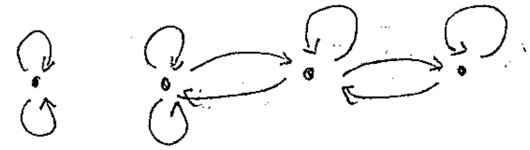

$$\mathbb{Z}/(I-A_E)\mathbb{Z} = 0 = \mathbb{Z}^2/(I-A_E)\mathbb{Z}^2$$

but $\det(I-A_E) = -1 \neq 1 = \det(I-A_F)$

$X_E \not\approx_{FE} X_F$ yet $C^*(E) \sim_{ME} C^*(F)$

if $\alpha = e_1 e_2$ then $S_\alpha S_\alpha^* = S_{e_1} S_{e_2} S_{e_2}^* S_{e_1}^*$

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[Ralf Meyer].

Classifying up to isomorphism is hard. Use weaker equiv. relation (e.g. Kasparov theory.)

Deep analysis (Kirchberg) says that for some nice objects,

weak equivalence $\Rightarrow$ isomorphism.

• if two C^*-alg. w. UCT have isomorphic K-theory, they are KK-equivalent.

We care about non-simple C^*-alg and therefore use a version

$KK(X)$ over a topological space X (the spectrum of our C^*-algs.)

When are two C^*-alg. over X isomorphic in $KK(X)$?

need a map from spectrum of $\mathfrak{A}$ to X.

$KK(X)$ has enough structure to support homological algebra.

$\{\mathfrak{A} \to \mathfrak{B} \to \mathfrak{C} \xrightarrow{\text{odd}} \mathfrak{A}\}$ exact triangles.

will determine

boundary map for long exact sequence.

$\Sigma\mathcal{A} := \mathcal{C}_0(\mathbb{R}, \mathcal{A})$ suspension odd map $\Sigma\mathcal{C} \to \mathcal{A}$.
or $\mathcal{C} \to \Sigma\mathcal{A}$.

These satisfy the axioms of a triangulated category.
We also need an "invariant" $F : KK(X) \to \mathfrak{A}$, homological functor to some abelian category, compatible with Σ : $\Sigma_{\mathfrak{A}} \circ F \cong F \circ \Sigma_{KK(X)}$

will replace this by $\mathcal{T}$ for a more general presentation

Def: $\mathcal{A} \to \mathcal{B} \to \mathcal{C} \xrightarrow{odd} \mathcal{A}$ is F-exact $\Leftrightarrow F(\mathcal{A}) \hookrightarrow F(\mathcal{B}) \twoheadrightarrow F(\mathcal{C})$ short exact

$\Leftrightarrow F(\mathcal{C} \to \mathcal{A}) = 0$.

Given an object M of the category $\mathfrak{A}$, does it lift to an object in the category $KK(X)$, and can we understand the non-uniqueness of the lift?

Look at more and more complicated M:

(-1): $M = 0$. need to restrict to UCT class (bootstrap class) to ensure

$F(M) = 0 \Rightarrow M = 0$ meaning $KK(X)$-equivalent.

(0): M projective (free).

example: F = K-theory functor $KK \to \mathcal{A}b^{\mathbb{Z}/2}$ countable.

$KK(\mathbb{C}, \mathcal{B}) \cong K_0(\mathcal{B}) = \mathrm{Hom}_{\mathfrak{A}}(\mathbb{Z}, F(\mathcal{B}))$

If F is "universal" and has enough projectives, then on projective objects of $\mathfrak{A}$, there is a functor $F^* : \mathrm{proj}\,\mathfrak{A} \to \mathcal{T}$ s.t.

$F(F^*(P)) \cong P$ and left adjoint to F: $\mathcal{T}(F^*(P), X) \cong \mathcal{A}(P, F(X))$.

(1) M has projective resolution of length 1.

$$M \twoheadleftarrow P_0 \xleftarrow{f} P_1 \leftarrow 0.$$

Then $F^*(P_0) \xleftarrow{F^*(f)} F^*(P_1)$ is part of an exact triangle (by an axiom of the category)

$$\hat{M} \leftarrow F^*(P_0) \leftarrow F^*(P_1) \xleftarrow{odd} \hat{M}$$

Fact: $F(\hat{M}) = M$, and there is a UCT:

$$\mathrm{Ext}^1_{\mathcal{A}}(\Sigma F(\hat{M}), F(X)) \rightarrowtail \mathcal{T}(\hat{M}, X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(F(\hat{M}), F(X))$$

So any map $F(\hat{M}) \to F(X)$ lifts to $\hat{M} \to X$, for any X in $\mathcal{T}$,

If both M, X have such a UCT, can further show that an isomorphism $F(\hat{M}) \to F(X)$ lifts to an isomorphism $\hat{M} \to X$.

So F classifies objects of $\mathcal{T}$ for which $F(-)$ has a length-1-projective resolution. (see Meyer-Nest).

(2) length-2-projective resolutions

$$\begin{tikzcd} M & P_0 \arrow[l, two heads, "\pi"'] & P_1 \arrow[l] \arrow[d] & P_2 \arrow[l] & 0 \arrow[l] \\ & & \ker \pi \arrow[ul] & & \end{tikzcd}$$

ker π has a projective resolution of length 1.

- By previous step, can lift ker(π) uniquely with a UCT.

UCT allows us to lift ker(π) $\to P_0$ to ker(π) $\xrightarrow{\varphi} F^*(P_0)$;

take a "cone" in $\mathcal{T}$ again, this lifts M (but not unique).

can prove that all lifts of M arise in this way for some φ;

isomorphism classes of lifts $\hat{M}$ of M are $\mathrm{Ext}^2_{\mathcal{A}}(M, \Sigma M)$

New invariant $F(A)$ together with a class in $\mathrm{Ext}^2_{\mathcal{A}}(F(A), \Sigma F(A))$

describing which lift we picked.

- can use this for graph algebras

for length 3 projective resolution, idea breaks down

because of non-uniqueness in 2^{nd} step.