[Mike Boyle] May 11, 2015

Strong shift equivalence + alg. K-theory

Given S a semi-ring, $\{0,1\} \subset S$, square matrices A, B over S.

def'n: $A \overset{ESSE}{\underset{S}{\sim}} B$ (elementary strong shift equiv. over S)
(Williams, '73)
if $(\exists)$ matrices U, V over S s.t. $A = UV$, $B = VU$.

example:

$$(2) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$\overset{SSE}{\underset{S}{\sim}}$ is the transitive closure of $\overset{ESSE}{\underset{S}{\sim}}$
($(\exists)$ a finite chain of ESSE matrices)

A square matrix A over $\mathbb{Z}_+$ defines an edge shift of finite type (SF)

σ_A

(Williams '73) $A \overset{SSE}{\underset{\mathbb{Z}_+}{\sim}} B \iff \sigma_A \cong \sigma_B$

Suppose
$$\begin{tikzcd}
\cdot \arrow[r, "A"] \arrow[d, "u_1"'] & \cdot \arrow[r, "A"] \arrow[d, "u_1"] & \cdot \\
\cdot \arrow[r] \arrow[d, "u_2"'] \arrow[ur, "v_1"] & \cdot \arrow[d, "U_2"] \arrow[ur] & \\
\cdot \arrow[r, "B"'] \arrow[ur, "v_2"] & \cdot &
\end{tikzcd}$$

Let $U = U_1 U_2$, $V = U_2 U_1$

def'n: $A \overset{SE}{\underset{S}{\sim}} B$ if $(\exists)$ U, V over S and $l \in \mathbb{N}$ s.t.

$$A^l = UV, \quad B^l = VU, \quad AU = UB, \quad BV = VA.$$

$$A \overset{SSE}{\underset{S}{\sim}} B \Rightarrow A \overset{SE}{\underset{S}{\sim}} B$$

Suppose $S = R$, a ring.

if R is a field: $A \underset{R}{\overset{SE}{\sim}} B \iff$ the non-nilpotent parts of A and B define isomorphic endomorphisms.

(throw away the nilpotent part of the Jacob normal form)

QN: for R a ring, how does $\underset{R}{\overset{SSE}{\sim}}$ refine $\underset{R}{\overset{SE}{\sim}}$?

ANS: in terms of algebraic K-theory.

Def'n: $GL_n(R) :=$ gp of invertible $n \times n$ matrices over R.

$EL_n(R) :=$ subgp of $GL_n(R)$ gen. by matrices equal to the identity except in one off-diag. entry.

$GL(R) = \lim_n GL_n(R)$ $\quad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

can view $GL(R)$ as a gp of infinite matrices

$\left(\begin{array}{c|c} U & 0 \\ \hline 0 & I_\infty \end{array}\right)$ where U is finite, square + invertible over R.

Whitehead Lemma: commutator of $GL(R) = EL(R)$

$K_1(R) := GL(R)/EL(R)$.

$K_1(R[t]) = K_1(R) \oplus NK_1(R)$

where $NK_1(R) = \{[U] : U \text{ has constant term } 1\}$

$NK_1(R) \cong Nil_0(R)$ where

$Nil_0(R)$ can be presented as the abelian group given by generators and relations:

generators $= \{[N] : N$ nilpotent matrix over $R\}$

relations: $[N] \sim [M]$ if $(\exists)$ n and $U \in GL_n(R)$ s.t. $U^{-1}NU = M$

$$[N] \sim \left[\begin{array}{c|c} N & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \sim [A] \oplus [B].$$

Farrell '77: If $Nil_0(R) \neq \{0\}$, then it is not finitely gen. as a gp.

if $G = \mathbb{Z}/n$ then $Nil_0(\mathbb{Z}G) = \{0\}$ iff n is square-free.

For many rings (e.g. regular noetherian) $Nil_0(R) = \{0\}$.

Def'n: Given A square over R

$$[A]_{SE(R)} = \{B : B \overset{SE}{\underset{R}{\sim}} A\}, \quad [A]_{SSE(R)} = \{B : B \overset{SSE}{\underset{R}{\sim}} A\}.$$

Thm (B-S) There is a bijection (for every square matrix A over R)

$$Nil_0(R) \longrightarrow \{[B]_{SSE\text{-}R} : B \overset{SE}{\underset{R}{\sim}} A\}$$ induced by

$$[N] \longrightarrow \begin{bmatrix} A & 0 \\ 0 & N \end{bmatrix}$$

$\uparrow$

nilpotent

$$A \overset{SE}{\underset{R}{\sim}} B \iff \operatorname{coker}(I-tA) \cong \operatorname{coker}(I-tB)$$
as $R[t]$-modules

$$\iff (I-tA) \underset{GL(R[t])}{\sim} (I-tB)$$

$$\Leftarrow (I-tA) \underset{EL(R[t])}{\sim} (I-tB) \iff A \overset{SSE}{\underset{R}{\sim}} B.$$

for the ring R

$\operatorname{coker}(I-A) \cong \operatorname{coker}(I-B)$ as R-mod $\overset{\Longrightarrow}{\Leftarrow}$ $(I-A) \underset{GL(R)}{\sim} (I-B)$

$\Longrightarrow$ only true if $I-A$ is injective

$\Uparrow$

$(I-A) \underset{EL(R)}{\sim} (I-B)$

refinement of $GL(R)$ equiv. by $EL(R)$-equiv

$$\longleftrightarrow K_1(R) \Big/ \underbrace{ELStab(I-A)}_{\text{stabiliser}}$$

$$= \{[U] \in K_1(R) : U(I-A) = E_1(I-A)E_2,\ E_1, E_2 \in EL(R)\}.$$

$ELStab(I-tA)$ is trivial in $K_1(R[t])$.

the refinement of $GL(R[t])$-equiv. by $EL(R[t])$-equiv

$$\longleftrightarrow NK_1(R) \Big/ ELStab(I-tA) = NK_1(R)$$

for an integral domain R: for every $U \in GL(R)$ there exists a matrix $I-A$ over R s.t. $[U] \in ELStab(I-A)$