[Adam Sørensen] May 14, 2015

Embedding into L_2
(joint work w. Nathan Brownlowe)

Kirchberg's Geneva Theorems:

1. A separable C^*-alg is exact iff it is a subalg. of $\mathcal{O}_2$.
2. $\mathfrak{A} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ iff $\mathfrak{A}$ is unital, simple, separable and nuclear.
3. for $\mathfrak{A}$ a separable, simple, nuclear C^*-alg $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_\infty$ iff $\mathfrak{A}$ is purely infinite.

QN: are there algebraic analogues of the Geneva thms (GT)?

(2) NO, since $L_{2,K} \otimes L_{2,K} \not\cong L_{2,K}$ [Ara-Cortiñas, Bell-Bergman, ...]

(3) NO, since $L_{\infty,K} \otimes L_{\infty,K} \not\cong L_{\infty,K}$.

(1) is what this talk is about.

Naive Algebraic GT1: Let R be a commutative ring with unit, $\mathfrak{A}$ an R-algebra. If $\mathfrak{A}$ has a countable basis as a vector space and $\mathfrak{A}$ is flat then $\mathfrak{A}$ embeds into $L_{2,R}$.

corresp. to separable

corresp. to exact

(translating the language to the algebra context)

Problem: statement is false: $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \not\hookrightarrow L_{2,\mathbb{Z}}$

- if R is a field, any algebra over R is flat.
- $L_R(E)$ is flat for all E, R

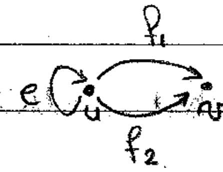
  (Leavitt path alg)

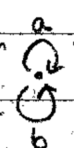
Qn: for which R-algebras $\mathcal{O}$ does $\mathcal{O}$ embed into $L_{2,R}$?
(*-algebraic, unital embedding desired, if possible)

Theorem (B.-S.) Let E be a countable graph. Then $L_R(E)$ embeds into $L_{2,R}$ (*-alg., unital)

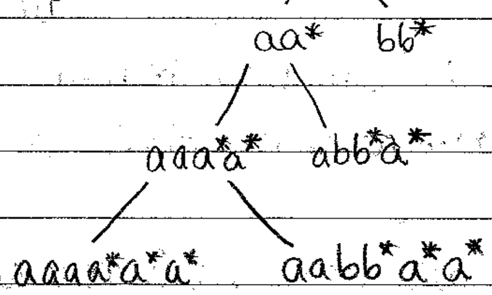
proof by example (part I)

- first assume condition (L):

  Consider the graph E 

$L_2 \cong L\Big($ 

$\Big)$



a, b - partial isometries

Define $\varphi(u) = aa^*$ $\quad \varphi(v) = bb^*$

$\varphi(e)$ = partial isometry connecting $aaaa^*a^*a^*$ and aa^*, e.g. $aaaa^*$

→

$\varphi(f_1)$ = partial isometry connecting abb^*a^* to bb^*
$= abb^*$

This is injective.

(q.e.d.)

Thm (Generalized Cuntz-Krieger uniqueness thm):
Let $\varphi: L_R(E) \to S$ be a ring homomorphism. Then φ is injective iff
(1) $\varphi(rv) \neq 0$ for $r \in R \setminus \{0\}$, $v \in E^0$.
(2) $\varphi(q(\alpha)) \neq 0$ for all α - cycle without exit, and $q \in R[x]$, q non-zero.

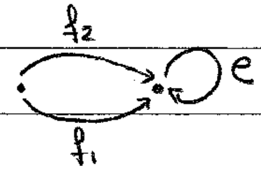
Remarks: * R-field, it follows from work of Rangaswamy
* "C^*-version" is due to Szymanski, Reznikoff-Nagy
* $uu^* = p = u^*u$ (i.e. u-partial unitary)
If $q(u) \neq 0$ then we say u has full spectrum.
(2) $\approx$ $\varphi(\alpha)$ has full spectrum.

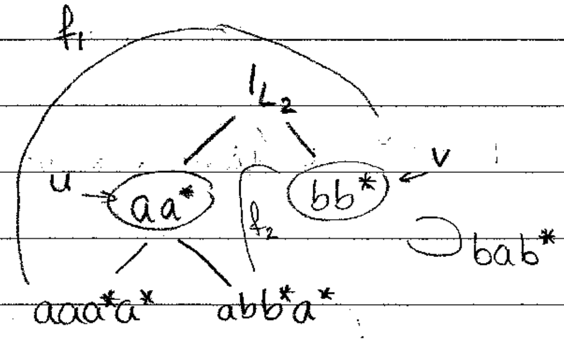
Prop'n: L_2 contains a full spectrum unitary.
Proof:
$u = aaa^* + aba^*b^* + bb^*b^*$ works (hw.!)

(q.e.d.)

Proof by example (part II)

· condition (L) does not hold.

E: 



Aim: investigate if $L_{2,\mathbb{R}} \otimes L_{2,\mathbb{R}} \hookrightarrow L_{2,\mathbb{R}}$?

we know that $\mathbb{R}[z, z^{-1}] \hookrightarrow L_{2,\mathbb{R}}$.

By flatness

$$\mathbb{R}[w, w^{-1}, z, z^{-1}] \hookrightarrow L_{2,\mathbb{R}} \otimes L_{2,\mathbb{R}}.$$

need to find two commuting unitaries w. joint full spectrum.

Def: $\mathcal{U} = \{u \in L_{2,\mathbb{R}} \mid uu^* = u^*u = 1\}$

$\mathcal{U}_1 = \{u \in \mathcal{U} \mid u = \sum_{i=1}^{n} \alpha_i \beta_i^* \text{ for distinct } (\alpha_i, \beta_i)\}$

$Z(\alpha) = \{\alpha\xi \mid \xi \in \{a,b\}^{\mathbb{N}}\}$

$\mathcal{U}_V = \{u \in \mathcal{U} \mid u = \sum_{i=1}^{n} \alpha_i \beta_i^* \text{ s.t. } \bigsqcup_{i=1}^{n} Z(\alpha_i) = \{a,b\}^{\mathbb{N}} = \bigsqcup_{i=1}^{n} Z(\beta_i)\}$

- Lemma (Nekrashevych):
  $\mathcal{U}_V \cong$ Thompson's group V.

- If R has characteristic zero, $\mathcal{U}_1 = \mathcal{U}_V$ (Pardo)

- $1 + aa^* \in \mathcal{U}(L_{2,\mathbb{Z}/3})$

Prop'n (build on work of Bleak-Salazar-Diaz)
If $u, v \in \mathcal{U}_V$ commute then $(\exists)$ a nonzero $q \in R[w, z]$ s.t. $q(u, v) = 0$.
$\hookrightarrow$ can't have joint full spectrum

Corollary: There does not exist a $*$-alg. embedding
$\varphi: R[z, z^{-1}, w, w^{-1}] \to L_{2,R}$ with $\varphi(w), \varphi(z) \in \mathcal{U}_V$.

Thm: There is no unital $*$-alg embedding of $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ into $L_{2,\mathbb{Z}}$.

Proof idea:
would get u, v full spectrum commuting unitaries.

Write $u = \sum_{i=1}^{n} \lambda_i \alpha_i \beta_i^*$; can ensure $\lambda_i \in \{-1, 1\}$ $\circledast$
and $\sqcup Z(\alpha_i) = \{a, b\}^{\mathbb{N}} = \sqcup Z(\beta_i)$

Let $u_{+,1} = \sum_{i=1}^{n} \alpha_i \beta_i^*$; do the same for v.

Then u_+, v_+ are commuting unitaries, so $(\exists)$ q s.t.
$q(u_+, v_+) = 0$

Modify q to give $\tilde{q}$ for which $\tilde{q}(u, v) = 0$ #

(q.e.d.)

NOTE: relies on the ring being $\mathbb{Z}$ for ⊛

Embedding into L_2 (joint w/ Nathan Brownlowe)

Set up

Kirchberg's Geneva Theorems

① A separable C^*-alg is exact if and only if it is a subalg of $\mathcal{O}_2$

② $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ if and only if A is a unital simple, separable and nuclear C^*-alg

③ ~~For~~ A is a separable, simple, nuclear C^*-alg ~~then~~ $A \cong A \otimes \mathcal{O}_\infty$ if and ~~only~~ only if A is purely infinite.

Question: Are there purely algebraic analogues?

② No, since $L_{2,k} \otimes L_{2,k} \not\cong L_{2,k}$ [Ara-Cortiñas, Bell-Bergman, Dicks]

③ No, since $L_{\infty,k} \otimes L_{\infty,k} \not\cong L_{\infty,k}$

① What this talk is about

Naive Algebraic CT 1: Translation:

Let R be a commutative ring w/ unit. Let A be an R-alg. If A has a countable basis ~~of~~ as a vector space and A is flat then A embeds into $L_{2,R}$.

Remark

* False for $R = \mathbb{Z}$ (~~on~~ unital $*$-alg. embedding)
* When R is a field K, ~~all~~ algebras are flat. I don't have a counter example but I don't believe it should be true.
* $L_R(E)$ is flat for all E, R. Countable basis if E is countable.

Question

For which R-algebras A does A embed into $L_{2,R}$? ($*$-alg embeddings unital)

A positive Result

Thm (B-S)

Let E be a countable graph. Then $L_R(E)$ embeds into $L_{2,R}$ ($*$-algebraic (unital)).

Pf by example (Condition (L) case)

Consider the graph

$$E \quad \begin{tikzcd} u \arrow[loop left, "e"] \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & v \end{tikzcd}$$

Think of $L_{2,R}$ as $L_R\left(\begin{tikzcd} \bullet \arrow[loop above, "a"] \arrow[loop below, "b"] \end{tikzcd} \right)$

$$\begin{tikzcd}[column sep=small] & & 1_{L_2} \arrow[dl, dash] \arrow[dr, dash] & \\ & aa^* \arrow[dl, dash] \arrow[dr, dash] & & bb^* \\ aaa^*a^* \arrow[d, dash] \arrow[dr, dash] & & abb^*a^* & \\ aaaa^*a^*a^* & aabb^*a^*a^* & & \end{tikzcd}$$

Define $\varphi : L_R(E) \to L_{2,R}$ by

$$\varphi(u) = aa^* \qquad \varphi(v) = bb^*$$

$\varphi(e)$ = partial isometry connecting

$aaaa^*a^*a^*$ to aa^* :

$t = aaaa^*$

$\varphi(f_1)$ = partial iso connecting

$aabb^*a^*a^*$ to $\cancel{aa^*}\ bb^*$

$\varphi(f_2)$ = partial iso connecting

abb^*a^* to bb^*

$t = abb^*$

One verifies that this defines a $*$-hom.

It is injective by the Cuntz-Krieger uniqueness theorem [Tomforde] ∎

GCUT (Generalized Cuntz-Krieger Uniqueness Theorem)

Let $\varphi : L_R(E) \to S$ be a ring homomorphism.

Then φ is injective if and only if

① $\varphi(rv) \neq 0$ for ~~any~~ all $v \in E^0$, $r \in R \setminus \{0\}$, and

② $\varphi(q(\alpha)) \neq 0$ for all cycles α w/o exit

and all nonzero $q \in R[x]$.

Remark

* When R is a field, the theorem can be derived from work of Rangaswamy or Aranda Pino - Martín Barquero - Martín González - Siles Molina.

* C^*-version is due to Szymański, Reznikoff-Nagy.

* An element u s.t. $uu^* = p = u^*u$ is called a partial unitary. If $q(u) \neq 0$ for all

If φ is a $*$-hom, then (2) says that $\varphi(\alpha)$ is a full spectrum partial unitary.

<u>Prop</u>

There exists a full spectrum unitary in $L_{2,R}$

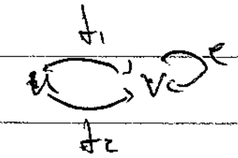
<u>Pf</u>

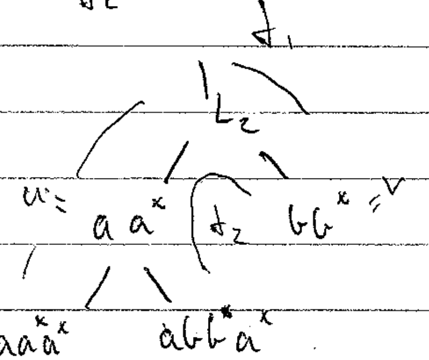
$$u = aaa^* + aba^*b^* + bb^*b^*$$

~~works~~ ~~is a full spe~~ works $\boxtimes$

<u>Pf by example</u> <u>(non condition (L))</u>:

Consider

E : 



$e = bub^*$

Apply generalised UT to get injection

Part 3 Negative results / Tensor products

Aim: Investigate ~~when~~ if $L_{z,\mathbb{R}} \otimes L_{z,\mathbb{R}} \hookrightarrow L_{z,\mathbb{R}}$.

Approach:

We know $L_{\mathbb{R}}[z, z^{-1}] \hookrightarrow L_{z,\mathbb{R}}$. By flatness

$$L_{\mathbb{R}}[z, z^{-1}, w, w^{-1}] \cong L_{\mathbb{R}}[z, z^{-1}] \otimes L_{\mathbb{R}}[w, w^{-1}]$$

$$\hookrightarrow L_{z,\mathbb{R}} \otimes L_{z,\mathbb{R}}$$

So ~~to prove a non~~ if $L_{z,\mathbb{R}} \otimes L_{z,\mathbb{R}}$ embeds into $L_{z,\mathbb{R}}$ so will $L_{\mathbb{R}}[z, z^{-1}, w, w^{-1}]$.

Def

* $\mathcal{U}(L_{z,\mathbb{R}}) = \{ u \in L_{z,\mathbb{R}} \mid uu^* = 1 = u^* u \}$
* $\mathcal{U}_1 = \{ u \in \mathcal{U}(L_{z,\mathbb{R}}) \mid u = \sum_{i=1}^{n} \alpha_i \beta_i^*$ for distinct pairs $\alpha_i, \beta_i \}$
* For $\alpha \in \{a,b\}^*$ let $Z(\alpha) = \{ \alpha \xi \mid \xi \in \{a,b\}^{\mathbb{N}} \}$

* $\mathcal{U}_V = \left\{ u \in \mathcal{U}(L_{Z,R}) \;\middle|\; u = \sum_{i=1}^{n} \alpha_i p_i^* \quad \bigsqcup_{i=1}^{n} Z(\alpha_i) = \{a,b\}^{\infty} = \bigsqcup_{i=1}^{n} Z(p_i) \right\}$

Lemma

* $\mathcal{U}_V \cong$ Thompson's grp V [Nekrashevych]
* If R has characteristic 0 then

$$\mathcal{U}_1 = \mathcal{U}_V \quad \text{[Pardo].}$$

Prop (Build on work of Bleak - Salazar Díaz)

If $u, v \in \mathcal{U}_V$ commute then there exists a nonzero polynomial $q \in R[w, z]$ s.t. $q(u, v) = 0$

THM

There does not exist a $*$-alg embedding

$$\varphi : L_R[w, w^{-1}, z, z^{-1}] \to L_{Z,R}$$

w/ $\varphi(w), \varphi(z) \in$ [illegible] $\cdot\, \mathcal{U}_V$.

Remark

~~This implies that any embedding~~ of $L_{Z,R} \otimes L_{Z,R}$ into

Thm

There is no unital $*$-alg embedding

$$L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}} \longrightarrow L_{2,\mathbb{Z}}.$$

Pf idea

* Such an embedding would give rise to two commuting unitaries u, v in $L_{2,\mathbb{Z}}$ (joint full sp)
* Can write

$$u = \sum_{i=1}^{n} \lambda_i \alpha_i \beta_i^*$$

when $\bigsqcup_{i=1}^{n} Z(\alpha_i) = \{a,b\}^{\mathbb{N}} = \bigcup_{i=1}^{n} Z(\beta_i)$, $\lambda_i \in \{-1,1\}$.

(same for v).

* Let

$$u_+ = \sum_{i=1}^{n} \alpha_i \beta_i^* \in \mathcal{U}_V$$

* Find nonzero q s.t. $q(u_+, v_+) = 0$.
* Modify to nonzero $\tilde{q}$ s.t. $\tilde{q}(u, v) = 0$