One Step Shift spaces over infinite alphabets

(O.T.W.)

Let $A^{=\{a_1, a_2, \dots\}}$ be an infinite alphabet.

$\tilde{A} \to$ one pt. cpctification, $\tilde{A} = A \cup \{\infty\}$

$\prod \tilde{A}$ is cpct.

have $(x_1, \infty, x_2, x_3, \infty, \dots)$.

Define: $\Sigma_A^{fin} := \{\phi\} \cup \bigcup_{n \in \mathbb{N}} A^n \qquad \Sigma_A^{inf} := \{(x_i) : x_i \in A\}$,

$$\Sigma_A = \Sigma_A^{fin} \cup \Sigma_A^{inf}.$$

Quotient map: $q : \prod \tilde{A} \longrightarrow \Sigma_A$ (surjective).

$(x_i) \longmapsto (x_i)$ if $x_i \neq \infty \ \forall i$.

$(x_j) \longmapsto x_1 \dots x_n$ if $x_i \neq \infty$ $1 \le i \le n$, $x_{n+1} = \infty$.

$(\infty, **) \longmapsto \phi$.

Def: give Σ_A the quotient topology.

Obs: In $\prod \tilde{A}$ $x \sim y$ iff $q(x) = q(y)$. $\to$ Put $\underset{\sim}{\prod \tilde{A}}$ quotient topology.

Corollary: Σ_A is cpct. (q is continuous, $\prod \tilde{A}$ is cpct)

Basis topology: $x = (x_1, \dots, x_n) \in \Sigma_A^{fin}$, $x \neq \phi$, $F \overset{finite}{\subseteq} A$.

$$Z(x,F) := \{y \in \Sigma_A : y_i = x_i, \ i = 1, \dots, n, \ y_{n+1} \notin F\}$$

$x = \phi$, $Z(\phi, F) = \{y \in \Sigma_A : y_1 \notin F\}$.

Convergence of sequences:

$(x^i) \to y$, $l(y) < \infty$ iff given $F \overset{\text{finite}}{\subseteq} A$, $\exists\, n$ s.t. $\forall\, i > n$, $l(x^i) \geq l(y)$, $x^i_{l(y)+1} \notin F$ and $x^i_j = y_j$ for $j = 1, \dots, l(y)$.

y

Obs: Shift map is not cont. in $\emptyset$: 2, 3, 4, 5, 6 $\to \emptyset$!!

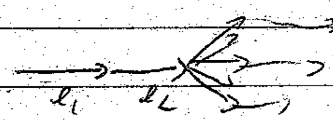
Shift Space: $\Lambda \subseteq \Sigma_A$ is a shift space if:

i) Λ is closed
ii) $\sigma(\Lambda) \subseteq \Lambda$.
iii) Λ satisfies the "infinite extension property", i.e., $\forall\, x \in \Lambda$, $l(x) < \infty$, the set

$$\{a \in A : xay \in \Lambda \text{ for some } y \in \Sigma_A\} \text{ is infinite}$$

graph $\leadsto$ infinite emitter.

E graph, $X_G = \{(e_1 e_2 e_3 \dots : s(e_{i+1}) = r(e_i))\}$



$e_1 e_2 \in \Lambda \Rightarrow$

Prop: Infinite Ext. Prop. $\Rightarrow$ Λ^{inf} is dense in Λ.

Shifts via Forbidden Words: Let $F \subseteq \bigcup_{k \geq 1} A^k$ and define

$$X_F^{\text{inf}} = \{x \in \Sigma_A^{\text{inf}} : \text{no subblock of } x \text{ is in } F\}$$

$$X_F^{\text{fin}} = \{x \in \Sigma_A^{\text{fin}} : \exists \text{ infinitely many } a \in A \text{ for which } \exists\, y \in \Sigma^{\text{inf}} \text{ and } xay \in X^{\text{inf}}\}$$

<u>Prop</u>: Λ is a shift space iff $\Lambda = X_F$ for some F.

<u>Def</u>: Λ is a SFT if we can take F having only finitely many elements and an M-step shift if

$$\ell(x) = M+1 \quad \forall x \in F.$$

<u>Def</u>: Let (E^0, E^1) be a graph, $E^\infty \to$ infinite paths.
X_E = Edge shift $:= \overline{E^\infty}$

<u>Prop</u>: If E^1 is infinite.

$$X_E = E^\infty \cup \{\alpha : \alpha \in \bigcup_{n \geq 1} E^n \text{ and } r(\alpha) \text{ is infinite emitter}\} \cup \{\emptyset\}.$$

<u>Conjugacy</u>: A map $\phi : \Lambda \longrightarrow Y$, $\Lambda \overset{\text{shift}}{\subseteq} \Sigma_A$, $Y \overset{\text{shift}}{\subseteq} \Sigma_B$

is a conjugacy iff is cont., biject.

$$\begin{tikzcd} \Lambda \arrow[r, "\sigma"] \arrow[d, "\phi"'] & \Lambda \arrow[d, "\phi"] \\ Y \arrow[r, "\sigma"] & Y \end{tikzcd}$$

<u>and</u>

$$\ell(\phi(x)) = \ell(x) \quad \forall x \in \Lambda$$

Conjectures

Q1: Is every SFT a edge shift? (No?)

Q2: For each $M \in \mathbb{N} \cup \{0\}$, does there $\exists$ a $(M+1)$-step shift, not conjugate to any M-step shift? (Yes!)

<u>Theor</u>: There exists a 2-step shift not conjugate to an 1-step shift.

D

Fix $x_1 \in A = \{x_1, x_2, x_3, \dots\}$

Let $F \subset A^3$ be such that $A^3 \setminus F = \{ (x_1, x_j, x_1), (x_j, x_1, x_j) : x_j \in A \}$

<u>Pf</u>: X_F is a 2-step shift that is not conj. to a 1-step shift

<u>Remark</u>: Notice that X_F contains the finite seq. (x_1), but does not contain any seq. of length 2.$^{\circledast}$

$\hookrightarrow \left((\forall j)\ \underset{\downarrow x_2}{x_1} x_j x_1 x_j x_1 \dots \in X_F \right)$

Suppose that $\Phi : X_F \to Y_G$ $\xrightarrow{}$ 1-step. is a conjugacy.

$z_1 z_2 \in Y_G$ ∅
$z_1 z_2 = x_1 x_j$ or $= x_j x_1$
has follower set ... $x_1 x_j \dots$ finite!!

$\circledast$ If $z_1 z_2 \in Y_F$ then
$z_1 z_2 = \lim g^n$, $\ell(g^n) = \infty$
$\exists N_0$ s.t. $\forall n \ge N_0$ $g^n_i = z_i$

$g^n \in X_F \Rightarrow z_1 = x_1$ or $z_2 = x_1$.
(So z_1 & $z_2 \ne x_1$ then $g^n \notin X_F$).
If $z_1 = x_1$, then take $F = \{x_j\}$
$\Rightarrow z[z_1, F]$ ... $z \in$ ... $\in z[z_1, F]$
So $z_2 = x_1$ ... $g^n_3 = z_1$ ... $g^n \to z_1 z_2$.

Consider the sequence,

$$\xi^j = x_1 x_j x_1 x_j x_2 x_j \dots$$

Clearly $\xi^j \longrightarrow x_1$

Then $y^j := \Phi(\xi_j) \longrightarrow y = \Phi(x_1)$

Notice that $\forall j$ $\ell(y^j) = \infty$ and $\ell(y) = 1$.

Since ξ^j <u>has period</u> 2, so does y^j, and since $y^j \to y$ we may pass to a subsequence s.t.

$$y^j = y\, y_2^j\, y\, y_2^j\, y\, y_2^j \dots \quad \text{and} \quad y_2^j \ne y_2^k \ \forall j \ne k.$$

Now, since each $y^j \in Y_G \to$ 1-step, we have that.

$$\{ y y_2^j, y_2^j y : j \in \mathbb{N} \} \subseteq A^2 \setminus G.$$

Fix j_0 and define

$$z_k = y\, y_2^{j_0}\, y\, y_2^k\, y\, y_2^k \dots \quad \in Y_G. \quad \forall k \in \mathbb{N}.$$

z_n converges to $y\, y_2^{j_0}\, y$. $\to$ length 3 & $(2M+1)$.

Theor: There exists a 2-step shift not conjugate to an 1-step shift.

D

Fix $x_1 \in A = \{x_1, x_2, x_3, \dots\}$

Let $F \subset A^3$ be such that $A^3 \setminus F = \{ (x_1, x_j, x_1), (x_j, x_1, x_j) : x_j \in A \}$.

Af: X_F is a 2-step shift that is not conj. to a 1-step shift

Remark: Notice that X_F contains the finite seq. (x_1), but does not contain any seq. of length 2.$^{\circledast}$

($(\forall j)\ x_1 x_j x_1 x_j x_1 \dots \in X_F$, $\downarrow x_2$)

[illegible] $x_1 x_j$ or $x_j x_1$ [illegible]

Suppose that $\Phi : X_F \to Y_G$ is a conjugacy. (1-step.)

$\circledast$ If $z_1 z_2 \in X_F$ then $z_1 z_2 = \lim y^n$, $\ell(y^n) = \infty$. $\exists n_0$ s.t. $\forall n \geq n_0$ $y^n_i = z_i$.

$y^n \in X_F \Rightarrow z_1 = x_1$ or $z_2 = x_1$. (ie z_1 & $z_2 \neq x_1$ then $y^n \notin X_F$).

If $z_1 = x_1$, then take $F = \{x_2\}$ $\Rightarrow z[z_1, F]$ [illegible] $z \in$ [illegible] $y^n \in z[z_1, F]$. So $z_2 = x_1$ & [illegible] $y^n_3 = z_1$ $\forall n$ & $y^n \not\to z_1 z_2$.

Consider the sequence,

$$\xi^j = x_1 x_j x_1 x_j x_2 x_j \dots$$

Clearly $\xi^j \longrightarrow x_1$

Then $y^j := \Phi(\xi_j) \longrightarrow y = \Phi(x_1)$

Notice that $\forall j$ $\ell(y^j) = \infty$ and $\ell(y) = 1$.

Since ξ^j has period 2, so does y^j, and since $y^j \to y$ & we may pass to a subsequence s.t.

$$y^j = y\, y_2^j\, y\, y_2^j\, y\, y_2^j \dots \quad \text{and} \quad y_2^j \neq y_2^k \ \forall j \neq k.$$

Now since each $y^j \in Y_G \to$ 1-step, we have that.

$$\{ y y_2^j, y_2^j y : j \in \mathbb{N} \} \subseteq A^2 \setminus G.$$

Fix j_0 and define

$$z_n = y\, y_2^{j_0}\, y\, y_2^{n}\, y\, y_2^{n} \dots \quad \in Y_G \quad \forall n \in \mathbb{N}.$$

z_n converges to $y\, y_2^{j_0}\, y$. $\to$ length 3 & $(2M+1)$.

But X_F has no elements of length 3, since it

$\forall n$, let

$$z^n = y y_2^{j_0} y y_2^n y y_i^n y y_i^n \cdots \quad y_i^n z_i y_i^j, \; j \neq n$$

Notice that $z^n \in X_G$ (1-step)

Now, $z^n \longrightarrow y y_i^{j_0} y$ $\Rightarrow$ X_G has a sequence of length 3 $\longrightarrow \in$ Since X_F has
$\Rightarrow$ has seq. " " 2 $\longrightarrow$ no element of length 2.

C^*-alg.:

OTW: E, F countable graphs with no sinks and no sources. If $X_E \cong X_F$ then $\mathcal{G}_E \cong \mathcal{G}_F$

$$\mathcal{G}_E := \{(\alpha\gamma, \ell(\alpha)-\ell(\beta), \beta\gamma) : \alpha, \beta \in \bigcup_{n=0}^{\infty} E^n, \gamma \in \partial E \text{ e } r(\alpha) = r(\beta) = s(\gamma)\}$$

$$\partial E := E^{\infty} \cup \{\alpha \in \bigcup_{n=0}^{\infty} E^n : r(\alpha) \text{ is an infinite emitter}\}$$

$$\forall \alpha \in \bigcup_{n=0} E^n, \quad F \overset{\text{finite}}{\subseteq} E^1$$

$$Z(\alpha, F) := \{\alpha\gamma : \gamma \in \partial E : r(\alpha) = s(\gamma) \text{ e } \gamma_1 \notin F\}$$

E-graph. X_E = edge shift.

As sets $\quad \partial E = (X_E \setminus \{\emptyset\}) \cup E^0_{inf}$ ← infinite emitters.

$e_1, e_2, e_3, \longrightarrow v$ in ∂E (out of vertices and if converges is

$\longrightarrow \emptyset$, X_E