May 13, 2015

# Localizing the classification problem for Kirchberg X-algebras at the universal UHF alg.

(A) the result
(B) steps in the proof.
(C) consequences in special cases.

(A) Let $X$ be a finite $T_0$-space.

Def: A Kirchberg X-alg. is a separable, nuclear, $\mathcal{O}_\infty$-absorbing $C^*$-algebra $\mathfrak{A}$ w. an identification $\mathrm{Prim}(\mathfrak{A}) \xrightarrow{\approx} X$ (then ideal structure $\mathbb{I}(\mathfrak{A}) \longleftrightarrow \mathrm{open}(X)$).

Thm (Kirchberg): For stable Kirchberg X-algebras $\mathfrak{A}, \mathfrak{B}$.

$$\mathrm{Iso}_X(\mathfrak{A},\mathfrak{B}) \Big/ \substack{\text{unitary} \\ \text{homotopy}} \longleftrightarrow KK(X;\mathfrak{A},\mathfrak{B})^{-1}.$$

classification up to $KK(X)$-equivalence is hard; there are results by Bonkat, Restorff, Meyer-Nest, B-Köhler, B-Meyer only for certain spaces $X$.

idea: localize at $M_{\mathbb{Q}}$!

NOTE: $\left.\begin{matrix} K_0(M_{\mathbb{Q}}) = \mathbb{Q} \\ K_1(M_{\mathbb{Q}}) = 0 \end{matrix}\right\}$ sufficient for our purposes.

Def'n: Let $\mathbb{Q}K_i(X)$ be the full subcategory of $KK(X)$ (i.e. same morphisms) w. objects stable Kirchberg $X$-algebras $\mathfrak{A}$ s.t.

- $\mathfrak{A} \in \mathcal{B}(X)$ (corresp. bootstrap class) ($\Leftrightarrow$ all simple subquotients $\mathfrak{A}(\{x\})$, $x \in X$, satisfy the UCT)
- $\mathfrak{A} \otimes M_{\mathbb{Q}} \cong \mathfrak{A}$ ($\Leftrightarrow K_*(\mathfrak{A}(\{x\}))$ rational $(\forall)\, x \in X$)

We have a canonical invariant

$$XK : \mathbb{Q}K_i(X) \longrightarrow \left(\mathrm{Vect}_c^{\mathbb{Z}_2}(\mathbb{Q})\right)^X = \mathrm{Mod}_c^{\mathbb{Z}/2}(\mathbb{Q}X)$$

$$A \longmapsto K_*(A(U_x))_{x \in X} + \text{maps induced by ideal inclusion}$$

where $U_x$ is the smallest open neighbourhood of $x$ in $X$.

$x \geq y \Leftrightarrow U_x \subseteq U_y$, $\quad x \longrightarrow y \Leftrightarrow x > y$, $(\nexists)\, z : x > z > y$

example:

$$X = \begin{tikzcd} & 3 \arrow[dr] & \\ 4 \arrow[ur] \arrow[dr] & & 1 \\ & 2 \arrow[ur] & \end{tikzcd}$$

open nbd: if $x \in U$ then for every $y \to x$, $y \in U$.

$$XK(A) = \begin{tikzcd} & K_*(A(U_3)) \arrow[dr] & \\ K_*(A(U_4)) \arrow[ur] \arrow[dr] & & K_*(A(U_1)) \\ & K_*(A(U_2)) \arrow[ur] & \end{tikzcd}$$

$\mathbb{Q}X$ = rational incidence algebra of $X$.

generators $\{c_x^y,\ x \geq y\}$ s.t. $c_y^z c_x^y = c_x^z$, $x \geq y \geq z$

finite-dimensional, finite global dimension.

Thm:

$$\begin{tikzcd} \mathbb{Q}Ki(X) \arrow[rr, "\sim", "\overline{XK}"'] \arrow[dr, "XK"'] & & Der_c^{\mathbb{Z}/2}(\mathbb{Q}X) \arrow[dl, "H_*"] \\ & Mod_c^{\mathbb{Z}/2}(\mathbb{Q}X) & \end{tikzcd}$$

Corollary: for $A, B \in \mathbb{Q}Ki(X)$

$Iso_X(A,B)/$unitary homotopy $\longleftrightarrow$ $\{$zig-zags of quasi-isomorphisms $\overline{XK(\mathcal{A})} \leftarrow \mathcal{C} \rightarrow \overline{XK(\mathcal{B})}\}/\sim$

objects $\quad M_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} M_1 \qquad M_i \in Mod_c^{\mathbb{Z}/2}(\mathbb{Q}X)$

$d_0 \circ d_1 = 0 = d_1 \circ d_0$

chain maps

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] \arrow[d, "\varphi_0"'] & M_1 \arrow[l, shift left, "d_1"] \arrow[d, "\varphi_1"] \\ N_0 \arrow[r, shift left, "e_0"] & N_1 \arrow[l, shift left, "N_1"] \end{tikzcd}$$

s.t. all possible squares commute.

morphisms $M_\bullet \longrightarrow N_\bullet$ zig-zags $M_\bullet \xleftarrow{\sim} C_\bullet \longrightarrow N_\bullet$

w. equivalence relation.

(B). topological model

(dell'Ambrogio - Emerson - Kandelaki - Meyer)

$$\mathbb{K} : \mathcal{B} \xrightarrow{\sim} \mathcal{H}o\,(\mathrm{Mod}(K))_c$$

commutative symmetric ring spectrum

generalize to

$$\mathcal{B}(X) \xrightarrow{\sim} \mathcal{H}o\,(\mathrm{Mod}(K)^X)_c$$

$$A \longmapsto (\mathbb{K}(A(U_x)))_{x \in X}$$

$$\mathbb{Q}K_i(X) \cong \mathbb{Q}\mathcal{B}(X) \cong \mathcal{H}o\,(L_{\substack{\text{pointwise}\\ \text{rational}\\ \text{equiv.}}}(\mathrm{Mod}(K)^X))_c$$

Rørdam (simple case),

Meyer-West (non-simple case)

$$\cong \mathcal{H}o\,((\underbrace{L_{\substack{\text{rational}\\ \text{equivalence}}}(\mathrm{Mod}(K))}_{\text{12 Quillen equiv.}})^X)_c$$

$\mathrm{Mod}(K \wedge H\mathbb{Q})$

due to Shipley

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$\mathrm{Mod}(\mathbb{A})$

$\mathbb{A}$ - commutative differentiable graded alg with $H_*(\mathbb{A}) = \underbrace{\mathbb{Q}[\beta,\beta^{-1}]}_{\text{intrinsically formal}}$

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$\mathrm{Mod}(\mathbb{Q}[\beta,\beta^{-1}])$

$\blacksquare$.

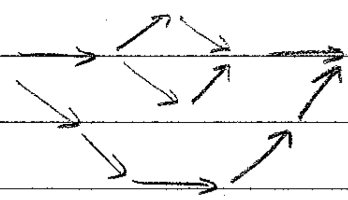
$\longrightarrow$

(c) * if $X = \bullet$ then $\mathbb{Q}X = \mathbb{Q}$ and $\mathcal{P}er_c^{\mathbb{Z}/2}(\mathbb{Q}X) = \mathrm{Vect}_c^{\mathbb{Z}/2}(\mathbb{Q})$.

* If $X$ is a unique path space, then global dim $(\mathbb{Q}X) = 1$.
$\Rightarrow$ classification by $XK$.

Thm (Igusa-Zacharia, 2007): global dim $(\mathbb{Q}X) \leq 2$
$\Leftrightarrow (\forall)\ x, y \in X$, $|\{z : x < z < y\}|$ (geometric realisation) is a disjoint union of contractible spaces.

e.g.

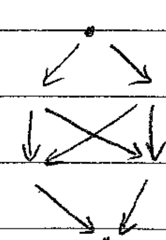


In this case, we can refine $XK$ by an $\mathrm{Ext}^2$-obstruction class.

Ladkani ($\sim$2006) constructed many derived equivalences $\mathcal{D}(\mathbb{Q}X) \cong \mathcal{D}(\mathbb{Q}X')$. Thm (Zhao, 2014) then implies $\mathcal{D}_f^{\mathbb{Z}/2}(\mathbb{Q}X) \simeq \mathcal{D}_f^{\mathbb{Z}/2}(\mathbb{Q}X')$

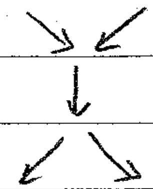
example:

$X =$



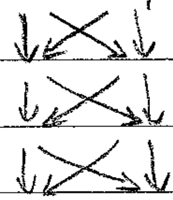
$\mathbb{Q}X$ has global dimension 3,
but

$\longrightarrow$

$X' =$ 

$QX'$ has global dimension 1.
can translate invariant from $X'$

non-example:



has global dimension 3; will not be equivalent to something of smaller global dimension