On the K-theoretic classification of graph C^*-algebras (1)

Graph convention

$C^*(\circlearrowleft\!\bullet \longrightarrow \bullet) \cong \mathcal{T}$, $\quad C^*(\bullet \longrightarrow \bullet\!\circlearrowright) \cong M_2(C(S^1))$

$C^*(\bullet\!\circlearrowright\infty) = \mathcal{O}_\infty \otimes \mathbb{K}$

Thm (Kirchberg–Phillips $\approx$ '94) $\boxed{\text{All graph } C^*\text{-alg's satisfy this}}$

$A, B =$ sep, nuc., UCT, simple C^*-algebras.

$$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K} \iff K_*(A) \cong K_*(B).$$

(without positivity).

Thm (Cuntz, Raeburn–Szymański, Drinen–Tomforde)

$E =$ graph, $E^0 = E^0_{reg} \sqcup E^0_{sing}$, $\begin{pmatrix} A & \alpha \\ * & * \end{pmatrix}$ adjacency matrix.

Then there is an exact seq.

$$K_1(C^*(E)) \rightarrowtail \mathbb{Z}^{E^0_{reg}} \xrightarrow{\begin{pmatrix} A^t - I \\ \alpha^t \end{pmatrix}} \mathbb{Z}^{E^0} \twoheadrightarrow K_0(C^*(E))$$

<u>Recall</u> $H \subseteq E^0$ is <u>hereditary</u> if
$H \ni v \geq w \Rightarrow w \in H$

H is <u>saturated</u> if $r(s^{-1}(v)) \subseteq H \Rightarrow v \in H$.

v is <u>breaking for H</u> if $v \xrightarrow{\exists \text{ fin (non-zero)}} H$

<u>Notation</u> Say that E has <u>condition (NB)</u> if no her+sat set has any br. vert.

<u>Thm</u> (Bates-Pask-Raeburn-Szymański)
E = cond (NB)+(K) Then there is lattice iso

$$\left\{ \begin{array}{c} \text{her+sat} \\ \text{sets} \end{array} \right\} \longrightarrow \mathbb{I}(C^*(E)) := \left\{ \begin{array}{c} \text{~~gauge inv.~~} \\ \text{ideals} \\ \text{in } C^*(E) \end{array} \right\}$$

$$H \longmapsto I_H.$$

$\star$

Thm (Rørdam)

A, B = sep, nuc, UCT C^*-alg's with exactly one non-trivial UCT ideal. Then
$A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K} \iff A$ and B have isomorphic six-term exact seq's in K-theory.

$\star$: $I \triangleleft A$ ideal $\rightsquigarrow$

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[dll, bend left] \\
K_1(I) \arrow[r] & K_1(A) \arrow[r] & K_1(A/I) \arrow[ull, bend right]
\end{tikzcd}$$

six-term exact seq.

Thm (Carlsen-Eilers-Tomforde)

$H \subseteq E^0$ her + sat, $\begin{pmatrix} A & \alpha \\ * & * \end{pmatrix}$ adj. matrix.

$$\begin{tikzcd}
K_0(I_H) \arrow[r] & K_0(C^*(E)) \arrow[r, two heads] & K_0(C^*(E)/I_H) \\
\mathbb{Z}^H \arrow[u, two heads] \arrow[r, hook] & \mathbb{Z}^{E^0} \arrow[u, two heads] \arrow[r, two heads] & \mathbb{Z}^{E^0 \setminus H} \arrow[u, two heads] \\
\mathbb{Z}^{H_{reg}} \arrow[u, "\begin{pmatrix} A^t - I \\ I \end{pmatrix}"'] \arrow[r, hook] & \mathbb{Z}^{E^0_{reg}} \arrow[u, "\begin{pmatrix} A^t - I \\ \alpha^t \end{pmatrix}"'] \arrow[r, two heads] & \mathbb{Z}^{E^0_{reg} \setminus H_{reg}} \arrow[u, "\exists !"'] \\
K_1(I_H) \arrow[u, tail] \arrow[r, tail] & K_1(C^*(E)) \arrow[u, tail] \arrow[r] & K_1(C^*(E)/I_H) \arrow[u, tail]
\end{tikzcd}$$

"Outer part" is six-term exact seq.

Recall $\mathbb{I}(A) \to \mathbb{I}(A \otimes \mathcal{O}_\infty \otimes \mathbb{K})$, $J \mapsto J \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ is iso. (3.5)

Deep Thm (Kirchberg '00)

$A, B =$ sep, nuc C^*-alg's, $\Phi : \mathbb{I}(A) \xrightarrow{\cong} \mathbb{I}(B)$.

Then $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathbb{K}$ ~~implementing~~ inducing Φ,

iff $A \sim_{KK(\Phi)} B$.

Goal: Classify graph C^*-algebras up to ideal-related KK-theory.

~~**Deep Thm** (Kirchberg '00)~~
~~Sep, nuc, ~~strongly purely~~ inf. $\mathcal{O}_\infty$-abs. C^*-alg's are classified (up to stable iso) by ideal-related KK-theory.~~

~~**Goal** Classify graph C^*-algebras up to ideal-related KK-theory.~~

2 ~~Main theorem~~ New things

$A = C^*$-alg. $I \lhd A$ is <u>compact</u> if whenever $I_1 \subseteq I_2 \subseteq \cdots$ is an increasing seq. of ideals in A s.t. $I \subseteq \bigcup I_n$ then $I \subseteq I_N$ for some N.

Let $\mathbb{I}_c(A) =$ lattice of cpct ideals in A.

<u>Fact</u> $E =$ cond. (NB) + (K) then

$$\mathbb{I}_c(C^*(E)) \longleftrightarrow \left\{\begin{array}{c}\text{her + sat sets gen by} \\ \text{fin. many vertices}\end{array}\right\}.$$

<u>In gen</u>: any ideal gen. by fin. many projections is compact.

Define the ring ~~Let~~ $R := \mathbb{Z}\mathbb{I}_c(A)$ = free abelian group with generators i^J_I with $I, J \in \mathbb{I}_c(A)$, $I \subseteq J$, equipped with multiplication

$$i^J_I \, i^{J'}_{I'} = \delta_{J,I'} \, i^{J'}_I .$$

We want non-deg, right R-modules M.

What is M?

Any such M is $\bigoplus_{I \in \mathbb{I}_c(A)} G_I$, G_I ab. grp's together with hom's.

$$i^J_I : G_I \longrightarrow G_J \text{ whenever } I \subseteq J, \text{ s.t. } i^K_J \circ i^J_I = i^K_I.$$

In this way, if $x \in G_{I_0} \subseteq \bigoplus G_I = M$

then $x \cdot i^J_I = \delta_{I_0, I} \, i^J_I(x) \in G_J \subseteq M$

☆ > An R-hom $\varphi : M \longrightarrow M'$ consists of grp hom's $\varphi_I : G_I \longrightarrow G'_I$ s.t.

$$\begin{tikzcd} G_I \arrow[r, "\varphi_I"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G'_I \arrow[d] \\ G_J \arrow[r, "\varphi_J"] & G'_J \end{tikzcd}$$

☆ Insert K-th. example!

(6)

Exm $R = \mathbb{Z}\mathbb{I}_c(A)$. Then

$$\mathbb{C}K_i^R(A) = \bigoplus_{I \in \mathbb{I}_c(A)} K_i(I)$$

with $\iota_I^J : K_i(I) \to K_i(J)$ induced by $I \hookrightarrow J$.

Exm E = cond. (NB)+(K), $A = C^*(E)$.
$R = \mathbb{Z}\mathbb{I}_c(C^*(E))$. Let $H(I)$ be hereditary set corr. to $I \in \mathbb{I}_c(A)$. Then

$$M_E := \bigoplus_{I \in \mathbb{I}_c(A)} \mathbb{Z}^{H(I)}$$

with $\iota_I^J : \mathbb{Z}^{H(I)} \longrightarrow \mathbb{Z}^{H(J)}$ canonical incl. (using $H(I) \subseteq H(J)$). Similarly

$$M_{E_{reg}} := \bigoplus_{I \in \mathbb{I}_c(A)} \mathbb{Z}^{H(I)_{reg}}$$

Fact $M_{E_{reg}}$ and M_E are projective R-modules.

Prop: E = Cond. (NB)+(K), $A = C^*(E)$, $R = \mathbb{Z}\mathbb{I}_c(A)$.
$\begin{pmatrix} A & \alpha \\ * & * \end{pmatrix}$ adj. matrix. Then

$$\mathbb{C}K_1^R(C^*(E)) \rightarrowtail M_{E_{reg}} \xrightarrow{\begin{pmatrix} A^t - I \\ \alpha^t \end{pmatrix}} M_E \twoheadrightarrow \mathbb{C}K_0^R(C^*(E))$$

~~8~~ (7)

<u>Thm</u> (G)

E, F graphs, Cond (NB)+(K),
suppose $\Phi : \mathbb{I}(C^*(E)) \xrightarrow{\cong} \mathbb{I}(C^*(F))$,
$R = \mathbb{Z}\mathbb{I}_c(C^*(E))$. TFAE:

(1) $C^*(E) \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong C^*(F) \otimes \mathcal{O}_\infty \otimes \mathbb{K}$
(inducing Φ)

(2) $C^*(E) \sim_{KK(\Phi)} C^*(F)$

(3) $\exists$ commuting diagram

$$\begin{tikzcd}
\mathbb{C}K_1^R(C^*(E)) \arrow[r, tail] \arrow[d, "\cong"'] & M_{E_{reg}} \arrow[r] \arrow[d] & M_E \arrow[r, two heads] \arrow[d] & \mathbb{C}K_0(C^*(E)) \arrow[d, "\cong"] \\
\mathbb{C}K_1(C^*(F)) \arrow[r, tail] & M_{F_{reg}} \arrow[r] & M_F \arrow[r, two heads] & \mathbb{C}K_0(C^*(F))
\end{tikzcd}$$

(4) $\forall$ diagram (as above)

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "G"] & \bullet \arrow[r] \arrow[d, dotted, "\exists"'] \arrow[dr, phantom, "G"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & \bullet \arrow[d, "\cong"] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Remarks: • The invariant does not depend on the graph! I.e. if we did not know the graph we could still construct the invariant. (basically from knowing that $\exists \varphi \in \mathrm{End}(\bigoplus \mathbb{K})$ s.t. $A \cong (\bigoplus \mathbb{K}) \rtimes_\varphi \mathbb{N}$.)

- The invariant is only functorial on isomorphisms
- If $C^*(E)$ has finitely many idals, then one can apply the Bentmann-Meyer invariant, which is basically the same but much "smaller".

(Jamie Gabe)

On the K-theoretic classification of graph C^*-alg. May 11, 2015 (1)

$C^*(\circlearrowleft\bullet \to \bullet) \cong \mathcal{T}$.

$C^*(\bullet \to \bullet\circlearrowleft) \cong M_2(\mathcal{C}(S^1))$.

$C^*(\cdots \to \cdots \to \bullet \circlearrowleft^{\infty}) \cong \mathcal{O}_\infty \otimes \mathcal{K}$.

Thm (Kirchberg-Phillips): $\mathfrak{A}, \mathfrak{B}$ = separable, nuclear, UCT, simple.

$$\mathfrak{A}\otimes\mathcal{O}_\infty\otimes\mathcal{K} \cong \mathfrak{B}\otimes\mathcal{O}_\infty\otimes\mathcal{K} \iff K_*(\mathfrak{A}) \cong K_*(\mathfrak{B})$$

Thm (Cuntz, Raeburn-Szymański, ...).

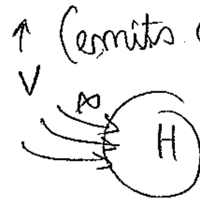
E-graph, $E^0 = E^0_{reg} \cup E^0_{sing}$ $\begin{pmatrix} A & \alpha \\ * & * \end{pmatrix}$ adjacency matrix.

Then there is an exact sq.

$$K_1(C^*(E)) \longrightarrow \mathbb{Z}^{E^0_{reg}} \xrightarrow{\begin{pmatrix} A^t - I \\ \alpha^t \end{pmatrix}} \mathbb{Z}^{E^0} \longrightarrow K_0(C^*(E))$$

Recall: $H \subseteq E^0$ hereditary if $H \ni v \geq w \Rightarrow w \in H$.

$H \subseteq E^0$ saturated if $(\forall)\ v \in E^0_{reg}$ $r(s^{-1}(v)) \subseteq H \Rightarrow v \in H$.

$v \in E^0$ is breaking for H if  (emits at least one edge outside H) + infinitely many into H.

Def: E has condition (NB) if no hereditary + saturated set has any breaking vertices.

E has condition K if any vertex which lies on a cycle lies on at least 2 simple cycles.

fact: E has condition (K) $\Leftrightarrow$ every ideal in $C^*(E)$ is gauge invariant

Thm: E has condition (NB) + (K) Then there is a lattice isom.

$$\left\{ \begin{array}{c} \text{her.+ sat. sets} \\ \text{in } E \end{array} \right\} \longrightarrow \mathbb{I}(C^*(E)) = \text{closed 2-sided ideals in } C^*(E).$$

$$H \longmapsto I_H.$$

If $J \lhd \mathfrak{A}$ get

$$\begin{tikzcd} K_0(J) \arrow[r] & K_0(\mathfrak{A}) \arrow[r] & K_0(\mathfrak{A}/J) \arrow[d] \\ K_1(\mathfrak{A}/J) \arrow[u] & K_1(\mathfrak{A}) \arrow[l] & K_1(J) \arrow[l] \end{tikzcd}$$

Theorem (Rordam): $\mathfrak{A}, \mathfrak{B}$ - separable, nuclear, UCT and have exactly one non-trivial UCT ideal.

then $\mathfrak{A} \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong \mathfrak{B} \otimes \mathcal{O}_\infty \otimes \mathcal{K}$

<=> $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic six-term exact sequences. (where J is the non-trivial UCT ideal mentioned)

Thm (Carlsen-Eilers-Tomforde): $H \subseteq E^0$ hereditary + saturated.

$\begin{pmatrix} A & \alpha \\ * & * \end{pmatrix}$ adjacency matrix.

$$\begin{tikzcd}
K_0(I_H) \arrow[r] & K_0(C^*(E)) \arrow[r, two heads] & K_0\left(C^*(E)/I_H\right) \\
\mathbb{Z}^H \arrow[u] \arrow[r, hook] & \mathbb{Z}^{E^0} \arrow[u] \arrow[r, two heads] & \mathbb{Z}^{E^0 \setminus H} \arrow[u] \\
\mathbb{Z}^{H_{reg}} \arrow[u, "\begin{pmatrix} A^t - I \\ \alpha^t \end{pmatrix}"] \arrow[r, hook] & \mathbb{Z}^{E^0_{reg}} \arrow[u, "\begin{pmatrix} A^t - I \\ \alpha^t \end{pmatrix}"] \arrow[r, two heads] & \mathbb{Z}^{E^0_{reg} \setminus H_{reg}} \arrow[u, dashed, "\exists !"] \\
K_1(I_H) \arrow[u] \arrow[r] & K_1(C^*(E)) \arrow[u] \arrow[r] & K_1\left(C^*(E)/I_H\right) \arrow[u]
\end{tikzcd}$$

snake lemma.

The part highlighted in blue is the 6 term exact sequence of $I_H \triangleleft C^*(E)$

Fact: $\mathfrak{A} = C^*$-alg $\quad \mathbb{I}(\mathfrak{A}) \to \mathbb{I}(\mathfrak{A} \otimes \mathcal{O}_\infty \otimes \mathcal{K})$
$\qquad J \mapsto J \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is a lattice iso.

Deep thm (Kirchberg).
$\mathfrak{A}, \mathfrak{B}$ = separable, nuclear $\quad \Phi : \mathbb{I}(\mathfrak{A}) \xrightarrow{\cong} \mathbb{I}(\mathfrak{B})$
Then $\mathfrak{A} \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong \mathfrak{B} \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ (induces Φ)
iff $\mathfrak{A} \sim_{KK(\Phi)} \mathfrak{B}$ (ideal-related — i.e. respect ideals in the KK-construction, vaguely)

goal: classify graph C^*-alg.'s up to ideal-related KK-equiv.

$\mathfrak{A}$ - C^*-alg. $J \triangleleft \mathfrak{A}$, then J is compact if $(\forall)$

$J_1 \subseteq J_2 \subseteq J_3 \subseteq \dots$ ideals in $\mathfrak{A}$ s.t. $J \subseteq \bigcup_{n=1}^{\infty} J_n$ then

$J \subseteq J_N$ for large n.

(related to topology on primitive ideal space)

notation: let $\mathbb{I}_c(\mathfrak{A})$ = lattice of compact ideals (ordered by inclusion)

fact: if E is a graph w. conditions (NB) and (K) then

$$\mathbb{I}_c(E) \longleftrightarrow \left\{ \begin{array}{l} \text{hereditary + saturated sets which are} \\ \text{generated by finitely many vertices} \end{array} \right\}$$

$$J \longmapsto H(J)$$

Define a ring $R = \mathbb{Z}\,\mathbb{I}_c(\mathfrak{A})$

:= free abelian gp on generators $i_J^{\mathcal{J}}$ where

$J \subseteq \mathcal{J}$ in $\mathbb{I}_c(\mathfrak{A})$

multiplication gen. by

$$i_J^{\mathcal{J}}\, i_{J'}^{\mathcal{J}'} = \delta_{J',\mathcal{J}}\, i_J^{\mathcal{J}'}$$

Want: non-degenerate right R-modules. M.

for any elem. of M $(\exists) r$ s.t. $rm = m$.

G_J = abelian group.

$$M = \bigoplus_{J \in \mathbb{I}_c(\mathfrak{A})} G_J$$

together with homomorphisms $\imath_J^{\mathcal{J}} : G_J \to G_{\mathcal{J}}$ for $J \subseteq \mathcal{J}$

s.t. $\imath_{\mathcal{J}}^{\mathcal{K}} \circ \imath_J^{\mathcal{J}} = \imath_J^{\mathcal{K}}$.

$$\begin{tikzcd} G_{\mathcal{J}} \arrow[r] & G_J \arrow[r] & G_{\mathcal{K}} \arrow[from=1-1, to=1-3, bend left, "\imath_J^{\mathcal{K}}"] \end{tikzcd}$$

example:

$\mathfrak{A}$ - C^*-alg., $R = \mathbb{Z}\mathbb{I}_c(\mathfrak{A})$ K-theory

$\mathcal{C}K_i(\mathfrak{A}) = \bigoplus_{J \in \mathbb{I}_c(\mathfrak{A})} K_i(J)$

$\iota_J^{\mathcal{J}} : K_i(J) \to K_i(\mathcal{J})$ is the hom. induced by $J \subseteq \mathcal{J}$.

$\varphi : M \to M'$ R-module homomorphism

$M = \bigoplus G_J$, $M' = \bigoplus G_{J'}$

meaning

$$\begin{tikzcd} G_J \arrow[r, "\varphi_i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G_{J'} \arrow[d] \\ G_{\mathcal{J}} \arrow[r, "\varphi_{\mathcal{J}}"'] & G_{\mathcal{J}'} \end{tikzcd}$$

example: $\star$ E-graph w. conditions (NB) and (K).

$R = \mathbb{Z}\mathbb{I}_c(C^*(E))$

$M_E := \bigoplus_{J \in \mathbb{I}_c(C^*(E))} \mathbb{Z}^{H(J)}$ $\qquad \iota_J^{\mathcal{J}} : \mathbb{Z}^{H(J)} \to \mathbb{Z}^{H(\mathcal{J})}$ canonical inclusion.

$M_{E\,reg} := \bigoplus_{J \in \mathbb{I}_c(C^*(E))} \mathbb{Z}^{H(J)_{reg}}$.

here A^t means A transpose
α^t — " — α — " —

Prop'n: E satisfying conditions (NB) + (K).

$$\mathcal{C}K_1(C^*(E)) \to M_{E\,reg} \xrightarrow[\iota]{\begin{pmatrix} A^t - 1 \\ \alpha^t \end{pmatrix}} M_E \to \mathcal{C}K_0(C^*(E))$$

fact: $M_{E_{reg}}$, M_E are projective R-modules.

Thm: E, F - graphs of condition (NB) and (K)

Suppose $\Phi : \mathbb{I}(C^*(E)) \xrightarrow{\cong} \mathbb{I}(C^*(F))$. TFAE:

(1) $C^*(E) \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{O}_\infty \otimes \mathcal{K}$.

s.t. it induces lattice isom. Φ.

(2) $C^*(E) \sim_{KK(\Phi)} C^*(F)$

(3) ($\exists$) commuting diagram.

$$\begin{tikzcd}
\mathbb{C}K_1(C^*(E)) \arrow[r, tail] \arrow[d, "\cong"'] & M_{E_{reg}} \arrow[r, "\begin{pmatrix} A_E^t - 1 \\ \alpha_E^t \end{pmatrix}"] \arrow[d] \arrow[r, phantom, shift right=6ex, "\text{module hom.}"] & M_E \arrow[r] \arrow[d] & \mathbb{C}K_0(C^*(E)) \arrow[d, "\cong"] \\
\mathbb{C}K_1(C^*(F)) \arrow[r] & M_{F_{reg}} \arrow[r] & M_F \arrow[r] & \mathbb{C}K_0(C^*(F))
\end{tikzcd}$$

4) (same diagram) for any diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\cong"'] & \bullet \arrow[r] \arrow[d, dashed] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, "\cong"] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

$\exists$ map which makes this commutative