}(\mathcal{B}),$$

where the sum converges strictly. Hence, $P_{1,k}$ is a projection in $\mathcal{M}(\mathcal{B})$ such that $P_{1,k}$ commutes with every element of $Ran(\phi) \cup Ran(\phi_1)$, $\widehat{P}_{1,k} \in \chi(K_0(\mathcal{B}))$, and $\sum_{k=1}^{N_1} P_{1,k} = 1_{\mathcal{M}(\mathcal{B})}$. Note that by Definition 4 (2), $P_{1,k} \notin \mathcal{B}$.

Moreover, it is clear that for all $1 \leq k \leq N_1$, the kernel of the map

$$C(X) \to P_{1,k}\mathcal{M}(\mathcal{B})P_{1,k} : f \mapsto \phi_1(f)P_{1,k}$$

is $C_0(X - \overline{X_{1,k}})$, and we get an induced totally trivial unital extension

$$\phi_{1,k} : C(\overline{X_{1,k}}) \to P_{1,k}\mathcal{M}(\mathcal{B})P_{1,k}$$

where for all $f \in C(X)$,

$$\phi_{1,k}(f|_{\overline{X_{1,k}}}) = \phi_1(f)P_{1,k}.$$

And by Theorem 4.8 (2), since $\overline{X_{1,k}}$ is path-connected, the range of $K_0(\phi_{1,k})$ is the subgroup of $K_0(\mathcal{M}(\mathcal{B}))$ generated by $[P_{1,k}]$. Hence, $K_0(\pi \circ \phi_{1,k}) = 0$.

For all $1 \leq k \leq N_1$, let

$$\psi_{1,k} : C(E_{1,k}) \to P_{1,k}\mathcal{M}(\mathcal{B})P_{1,k}$$

be given by:

$$\psi_{1,k}(f) =_{df} \sum_{y=(\rho_{1,k})^{-1}(x),\ x\in X_{1,k},\ (x,p)\in\Lambda'} f(y)p$$

for all $f \in C(E_{1,k})$.

Note that $\psi_{1,k}$ is a totally trivial unital extension of $P_{1,k}\mathcal{B}P_{1,k}$ by $C(E_{1,k})$,

$$\psi_{1,k}(f \circ \rho_{1,k}) = \phi_{1,k}(f)$$

for all $f \in C(\overline{X_{1,k}})$,

$$\psi_{1,k}(f \circ \rho_{1,k}) = \phi_1(f)P_{1,k}$$

for all $f \in C(X)$, and $K_0(\pi \circ \psi_{1,k}) = 0$.

Let

$$\Lambda_{\psi_{1,k}} =_{df} \{(y,p) \in Int(E_{1,k}) \times \mathcal{B} : (\rho_{1,k}(y), p) \in \Lambda'\}.$$

Then $\Lambda_{\psi_{1,k}}$ is a clump affiliated with $\psi_{1,k}$ such that $(\Lambda_{\psi_{1,k}})_{\mathcal{B}} \subseteq (\Lambda_{\phi_1})_{\mathcal{B}}$. In particular, every projection in $(\Lambda_{\psi_{1,k}})_{\mathcal{B}}$ is a subprojection of a projection in Λ_{ϕ}.

Let

$$\mathcal{D}_1 =_{df} C^*(P_{1,1}, P_{1,2}, ..., P_{1,N_1}) = \mathbb{C}P_{1,1} + \mathbb{C}P_{1,2} + ... + \mathbb{C}P_{1,N_1} \cong \mathbb{C}^{N_1}.$$

This completes the basis step.

Induction step.

Suppose that $\mathcal{D}_n$, $\{X_{n,k}\}_{k=1}^{N_n}$, $\{E_{n,k}\}_{k=1}^{N_n}$, $\{\rho_{n,k}\}_{k=1}^{N_n}$, $\{P_{n,k}\}_{k=1}^{N_n}$, ϕ_n, Λ_{ϕ_n}, $\{\phi_{n,k}\}_{k=1}^{N_n}$, $\{\psi_{n,k}\}_{k=1}^{N_n}$, and $\{\Lambda_{\psi_{n,k}}\}_{k=1}^{N_n}$ have been constructed to satisfy the statements in (*).

We now construct $\mathcal{D}_{n+1}$, $\{X_{n+1,k}\}_{k=1}^{N_{n+1}}$, $\{E_{n+1,k}\}_{k=1}^{N_{n+1}}$, $\{\rho_{n+1,k}\}_{k=1}^{N_{n+1}}$, $\{P_{n+1,k}\}_{k=1}^{N_{n+1}}$, ϕ_{n+1}, $\Lambda_{\phi_{n+1}}$, $\{\phi_{n+1,k}\}_{k=1}^{N_{n+1}}$, $\{\psi_{n+1,k}\}_{k=1}^{N_{n+1}}$, and $\{\Lambda_{\psi_{n+1,k}}\}_{k=1}^{N_{n+1}}$.

By the induction hypothesis, for all n,k (with $1 \leq k \leq N_n$), $\rho_{n,k} : E_{n,k} \to \overline{X_{n,k}}$ is a continuous surjection such that $\rho_{n,k}(Int(E_{n,k})) = X_{n,k}$ and the map $\rho_{n,k}|_{Int(E_{n,k})} : Int(E_{n,k}) \to X_{n,k}$ is a homeomorphism.

Also, by the induction hypothesis, for all n,k $(1 \leq k \leq N_n)$,

$$\psi_{n,k} : C(E_{n,k}) \to P_{n,k}\mathcal{M}(\mathcal{B})P_{n,k} = \mathcal{M}(P_{n,k}\mathcal{B}P_{n,k})$$

is a totally trivial unital extension of $P_{n,k}\mathcal{B}P_{n,k}$ by $C(E_{n,k})$, $\Lambda_{\psi_{n,k}}$ is a clump over $E_{n,k} \times P_{n,k}\mathcal{B}P_{n,k}$ with $(\Lambda_{\psi_{n,k}})_{\mathcal{B}} \subseteq (\Lambda_{\phi_n})_{\mathcal{B}}$ and such that every projection in $(\Lambda_{\psi_{n,k}})_{\mathcal{B}}$ is a subprojection of a projection in $(\Lambda_{\phi})_{\mathcal{B}}$, and

$$\sum_{(x,p) \in \Lambda_{\psi_{n,k}}} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

Hence, by Lemma 4.6, for all $1 \leq k \leq N_n$, let $\Gamma_{n+1,k}$ be a clump over $E_{n,k} \times P_{n,k}\mathcal{B}P_{n,k}$ and let $\{E^g_{n+1,k,l}\}_{l=1}^{M_k}$ be a collection of pairwise disjoint subsets of $Int(E_{n,k})$ such that the following statements are true:

i. $E_{n,k} = \overline{\bigcup_{l=1}^{M_k} E^g_{n+1,k,l}}$.

ii. For all l, let $X_{n+1,k,l} =_{df} \rho_{n,k}(E^g_{n+1,k,l}) \subseteq X_{n,k} \subseteq X$.
Then $diam(\overline{X_{n+1,k,l}}) < \frac{1}{n+10}$.

iii. For all k,l, there exist a closed ball $E_{n+1,k,l}$ and a continuous surjection $\rho'_{n+1,k,l} : E_{n+1,k,l} \to \overline{E^g_{n+1,k,l}} \subseteq E_{n,k}$ such that $\rho'_{n+1,k,l}(Int(E_{n+1,k,l})) = E^g_{n+1,k,l}$ and the map $\rho'_{n+1,k,l}|_{Int(E_{n+1,k,l})} : Int(E_{n+1,k,l}) \to E^g_{n+1,k,l}$ is a homeomorphism.

iv. For all k, $\Lambda_{\psi_{n,k}} \sim \Gamma_{n+1,k}$.

v. For all k,l, $(\Gamma_{n+1,k})_{E_{n,k}} \cap E^g_{n+1,k,l}$ is a dense subset of $E^g_{n+1,k,l}$.

vi. For all k, $(\Gamma_{n+1,k})_{E_{n,k}} \subseteq \bigcup_{l=1}^{M_k} E^g_{n+1,k,l}$.

vii. Every projection in $(\Gamma_{n+1,k})_{\mathcal{B}}$ is a subprojection of a projection in $(\Lambda_{\psi_{n,k}})_{\mathcal{B}}$. Hence, every projection in $(\Gamma_{n+1,k})_{\mathcal{B}}$ is a subprojection of a projection in $(\Lambda_{\phi_n})_{\mathcal{B}} \cup (\Lambda_{\phi})_{\mathcal{B}}$.

viii. For all k,l,

$$\sum_{(x,p) \in \Gamma_{n+1,k},\ x \in E^g_{n+1,k,l}} \widehat{p} \in \chi(K_0(\mathcal{B})).$$

For all n, k, l, let

$$\rho_{n+1,k,l} : E_{n+1,k,l} \to \overline{X_{n,k}}$$

be the continuous map given by

$$\rho_{n+1,k,l} =_{df} \rho_{n,k} \circ \rho'_{n+1,k,l}.$$

Recall that $\rho'_{n+1,k,l}(Int(E_{n+1,k,l})) = E^g_{n+1,k,l} \subseteq Int(E_{n,k})$, $\rho'_{n+1,k,l}|_{Int(E_{n+1,k,l})} : Int(E_{n+1,k,l}) \to E^g_{n+1,k,l}$ is a homeomorphism, $\rho_{n,k}(Int(E_{n,k})) = X_{n,k}$, $\rho_{n,k}|_{Int(E_{n,k})} : Int(E_{n,k}) \to X_{n,k}$ is a homeomorphism, and that $X_{n+1,k,l} =_{df} \rho_{n,k}(E^g_{n+1,k,l})$.

Hence, since $E_{n+1,k,l}$ is compact, we have that $\rho_{n+1,k,l}(E_{n+1,k,l}) = \overline{X_{n+1,k,l}}$, and $\rho_{n+1,k,l} : E_{n+1,k,l} \to \overline{X_{n+1,k,l}}$ is a continuous surjection. We also have that $\rho_{n+1,k,l}(Int(E_{n+1,k,l})) = X_{n+1,k,l}$ and that the restricted map $\rho_{n+1,k,l}|_{Int(E_{n+1,k,l})} : Int(E_{n+1,k,l}) \to X_{n+1,k,l}$ is a homeomorphism.

We henceforth view $\rho_{n+1,k,l}$ as a map from $E_{n+1,k,l}$ to $\overline{X_{n+1,k,l}}$, i.e., we view $\rho_{n+1,k,l}$ as a map of the form

$$\rho_{n+1,k,l} : E_{n+1,k,l} \to \overline{X_{n+1,k,l}},$$

which is continuous and surjective.

Moreover, $\{X_{n+1,k,l}\}_{l=1}^{M_k}$ is a collection of pairwise disjoint subsets of $X_{n,k}$ such that

$$\overline{\bigcup_{l=1}^{M_k} X_{n+1,k,l}} = \overline{X_{n,k}}.$$

Hence, $\{X_{n,k,l}\}_{l=1}^{M_k}{}_{k=1}^{N_n}$ is a collection of pairwise disjoint subsets of X such that

$$X = \bigcup_{k=1}^{N_n} \bigcup_{l=1}^{M_k} \overline{X_{n,k,l}}.$$

Now for all $1 \leq k \leq N_n$, for all $1 \leq l \leq M_k$, let $P_{n+1,k,l} \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ be the projection given by

$$P_{n+1,k,l} =_{df} \sum_{(x,p) \in \Gamma_{n+1,k},\ x \in E^g_{n,k,l}} p.$$

Then for all k, l,

$$P_{n+1,k,l} \leq P_{n,k},$$

$P_{n+1,k,l}$ commutes with every element of $Ran(\phi) \cup Ran(\psi_{n,k})$, and

$$\widehat{P_{n+1,k,l}} \in \chi(K_0(\mathcal{B})).$$

Moreover, for every k,

$$P_{n,k} = \sum_{l=1}^{M_k} P_{n+1,k,l},$$

and thus,

$$1_{\mathcal{M}(\mathcal{B})} = \sum_{k=1}^{N_n} \sum_{l=1}^{M_k} P_{n+1,k,l}.$$

Let

$$\phi_{n+1} : C(X) \to \mathcal{M}(\mathcal{B})$$

be the unital *-monomorphism given by

$$\phi_{n+1}(f) =_{df} \sum_{x=\rho_{n,k}(y),\ (y,p)\in\Gamma_{n+1,k},\ 1\leq k\leq N_n} f(x)p$$

for all $f \in C(X)$.

Then

$$\pi \circ \phi = \pi \circ \phi_{n+1},$$

and for all k, l, $P_{n+1,k,l}$ commutes with every element of $Ran(\phi_{n+1})$.

Define

$$\Lambda_{\phi_{n+1}} =_{df} \{(x,p) : \exists 1 \leq k \leq N_n \exists y \in E_{n,k} \text{ s.t. } x = \rho_{n,k}(y) \text{ and } (y,p) \in \bigcup_{k=1}^{N_n} \Gamma_{n+1,k}\}.$$

Then $\Lambda_{\phi_{n+1}}$ is a clump over $X \times \mathcal{B}$ that is affiliated with ϕ_{n+1}, and every projection in $(\Lambda_{\phi_{n+1}})_{\mathcal{B}}$ is a subprojection of a projection in $(\Lambda_\phi)_{\mathcal{B}}$.

For every k, l, since the restriction map $\rho_{n,k}|_{E^g_{n+1,k,l}} : E^g_{n+1,k,l} \to X_{n+1,k,l}$ is a homeomorphism, the kernel of the map

$$C(X) \to P_{n+1,k,l}\mathcal{M}(\mathcal{B})P_{n+1,k,l} : f \mapsto P_{n+1,k,l}\phi_{n+1}(f)P_{n+1,k,l} = \phi_{n+1}(f)P_{n+1,k,l}$$

is $C_0(X - \overline{X_{n+1,k,l}})$. And thus, the above map induces a unital *-monomorphism

$$\phi_{n+1,k,l} : C(\overline{X_{n+1,k,l}}) \to P_{n+1,k,l}\mathcal{M}(\mathcal{B})P_{n+1,k,l}.$$

Also, for all k, l, let

$$\psi_{n+1,k,l} : C(E_{n+1,k,l}) \to P_{n+1,k,l}\mathcal{M}(\mathcal{B})P_{n+1,k,l}$$

be the totally trivial unital extension that is given by

$$\psi_{n+1,k,l}(f) =_{df} \sum_{(y,p)\in\Gamma_{n+1,k},\ x\in Int(E_{n+1,k,l}),\ \rho'_{n+1,k,l}(x)=y} f(x)p,$$

for all $f \in C(E_{n+1,k,l})$.

Clearly, for all $f \in C(X)$,

$$\pi \circ \psi_{n+1,k,l}(f \circ \rho_{n+1,k,l}) = \pi \circ \phi_{n+1,k,l}(f|_{\overline{X_{n+1,k,l}}}).$$

Also, by Theorem 4.8 (2), since $\overline{X_{n+1,k,l}}$ is path-connected, the range of $K_0(\phi_{n+1,k,l})$ is the subgroup of $K_0(\mathcal{M}(\mathcal{B}))$ generated by $[P_{n+1,k,l}]$. Hence, $K_0(\pi \circ \phi_{n+1,k,l}) = 0$. Similarly, $K_0(\pi \circ \psi_{n+1,k,l}) = 0$.

For every k, l, let

$$\Lambda_{\psi_{n+1,k,l}} =_{df} \{(x,p) : x \in Int(E_{n+1,k,l}) \text{ and } \exists y \text{ s.t. } \rho'_{n+1,k,l}(x) = y \text{ and } (y,p) \in \Gamma_{n+1,k}\}.$$

Then $\Lambda_{\psi_{n+1,k,l}}$ is a clump over $E_{n+1,k,l} \times P_{n+1,k,l}\mathcal{B}P_{n+1,k,l}$ which is affiliated with $\psi_{n+1,k,l}$. Moreover, $(\Lambda_{\psi_{n+1,k,l}})_{\mathcal{B}} \subseteq (\Lambda_{\phi_{n+1}})_{\mathcal{B}}$, and hence, every projection in $\Lambda_{\psi_{n+1,k,l}}$ is a subprojection of some projection in Λ_ϕ.

Let $\mathcal{D}_{n+1} \subseteq \mathcal{M}(\mathcal{B})$ be the commutative finite dimensional unital C*-subalgebra given by

$$\mathcal{D}_{n+1} =_{df} C^*(P_{n+1,k,l} : 1 \leq k \leq N_n,\ 1 \leq l \leq M_k).$$

Now, in a consistent manner, relabel

$\{X_{n+1,k,l}\}_{k=1}^{N_n}{}_{l=1}^{M_k}$, $\{E_{n+1,k,l}\}_{k=1}^{N_n}{}_{l=1}^{M_k}$, $\{\rho_{n+1,k,l}\}_{k=1}^{N_n}{}_{l=1}^{M_k}$, $\{P_{n+1,k,l}\}_{k=1}^{N_n}{}_{l=1}^{M_k}$, $\{\phi_{n+1,k,l}\}_{k=1}^{N_n}{}_{l=1}^{M_k}$, $\{\psi_{n+1,k,l}\}_{k=1}^{N_n}{}_{l=1}^{M_k}$

with

$$\{X_{n+1,k}\}_{k=1}^{N_{n+1}}, \{E_{n+1,k}\}_{k=1}^{N_{n+1}}, \{\rho_{n+1,k}\}_{k=1}^{N_{n+1}}, \{P_{n+1,k}\}_{k=1}^{N_{n+1}}, \{\phi_{n+1,k}\}_{k=1}^{N_{n+1}}, \{\psi_{n+1,k}\}_{k=1}^{N_{n+1}}$$

respectively.

This completes the inductive construction.

Let $\mathcal{D} \subseteq \mathcal{M}(\mathcal{B})$ be the commutative AF subalgebra given by

$$\mathcal{D} =_{df} \overline{\bigcup_{n=1}^{\infty} \mathcal{D}_n}.$$

Then $\mathcal{D}$ has the required properties. Firstly, property i. of Lemma 4.9 follows from (*) statement (5).

Next, property ii. of Lemma 4.9 follows from (*) statements (2), (5), (6) and (7). In more detail, let $f \in C(X)$ and $\epsilon > 0$ be given. By (*) statement (2), choose $n \geq 1$ sufficiently large so that for all k, for all $x, y \in \overline{X_{n,k}}$, $|f(x) - f(y)| < \epsilon$. Therefore, by (*) statement (7), for all k, $\phi_n(f)P_{n,k}$ is norm within ϵ of a scalar multiple of $P_{n,k}$. Hence, by (*) statements (5) and (6), for all k, $\pi \circ \phi(f)$ is norm within ϵ of an element of $\pi(\mathcal{D}_n) \subseteq \pi(\mathcal{D})$. Since ϵ was arbitrary, $\pi \circ \phi(f) \in \pi(\mathcal{D})$.

Finally, property iii. of Lemma 4.9, follows from (*) statement (5).

□

The last result, once more, works in both the unital and nonunital cases.

Theorem 4.10. *Let X be a finite CW complex and let $\mathcal{B}$ be a nonunital separable simple continuous scale C*-algebra with real rank zero, stable rank one and weakly unperforated K_0 group.*

Suppose that $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is an essential extension.

Then the following statements are equivalent:

1. *ϕ is a null extension.*
2. *ϕ is a totally trivial extension with $K_0(\phi) = 0$.*
3. *If, in addition, $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ then ϕ is self-absorbing, i.e., $\phi \oplus \phi \sim \phi$.*

Proof. Since X is metrizable, let us assume that there is a metric on X.

The equivalence of (1) and (3) follows from Theorem 3.4.

That (2) implies (1) follows from Lemma 4.9.

All that remains is to prove that (1) implies (2). Say that $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a null extension. It suffices to prove that ϕ is totally trivial.

One can easily construct a totally trivial extension $\psi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ with $K_0(\psi) = 0$ such that ψ is unital if and only if ϕ is unital. Here is a sketch of the proof: Suppose that ϕ is unital. (The proof of the nonunital case is similar and easier.) Since $K_0(\phi) = 0$, $[1_{\mathcal{M}(\mathcal{B})/\mathcal{B}}] = 0$ in $K_0(\mathcal{M}(\mathcal{B})/\mathcal{B})$. Hence, by Theorem 4.8, $\widehat{1_{\mathcal{M}(\mathcal{B})}} \in \chi(K_0(\mathcal{B}))$. Say that X is a disjoint union $X = \bigsqcup_{j=1}^{N} X_j$ where each X_j is path-connected. By Theorem 4.8, let $P_1, P_2, ..., P_N \in \mathcal{M}(\mathcal{B}) - \mathcal{B}$ be pairwise orthogonal projections such that for all j, $\widehat{P_j} \in \chi(K_0(\mathcal{B}))$, and $1_{\mathcal{M}(\mathcal{B})} = P_1 + P_2 + ... + P_N$. Since $\mathcal{B}$ has real rank zero, for all j, let $\{p_{j,l}\}_{l=1}^{\infty}$ be a sequence of pairwise orthogonal nonzero projections in $\mathcal{B}$ such that $P_j = \sum_{l=1}^{\infty} p_{j,l}$, where the sum converges strictly. Also, for all j, let $\{x_{j,l}\}_{l=1}^{\infty}$ be a dense sequence in X_j

such that each term repeats infinitely many times. Then $\psi_1 : C(X) \to \mathcal{M}(\mathcal{B}) : f \mapsto \sum_{j=1}^{N} \sum_{l=1}^{\infty} f(x_{j,l}) p_{j,l}$ induces a totally trivial unital extension. Also, since each X_j is path-connected, if we define $\psi =_{df} \pi \circ \psi_1$ then, by Theorem 4.8, we have that $K_0(\psi) = 0$.

By Lemma 4.9, ψ is a null extension. Hence, by Theorem 3.4, ψ is self-absorbing, i.e.,

$$\psi \oplus \psi \sim \psi.$$

But since ϕ is a null extension, by Theorem 3.4, ϕ is also self-absorbing, i.e.,

$$\phi \oplus \phi \sim \phi.$$

By Theorems 2.6 and 2.7, there exists a *-monomorphism $\sigma : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ where σ is unital if and only if ϕ is unital such that

$$\phi \sim \sigma \oplus \psi.$$

Hence,

$$\phi \oplus \psi \sim \sigma \oplus \psi \oplus \psi \sim \sigma \oplus \psi \sim \phi.$$

By a similar argument

$$\phi \oplus \psi \sim \psi.$$

Hence,

$$\phi \sim \psi.$$

Hence, since ψ is totally trivial, ϕ is totally trivial, as required. □

In [41], we will show that under the hypotheses of Theorem 4.10, the conditions in the conclusion of Theorem 4.10 are each equivalent to the condition that $KL(\phi) = 0$. We will then use this to classify all extensions of the form

$$0 \to \mathcal{B} \to \mathcal{D} \to C(X) \to 0.$$

References

[1] W. Arveson, *Notes on extensions of C*-algebras.* Duke Math. J. **44** (1977), no. 2, 329–355.
[2] B. Blackadar, K-theory for operator algebras. Second edition. Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.
[3] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras.* J. Funct. Anal. **45** (1982), no. 3, 297–340.
[4] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras.* Proceedings of a Conference on Operator Theorem (Dalhouse Univ., Halifax, N.S., 1973), pp58–128. Lecture Notes in Math., Vol 345, Springer, Berlin, 1973.
[5] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of C*-algebras and K-homology* Annals of Mathematics, Second Series **105** (1977), no. 2, 265–324.
[6] M. Dadarlat and S. Eilers, *On the classification of nuclear C*-algebras.* Proc. London Math. Soc. (3) **85** (2002), no. 1, 168–210.
[7] G. Elliott, *Derivations of matroid C*-algebras. II.* Annals of Mathematics **100** (1974), no. 2, 407–422.
[8] G. Elliott, G. Gong, H. Lin and Z. Niu, *The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension.* Preprint. A copy can be found at https://arxiv.org/abs/1712.09463.
[9] G. Elliott and D. Handelman, *Addition of C*-algebra extensions.* Pacific Journal of Mathematics **137** (1989), no. 1, 87–121.
[10] G. Elliott and D. Kucerovsky, *An abstract Voiculescu–Brown–Douglas–Fillmore absorption theorem.* Pacific J. Math. **198** (2001), no. 2, 385–409.
[11] G. Gong and H. Lin, *Classification of homomorphisms from C(X) to simple C*-algebras of real rank zero.* Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 2, 181–206.

[12] G. Gong and H. Lin, *Almost multiplicative morphisms and K-theory.* Internat. J. Math. **11** (2000), 983–1000.

[13] P. S. Guinand and L. Marcoux, *Between the unitary and similarity orbits of normal operators.* Pacific J. Math. **159** (1993), no. 2, 299–355.

[14] D. Hadwin, *Nonseparable approximate unitary equivalence.* Trans. Amer. Math. Soc. **266** (1981), no. 1, 203–231.

[15] S. Hu, H. Lin and Y. Xue, *Limits of homomorphisms with finite dimensional range.* Internat. J. Math. **16** (2005), no. 7, 807–821.

[16] X. Jiang and H. Su, *On a simple unital projectionless C*-algebra.* Amer. J. Math. **121** (1991), no. 2, 359–413.

[17] V. Kaftal, P. W. Ng and S. Zhang, *Strict comparison of positive elements in multiplier algebras.* Accepted for publication at the Canadian Journal of Mathematics. Published electronically at http://dx.doi.org/10.4153/CJM-2016-015-3.

[18] V. Kaftal, P. W. Ng and S. Zhang, *The minimal ideal in a multiplier algebra.* Preprint (2017).

[19] V. Kaftal, P. W. Ng S. Zhang, *Purely infinite corona algebras.* Preprint (2014).

[20] G. G. Kasparov, *Hilbert C*-modules: theorems of Stinespring and Voiculescu.* J. Operator Theory **4** (1980), no. 1, 133–150.

[21] D. Kucerovsky, *Extensions contained in ideals.* Trans. Amer. Math. Soc. **356** (2004), no. 3, 1025–1043.

[22] D. Kucerovsky and P. W. Ng, *Completely positive maps into corona algebras.* Mathematical Proceedings of the Cambridge Philosophical Society. **147** (2009), no. 2, 323–337.

[23] D. Kucerovsky, P. W. Ng and F. Perera, *Purely infinite corona algebras of simple C*-algebras.* Math. Ann. **346** (2010), no. 1, 23–40.

[24] H. Lin, *Simple C*-algebras with continuous scales and simple corona algebras.* Proceedings of the AMS. **112** (1991), no. 3, 871–880.

[25] H. Lin, *Notes on K-Theory of multiplier algebras.* (1992) University of Victoria Library Technical Reports. (Mathematics and Statistics) A copy is available at https://dspace.library.uvic.ca/handle/1828/2827
filename: DMS-603-IR.

[26] H. Lin, *Extensions by C*-algebras of real rank zero. II.* Proc. London Math. Soc. (3) **71** (1995), no. 3, 641–674.

[27] H. Lin, *Generalized Weyl–von Neumann theorems. II.* Math. Scand. **77** (1995), no. 1, 129–147.

[28] H. Lin, *Almost multiplicative morphisms and some applications.* J. Operator Theory **37** (1997), no. 1, 121–154.

[29] H. Lin, *Extensions of $C(X)$ by simple C*-algebras of real rank zero.* Amer. J. Math. **119** (1997), no. 6, 1263–1289.

[30] H. Lin, *Extensions by C*-algebras of real rank zero. III.* Proc. London Math. Soc. (3) **76** (1998), no. 3, 634–666.

[31] H. Lin, *Embedding and AH algebra into a simple C*-algebra with prescribed KK data.* K-theory **24** (2001), no. 2, 135–156.

[32] H. Lin, An introduction to the classification of amenable C*-algebras, World Scientific Publishing Company, Inc., River Edge, NJ, 2001.

[33] H. Lin, *Stable approximate unitary equivalence of homomorphisms.* J. Operator Theory **47** (2002), no. 2, 343–378.

[34] H. Lin, *Simple corona algebras.* Proceedings of the American Mathematical Society **132** (2004), no. 11, 3215–3224.

[35] H. Lin, *Extensions by simple C*-algebras: quasidiagonal extensions.* Canad. J. Math. **57** (2005), no. 2, 351–399.

[36] H. Lin, *Classification of homomorphisms and dynamical systems.* Trans. Amer. Math. Soc. **359** (2007), no. 2, 859–895.

[37] H. Lin and P. W. Ng, *The corona algebra of stabilized Jiang–Su algebra.* J. Funct. Anal. **270** (2016), no. 3, 1220–1267.

[38] P. W. Ng, *Approximate uniqueness for maps from $C(X)$ into simple real rank zero C*-algebras.* Preprint (2016). A copy is available at http://www.ucs.louisiana.edu/∼pwn1677/references.html.

[39] P. W. Ng, *A double commutant theorem for purely large C*-subalgebras of real rank zero corona algebras.* Studia Mathematica **190** (2009), no. 2, 135–145.

[40] P. W. Ng, *Real rank zero for purely infinite corona algebras.* Preprint (2015).

[41] P. W. Ng, *Purely infinite corona algebras and extensions.* Preprint (2017). A copy is available at http://www.ucs.louisiana.edu/∼pwn1677/references.html.

[42] N. C. Phillips, *Approximation by unitaries with finite spectrum in purely infinite C*-algebras.* J. Funct. Anal. **120** (1994), no. 1, 98–106.

[43] D. Voiculescu, *A noncommutative Weyl–von Neumann theorem.* Rev. Roumaine Math. Pures Appl. **21** (1976), 97–113.

[44] N. E. Wegge-Olsen, K-theory and C*-algebras. A friendly approach. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[45] S. Zhang, *A Riesz decomposition property and ideal structure of multiplier algebras.* J. Operator Theory **24** (1990), no. 2, 209–225.

[46] S. Zhang, *K_1-groups, quasidiagonality, and interpolation by multiplier projections.* Trans. Amer. Math. Soc. **325** (1991), no. 2, 793–818.

[47] S. Zhang, *Certain C*-algebras with real rank zero and their corona and multiplier algebras. Part I.* Pacific Journal of Mathematics **155** (1992), no. 1, 169–197.

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