[Ortega] May 14, 2015

Cyclic C^*-correspondences

program: A) Cuntz - Krieger uniqueness property (CKUP)
B) Cuntz - Pimsner algebras
C) n-cycle, acyclicity

A) $\mathcal{C}$ = category (C^*-alg, ring, alg.)

(G,R) - generators and relations
Say $\varphi : G \to A \in \mathcal{C}$ is a <u>representation</u>
if $\{\varphi(g)\}_{g\in G}$ satisfy relation R.
φ is <u>injective</u> if $\varphi(g) \neq 0$ $(\forall)\, g \in G$.

For φ a representation, define
$\mathcal{C}(\varphi)$ = the minimum object in $\mathcal{C}$ that contains $\{\varphi(g)\}$.

φ_α is <u>universal</u> if $(\forall)\, \varphi \in Rep(G,R)$ $(\exists)$
$\eta : \mathcal{C}(\varphi_\alpha) \to \mathcal{C}(\varphi)$
$\varphi_\alpha(g) \mapsto \varphi(g)$ homomorphism in $\mathcal{C}$.
$\mathcal{C}(G,R) := \mathcal{C}(\varphi_\alpha)$

(G,R) satisfies the CKUP in $\mathcal{C}$ if

$(\forall)$ φ injective rep'n of (G,R)

$$\eta : \mathcal{C}(G,R) \longrightarrow \mathcal{C}(\varphi)$$
$$\varphi_u(g) \longmapsto \varphi(g) \quad \text{is injective}$$

example:

☆ $C^*(\underbrace{u,1}_{G} : \underbrace{1 \text{ unit and } uu^* = u^*u = 1}_{R})$

$$\mathcal{C}(G,R) \cong \mathcal{C}(S^1) \longrightarrow C^*(v) \cong \mathcal{C}(\underbrace{sp(v)}_{\subseteq S^1})$$

$$t \longmapsto v \neq 0$$

☆ $C^*(u, 1 ; 1 \text{ unit and } u^*u = 1,\ uu^* \neq 1)$

$$\mathcal{C}(G,R) \cong \mathcal{T} \longrightarrow C^*(v) \quad \text{is an isomorphism}$$

Toeplitz alg

$$S \longmapsto v \quad \text{(hence injective)}$$

B) Cuntz - Pimsner algebras

graph C^*-alg. are examples

	generators	relations (C.K)
$C^*(E)$	$\{P_e, S_e\}$	$S_e^* S_f = \delta_{e,f} P_{r(e)}$
		$P_v = \sum_{s(e)=v} S_e S_e^*$ for v regular

$\overline{\text{span}}\{p_e\} = \mathcal{C}_0(E^0)$ is a C^*-subalg $\subseteq C^*(E)$

$\overline{\text{span}}\{s_e\} = \mathcal{C}_d(E^1)$ is a Banach space.

$p_v s_e p_w = \delta_{v,s(e)} \delta_{w,r(e)} s_e$

can view $\mathcal{C}_d(E^1)$ as a $\mathcal{C}_0(E^0)$-bimodule using this

$\langle s_e, s_f \rangle := s_e^* s_f \in \mathcal{C}_0(E^0)$ defines an inner prod $\langle \cdot, \cdot \rangle : \mathcal{C}_d(E^1) \times \mathcal{C}_d(E^1) \to \mathcal{C}_0(E^0)$

$\langle s_e, p_v s_f \rangle := s_e^* p_v s_f = \overline{s_e^* p_v^*} s_f$

$\varphi : \mathcal{C}_0(E^0) \to \mathcal{L}(\mathcal{C}_d(E^1))$

action by adjointable operators

Identify all the non-zero elements in $C^*(E)$ that induces by left multiplication the same action on $\mathcal{C}_d(E^1)$

$p_v \cdot : \mathcal{C}_d(E^1) \to \mathcal{C}_d(E^1)$ $\qquad \sum_{s(e)=v} s_e s_e^* : \mathcal{C}_d(E^1) \to \mathcal{C}_d(E^1)$

$s_e \mapsto p_v \cdot s_e$

identify

Given $\mathfrak{A}$ - C^*-alg
$X_{\mathfrak{A}}$ - Ban. sp. that has a right $\mathfrak{A}$-module structure
and $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{A}$ non degenerate inner prod.
and $\varphi : \mathfrak{A} \to \underbrace{\mathcal{L}(X_{\mathfrak{A}})}$ $*$-hom
adjointable operators

$\to$ C^* correspondence

C^* alg

$(t_0, t_1) : \mathfrak{A} \times X \to \mathcal{B}$ representation

1) $t_0 : \mathfrak{A} \to \mathcal{B}$ $*$-hom
2) $t_1 : X \to \mathcal{B}$ linear
3) $t_1(a \cdot \xi \cdot b) = t_0(a)\, t_1(\xi)\, t_0(b)$
4) $t_0(\langle \xi, \eta \rangle) = t_1(\xi)^* \, t_1(\eta)$

(t_0, t_1) is called <u>injective</u> if t_0 is injective ($\Rightarrow t_1$ is inj.)
There exists a universal representation (t_0, t_1) with
$\mathcal{T}_X = C^*(t_0(\mathfrak{A}), t_1(X))$ called the Toeplitz algebra

injective

We define $\pi : \mathcal{K}(X) \to \mathcal{T}_X$

$$\theta_{\xi,\eta} \mapsto t_1(\xi) t_1^*(\eta)$$

generalized compact operators

$$\theta_{\xi,\eta}(x) = \xi \cdot \langle \eta, x \rangle$$

$J_X := \varphi^{-1}(\mathcal{K}(X)) \cap (\ker \varphi)^\perp \triangleleft \mathcal{A}$

$\mathcal{T}(J_X) = \langle t_0(a) - \pi(\varphi(a)) : a \in J_X \rangle \triangleleft \mathcal{T}_X$

$\mathcal{T}_X / \mathcal{T}(J_X) =: \mathcal{O}_X$ the Cuntz-Pimsner algebra

(t_0, t_1) rep'n of $(\mathcal{A}, X)$ is <u>covariant</u> if
$$t_0(a) = \pi(\varphi(a)) \quad (\forall)\, a \in J_X.$$

examples:

- $C^*(E)$ for graph E
- $\mathcal{A} \times_\alpha \mathbb{Z}$ cross product, where $\alpha \in \mathrm{Aut}(\mathcal{A})$
- E is a topological graph
- topological quivers
- $(X, \sigma) \rightsquigarrow \mathcal{O}_{(X,\sigma)}$

c) n-cycle: fix $(\mathcal{A}, X)$ a C^*-correspondence

an ideal $J \triangleleft \mathcal{A}$ is X-invariant if $(\forall)\, \xi, \eta \in X$
$\langle \xi, a \cdot \eta \rangle \in J \quad (\forall)\, a \in J$.

Given $J \triangleleft \mathcal{A}$ define inductively ideals $J^{[n]}$ as follows:

$J^{[1]} := J$ and $J^{[n]} = \{a \in J : \langle \xi, a\cdot\eta\rangle \in J^{[n-1]}$
$(\forall)\ \xi, \eta \in X\}$

Define $J^{[\infty]} = \bigcap_{n=1}^{\infty} J^{[n]}$

Given $n \in \mathbb{N}$ and $(\mathcal{A}, X)$ C^*-corresp.
$X^{\otimes n}$ is a C^*-corresp with inner prod $\langle\cdot,\cdot\rangle^n$.

Fock representation

$(\Psi, T) : \mathcal{A} \times X \longrightarrow \mathcal{L}(\mathcal{A} \oplus X \oplus X^{\otimes 2} \oplus X^{\otimes 3} \oplus \dots)$

$a \longmapsto \Upsilon(a)\,(x_i)_i = (\varphi(a)\,x_i)_i$

$\xi \longmapsto T_\xi\,(x_i)_i = (\xi \otimes x_i)_{i+1}$

$\eta : J \to X^{\otimes n}$ is an n-cycle if

a) J is an X-invariant ideal contained in $J_X^{[\infty]}$

b) $\langle \eta(a), \eta(b)\rangle^n = a^* b$

c) $T_\xi^* T_{\eta(a)} T_\theta = T_{\eta(\langle \xi, a\theta\rangle)}$ $(\forall)\ a \in J,\ \xi, \theta \in X$

corresponds to various properties of the graph C^*-alg