The Operator System Generated by Cuntz Isometries

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May 15, 2015

CLASSIFICATION OF C*-ALGEBRAS, FLOW EQUIVALENCE OF SHIFT SPACES, AND GRAPH AND LEAVITT PATH ALGEBRAS
MAY 11-15, 2015
University of Louisiana at Lafayette

Outline:

In this talk, we consider the operator system $\mathcal{S}_n$ $(2 \leq n < \infty)$ generated by the Cuntz isometries $S_1, \ldots, S_n$, that is, $S_1, \ldots, S_n$ are isometries with $\sum_{i=1}^n S_i S_i^* = I$.

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We will:

1. show that $\mathcal{S}_n$ has a universal property;
2. define an operator system $\mathcal{E}_n \subseteq M_n$ and prove that $\mathcal{S}_n$ is complete order isomorphic to a quotient of $\mathcal{E}_n$;
3. study tensor products of $\mathcal{S}_n$ and examine various nuclearities of $\mathcal{S}_n$ in the operator system category;
4. discover some properties of $\mathcal{S}_n^d$;
5. discuss some open problems.

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1. The operator system generated $\mathfrak{S}_n$ by n universal unitaries can also be used to characterize WEP, Kirchberg's Conjecture, etc. Farenick, Kavruk and Paulsen have shown that

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C^*(F_2) \otimes_{\min} C^*(F_2) = & C^*(F_2) \otimes_{\max} C^*(F_2) \\
& \Longleftrightarrow \mathfrak{S}_2 \otimes_{\min} \mathfrak{S}_2 = \mathfrak{S}_2 \otimes_c \mathfrak{S}_2. \\
\mathcal{A} \text{ has WEP } & \Longleftrightarrow \mathfrak{S}_2 \otimes_{\min} \mathcal{A} = \mathfrak{S}_2 \otimes_{\max} \mathcal{A}.
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2. The operator system generated by n universal unitaries is completely order isomorphic to the quotient of an operator subsystem of the matrix algebra. Hence, the WEP can be characterized by a lifting property.

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$$C^*(F_2) \otimes_{\min} C^*(F_2) = C^*(F_2) \otimes_{\max} C^*(F_2) \iff \mathfrak{S}_2 \otimes_{\min} \mathfrak{S}_2 = \mathfrak{S}_2 \otimes_c \mathfrak{S}_2.$$

$$\mathcal{A} \text{ has WEP } \iff \mathfrak{S}_2 \otimes_{\min} \mathcal{A} = \mathfrak{S}_2 \otimes_{\max} \mathcal{A}.$$

2. The operator system generated by n universal unitaries is completely order isomorphic to the quotient of an operator subsystem of the matrix algebra. Hence, the WEP can be characterized by a lifting property.
3. Also, it is observed that if $\mathcal{S}_n$ and $\mathcal{T}_n$ are two operator systems generated by n isometries with Cuntz relation, then $\mathcal{S}_n$ being completely order isomorphic to $\mathcal{T}_n$ implies $C^*(\mathcal{S}_n) = C^*(\mathcal{T}_n)$.

Quotient of Operator Systems

Definition (Concrete Operator System)

A **concrete operator system** $\mathcal{S}$ is a unital $*$-closed subspace of a unital C^*-algebra $\mathcal{A}$, that is, $\mathcal{S} \subseteq \mathcal{A}$ is a subspace of $\mathcal{A}$ such that $1 \in \mathcal{S}$ and $a \in \mathcal{S} \Rightarrow a^* \in \mathcal{S}$.

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We write $M_n(\mathcal{S})^+$, $n \in \mathbb{N}$ for the positive cones of $\mathcal{S}$ and $(a_{ij}) \geq 0$ if $(a_{ij}) \in M_n(\mathcal{S})^+$.

Definition (Completely Positive Maps)

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A linear map $\phi : \mathcal{S} \to \mathcal{T}$ is called **completely positive** if

$$\phi^{(n)}((a_{ij})) := (\phi(a_{ij})) \geq 0, \quad \text{for each } (a_{ij}) \in M_n(\mathcal{S})^+ \text{ and for all } n \in \mathbb{N}.$$

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Let $\mathcal{S}$ and $\mathcal{T}$ be operators systems. A map $\phi : \mathcal{S} \to \mathcal{T}$ is called a **complete order isomorphism** if ϕ is a unital linear isomorphism and both ϕ and ϕ^{-1} are completely positive, and we say that $\mathcal{S}$ is completely order isomorphic to $\mathcal{T}$ if such ϕ exists.

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Theorem (Choi, Effros)

Let $\mathcal{S}$ be an abstract operator system, then there exists a Hilbert space $\mathcal{H}$, a concrete operator system $\mathcal{S}_1 \subseteq B(\mathcal{H})$, and a unital complete order isomorphism $\varphi : \mathcal{S} \to \mathcal{S}_1$. Conversely, a concrete operator system is also an abstract operator system.

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Due to this theorem, we can always identify an abstract operator system with a concrete one.

Definition (Kernel)

Given an operator system $\mathcal{S}$, we call $J \subseteq \mathcal{S}$ a *kernel*, if $J = \ker \phi$ for an operator system $\mathcal{T}$ and some (unital) completely positive map $\phi : \mathcal{S} \to \mathcal{T}$.

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Proposition (Kavruk, Paulsen, Todorov, Tomforde)

Let $\mathcal{S}$ be an operator system and $J \subseteq \mathcal{S}$ be kernel, if we define a family of matrix cones on $\mathcal{S}/J$ by setting

$$\begin{aligned} C_n = &\{(x_{ij} + J) \in M_n(\mathcal{S}/J) : \text{ for each } \epsilon > 0, \text{ there exists } (k_{ij}) \in M_n(J) \\ &\text{such that } \epsilon \otimes I_n + (x_{ij} + k_{ij}) \in M_n(\mathcal{S})^+\}. \end{aligned}$$

then $(\mathcal{S}/J, \{C_n\}_{n=1}^{\infty})$ is a matrix ordered $*$-vector space with an Archimedean matrix unit $1 + J$, and the quotient map $q : \mathcal{S} \to \mathcal{S}/J$ is completely positive.

Definition (Operator System Quotient)

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Definition (Complete Quotient Map)

Let $\mathcal{S}$, $\mathcal{T}$ be operator systems and $\phi : \mathcal{S} \to \mathcal{T}$ be a completely positive map, then ϕ is called a **complete quotient map** if $\mathcal{S}/\operatorname{Ker}\phi$ is complete order isomorphic to $\mathcal{T}$.

The Operator System Generated by Cuntz Isometries

Let $S_1, \ldots, S_n$ be n $(n \geq 2)$ isometries in $B(\mathcal{H})$ with $\sum_{i=1}^{n} S_i S_i^* = I$, where $B(\mathcal{H})$ denotes the space of all bounded linear operators on some Hilbert space $\mathcal{H}$, I denotes the identity on $\mathcal{H}$ and we set

$$\mathcal{S}_n = \text{span}\{I, S_1, \ldots, S_n, S_1^*, \ldots, S_n^*\},$$

so that $\mathcal{S}_n$ is the operator system generated by the Cuntz isometries.

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so that $\mathcal{S}_n$ is the operator system generated by the Cuntz isometries.

On the other hand, let $\hat{S}_1, \cdots, \hat{S}_n$ be n $(n \geq 2)$ isometries with $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* < I$ and set

$$\hat{\mathcal{S}}_n = \text{span}\{I, \hat{S}_1, \ldots, \hat{S}_n, \hat{S}_1^*, \ldots, \hat{S}_n^*\},$$

so that $\hat{\mathcal{S}}_n$ is the operator system generated by the Toeplitz-Cuntz isometries.

Proposition

Let $\hat{S}_1, \cdots, \hat{S}_n$ be n $(n \geq 2)$ isometries with $\sum_{i=1}^{n} \hat{S}_i \hat{S}_i^* < I$, then they can be dilated to n isometries $S_1, \cdots, S_n$ with $\sum_{i=1}^{n} S_i S_i^* = I$.

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Proposition

We have that $\mathcal{S}_n = \hat{\mathcal{S}}_n$ completely order isometrically.

Hence, we do not need to distinguish $\mathcal{S}_n$ and $\hat{\mathcal{S}}_n$.

Definition

An n-tuple $(a_1, \ldots, a_n) \in \mathcal{A}$ is called a **row contraction** if $\sum_{i=1}^n a_i a_i^* \leq 1$, where $\mathcal{A}$ is a unital C^*-algebra.

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Proposition (Bunce)

Let $\{A_i : i \in \Gamma\}$ be a family of bounded operators on a Hilbert space $\mathcal{H}$. Then the following two conditions are equivalent.

1. $\sum_{i \in \Gamma} A_i^* A_i \leq I$.
2. *There exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and coisometries $\{S_i : i \in \Gamma\}$ acting on $\mathcal{K}$ such that $S_i S_j^* = 0$ for $i \neq j$, and $S_i(\mathcal{H}) \subseteq \mathcal{H}$, $S_i|_{\mathcal{H}} = A_i$ for each i.*

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Proposition (Universal Property of $\mathcal{S}_n$)

For any row contraction $(A_1, \cdots, A_n) \in \mathcal{A}$, there exists a unital completely positive map $\phi : \mathcal{S}_n \to \mathcal{A}$ such that $\phi(S_i) = A_i$, $1 \leq i \leq n$.

In order to study $\mathcal{S}_n$, we construct an operator subsystem $\mathcal{E}_n \subseteq M_{n+1} := M_{n+1}(\mathbb{C})$ and prove that $\mathcal{S}_n$ is complete order isomorphic to a quotient of $\mathcal{E}_n$. This will allow us to study $\mathcal{S}_n$ via $\mathcal{E}_n$.

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Definition

We define $\mathcal{E}_n \subseteq M_{n+1}$ as the following,

$$\mathcal{E}_n = \operatorname{span}\{E_{00}, E_{0i}, E_{i0}, \sum_{i=1}^{n} E_{ii} : 1 \leq i \leq n\},$$

where E_{ij} denotes the elements in the canonical basis of M_{n+1}.

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We have that an element in $\mathcal{E}_n$ is of the following form,

$$\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0,n} \\ a_{10} & b & & \\ \vdots & & \ddots & \\ a_{n0} & & & b \end{pmatrix}.$$

We define an operator $R : \mathcal{H}^{(n+1)} \to \mathcal{H}$ by $R := (\frac{\sqrt{2}}{2} I, \frac{\sqrt{2}}{2} S_1^*, \ldots, \frac{\sqrt{2}}{2} S_n^*)$.

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$$R^* R = \begin{pmatrix} \frac{1}{2} I & \frac{1}{2} S_1^* & \cdots & \frac{1}{2} S_n^* \\ \frac{1}{2} S_1 & & & \\ \vdots & & \left(\frac{1}{2} S_i S_j^*\right) & \\ \frac{1}{2} S_n & & & \end{pmatrix},$$

is positive in $M_{n+1}(B(\mathcal{H}))$.

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Then, we define $\psi : M_{n+1} \to B(\mathcal{H})$ by

$$\left(\psi(E_{ij})\right)_{i,j=0}^{n} = R^* R,$$

and extend it linearly to M_{n+1}.

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Then, we define $\psi : M_{n+1} \to B(\mathcal{H})$ by

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and extend it linearly to M_{n+1}.
Since $\sum_{i=1}^{n} S_i S_i^* = I$, we have that ψ is unital. Also, it is easy to see that $\operatorname{Ker} \psi = \operatorname{span}\{E_{00} - \sum_{i=1}^{n} E_{ii}\}$. Henceforth, we denote $J := \operatorname{Ker} \psi$

Theorem (Choi)

Let $\mathcal{A}$ be a C^*-algebra, $\phi : M_n \to \mathcal{A}$ be linear, and $\{E_{ij}\}$ be the standard matrix units for M_n, then the following are equivalent:

1. *ϕ is completely positive.*
2. *ϕ is n-positive.*
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Thus, ψ is a unital completely positive map. Let $\phi = \psi|_{\mathcal{E}_n} : \mathcal{E}_n \to \mathcal{S}_n$, then we have that ϕ is also unital completely positive map with $\ker \phi = J$.

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Thus, ψ is a unital completely positive map. Let $\phi = \psi|_{\mathcal{E}_n} : \mathcal{E}_n \to \mathcal{S}_n$, then we have that ϕ is also unital completely positive map with $\ker \phi = J$. Indeed, we can prove that

Theorem

We have that $\mathcal{E}_n / J = \mathcal{S}_n$.

By using the following identification

$$M_p(\mathcal{E})/M_p(J) = M_p(\mathcal{E}/J) = M_p \otimes (\mathcal{E}/J).$$

We can prove that

Theorem

The matrix of operators $A_0 \otimes I + \sum_{i=1}^n A_i \otimes S_i + \sum_{i=1}^n A_i^* \otimes S_i^* \in M_p(\mathcal{S})$ is positive if and only if there exists $B \in M_p$ such that

$$\begin{pmatrix} A_0 & 2A_1^* & \cdots & 2A_n^* \\ 2A_1 & A_0 & & \\ \vdots & & \ddots & \\ 2A_n & & & A_0 \end{pmatrix} + \begin{pmatrix} B & & & \\ & -B & & \\ & & \ddots & \\ & & & -B \end{pmatrix} \geq 0$$

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Let$\mathcal{S}_\infty = \text{span}\{I, S_i, S_i^* : 1 \leq i < +\infty\}$, where $\{S_i\}$ are the generators of $\mathcal{O}_\infty$, then for $n < m \leq \infty$, $\mathcal{S}_n \subseteq_{\text{c.o.i}} \mathcal{S}_m$.

Let $\mathcal{S}$ be an operator system and $\mathcal{S}^d$ be the space of all bounded linear functionals on it. We define an order structure on $\mathcal{S}^d$ by

$$(f_{ij}) \in M_p(\mathcal{S}_n^d)^+ \Leftrightarrow (f_{ij}) : \mathcal{S}_n \to M_p \text{ is completely positive .}$$

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It is a well-known result by Choi and Effros that with the order structure defined above, the dual space of a finite dimensional operator system is again an operator system with an Archimedean order unit, and indeed, any strictly positive linear functional is an Archimedean order unit. Hence, $\mathcal{S}_n^d$ is an operator system with Archimedean order unit δ_0.

We choose a basis for $\mathcal{S}_n^d$ as the following,

$$\{\delta_0, \delta_i, \delta_i^* : 1 \leq i \leq n\},$$

where

$$\begin{aligned} \delta_0(I) := 1, \delta_0(S_i) := \delta_0(S_i^*) = 0, &\quad \text{for all } i; \\ \delta_i(I) = 0, \delta_i(S_j) = \delta_{ij}, \delta_i(S_k^*) = 0, &\quad \text{for all } k; \\ \delta_i^*(I) = 0, \delta_i^*(S_j^*) = \delta_{ij}, \delta_i(S_k) = 0, &\quad \text{for all } k, \end{aligned}$$

where δ_{ij} is the Kronecker delta notation.
So we have that $\mathcal{S}_n^d = \text{span}\{\delta_0, \delta_i, \delta_i^* : 1 \leq i \leq n\}$.

Proposition

An element $I_p \otimes \delta_0 + \sum_{i=1}^n A_i \otimes \delta_i + \sum_{i=1}^n A_i^* \otimes \delta_i^* \in M_p(\mathcal{S}_n^d)$ *is positive if and only if* $(A_1, \ldots, A_n)$ *is a row contraction.*

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Corollary

A unital linear map $\phi : \mathcal{S}_n^d \to \mathcal{A}$ is completely positive if and only if ϕ is self-adjoint and

$$w(S_1 \otimes \phi(\delta_1) + \cdots + S_n \otimes \phi(\delta_n)) \leq \frac{1}{2},$$

where $S_1, \ldots, S_n$ are Cuntz isometries.

Question

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The following proposition can be proved by using Choi's multiplicative domain techniques.

Proposition

If we have that

$$\mathcal{S}_n \otimes_{\min} \mathcal{A} = \mathcal{S}_n \otimes_{\max} \mathcal{A},$$

for a C^*-algebra $\mathcal{A}$, then without using the nuclearity of $\mathcal{O}_n$, we have that

$$\mathcal{O}_n \otimes_{\min} \mathcal{A} = \mathcal{O}_n \otimes_{\max} \mathcal{A}.$$

So if we can show that $\mathcal{S}_n \otimes_{\min} \mathcal{A} = \mathcal{S}_n \otimes_{\max} \mathcal{A}$, then we will be able to give an alternative proof of the nuclearity of $\mathcal{O}_n$.

So if we can show that $\mathcal{S}_n \otimes_{\min} \mathcal{A} = \mathcal{S}_n \otimes_{\max} \mathcal{A}$, then we will be able to give an alternative proof of the nuclearity of $\mathcal{O}_n$. Using the nuclearity of $\mathcal{O}_n$, we can show that this even by using the fact that $\mathcal{O}_n$ is exact. Without using the nuclearity of $\mathcal{O}_n$, we can show that $\mathcal{S}_n \otimes_{\min} B(\mathcal{H}) = \mathcal{S}_n \otimes_{\max} B(\mathcal{H})$ (OSLLP), for any Hilbert space $\mathcal{H}$. Finally, we turn our attention to $\mathcal{S}_n^d$ and we can see that $\mathcal{S}_n^d$ has the LP. In addition, we use the LP of $\mathcal{S}_n^d$ to prove a lifting result concerning the joint numerical radius.

Definition (The Min Tensor Product)

The **minimal operator system structure** on $\mathcal{S} \otimes \mathcal{T}$ is defined as

$$C_n^{\min} = \{(p_{ij}) \in M_n(\mathcal{S} \otimes \mathcal{T}) : \Big((\phi \otimes \psi)(p_{ij})\Big) \in M_{nkm}^+,$$
$$\text{for all } \phi \in S_k(\mathcal{S}), \psi \in S_m(\mathcal{T}), \text{ for all } k, m \in \mathbb{N}\},$$

where $S_k(\mathcal{S})$ denotes the set of all completely positive maps from $\mathcal{S}$ to M_k. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, (C_n^{\min})_{n=1}^{\infty}, 1 \otimes 1)$ the **minimal tensor product** of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\min} \mathcal{T}$.

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It can be shown that the min-tensor product is injective ,associative, symmetric and functorial. Moreover, it coincide with the operator system arising from the embedding $\mathcal{S} \otimes \mathcal{T} \subseteq_{\text{c.o.i}} B(\mathcal{H} \otimes \mathcal{K})$.

Definition (The Max Tensor Product)

The **maximal operator system structure** on $\mathcal{S} \otimes \mathcal{T}$ is defined as the Archimedeanization of the following cones:

$$D_n^{\max} = \{a(P \otimes Q)a^* : P \in M_k(\mathcal{S})^+, Q \in M_m(\mathcal{T})^+, a \in M_{n,km}, k, m \in \mathbb{N}\}.$$

We denote the Archimedeanization of $D_n^{\max}$ as $C_n^{\max}$, then the **maximal tensor product** of $\mathcal{S}$ and $\mathcal{T}$, denoted by $\mathcal{S} \otimes_{\max} \mathcal{T}$, is the operator system $(\mathcal{S} \otimes \mathcal{T}, (C_n^{\max})_{n=1}^{\infty}, 1 \otimes 1)$.

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The max-tensor product is symmetric, associative and functorial. We will also see later that it is projective.

Definition (The Commuting Tensor Product)

Let $\{\mathcal{S}, \mathcal{T}\}$ be operator systems. We set

$$\begin{aligned} \text{CP}(\mathcal{S}, \mathcal{T}) = &\{(\phi, \psi) : \phi \text{ is CP from } \mathcal{S} \text{ to } B(\mathcal{H}), \\ &\psi \text{ is CP from } \mathcal{T} \text{ to } B(\mathcal{H}), \text{ and } \phi(\mathcal{S}) \text{ commutes with } \phi(\mathcal{T})\} \end{aligned}$$

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We define $\phi \cdot \cdot \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to B(\mathcal{H})$ as $\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y)$.
The **commuting operator system structure** on $\mathcal{S} \otimes \mathcal{T}$ is defined as:

$$C_n^c = \{u \in M_n(\mathcal{S} \otimes \mathcal{T}) : (\phi \cdot \psi)^{(n)}(u) \geq 0, \text{ for all } (\phi, \psi) \in \mathrm{CP}(\mathcal{S}, \mathcal{T}).$$

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We call the operator system $(\mathcal{S} \otimes \mathcal{T}, (C_n^c)_{n=1}^{\infty}, 1 \otimes 1)$ the **commuting tensor product** of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_c \mathcal{T}$.

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We call the operator system $(\mathcal{S} \otimes \mathcal{T}, (C_n^c)_{n=1}^{\infty}, 1 \otimes 1)$ the **commuting tensor product** of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_c \mathcal{T}$.

The commuting-tensor product is symmetric and functorial. Also, if $\mathcal{A}$ is a C^*-algebra and $\mathcal{S}$ is an operator system, then we have
$\mathcal{S} \otimes_c \mathcal{A} = \mathcal{S} \otimes_{max} \mathcal{A}$.

Definition (The el and er Tensor Product)

We let $\mathcal{S} \otimes_{\text{el}} \mathcal{T}$ (resp. $\mathcal{S} \otimes_{\text{er}} \mathcal{T}$) be the operator system whose operator structure on $\mathcal{S} \otimes \mathcal{T}$ is induced by the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{I}(\mathcal{S}) \otimes_{\max} \mathcal{T}$ (resp. $\mathcal{S} \otimes_{\max} \mathcal{I}(\mathcal{T})$). Here, $\mathcal{I}(\mathcal{S})$ denotes the injective envelope of $\mathcal{S}$.

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We have that el, er-tensor product are both functorial but not symmetric.

1. We call $\tau_1 \leq \tau_2$ if $M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+$ for every $n \in \mathbb{N}$.

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1. We call $\tau_1 \leq \tau_2$ if $M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+$ for every $n \in \mathbb{N}$.
2. Let α and β be two operator system tensor products. An operator system $\mathcal{S}$ is called **(α, β)-nuclear** if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is completely order isomorphic for every operator system $\mathcal{T}$.

The order relations between these tensor products are:

$$\min \leq \text{ el, er } \leq c \leq \max .$$

Claim

The operator system $\mathcal{S}_n$ is not (min, max)-nuclear.

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On the other hand, let $\mathcal{S}_1$ be the operator system generated by a universal isometry, i.e. $\mathcal{S}_1 = \text{span}\{I, S_1, S_1^*\}$, where S_1 is an arbitrary isometry with $S_1 S_1^* < I$. We have that

$$\mathfrak{S}_1 = \mathcal{S}_1.$$

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However, $\Phi(X) = S_1^* X S_1$ is a completely positive projection from $\mathcal{S}_n$ onto $\mathcal{S}_1$. Thus,

$$\mathfrak{S}_1 \otimes_{\min} \mathcal{S}_1 \subseteq_{\text{c.o.i}} \mathfrak{S}_1 \otimes_{\min} \mathcal{S}_n$$
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So $\mathfrak{S}_1 \otimes_{\min} \mathcal{S}_n \neq \mathfrak{S}_1 \otimes_{\max} \mathcal{S}_n$.

Theorem (Farenick, Paulsen)

Let $\mathcal{S}$ and $\mathcal{T}$ be finite-dimensional operator systems. Then we have that

$$(\mathcal{S} \otimes_{\min} \mathcal{T})^d = \mathcal{S}^d \otimes_{\max} \mathcal{T}^d,$$

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Thus, it follows from the last slide that $\mathfrak{S}_1^d \otimes_{\min} \mathcal{S}_n^d \neq \mathfrak{S}_1^d \otimes_{\max} \mathcal{S}_n^d$

Definition (Left Exact, Right Exact, Exact, 1–exact)

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An operator system $\mathcal{S}$ is called *left exact* if for every unital C^*-algebra $\mathcal{B}$ and every ideal $I \subseteq \mathcal{B}$, we have that $I \hat{\otimes}_{\min} \mathcal{S}$ is the kernel of the map $q \otimes \mathrm{id}_{\mathcal{S}} : \mathcal{B} \hat{\otimes}_{\min} \mathcal{S} \to (\mathcal{B}/I) \hat{\otimes}_{\min} \mathcal{S}$, where $q : \mathcal{B} \to \mathcal{B}/I$ is the quotient map.

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$$\mathcal{B}\hat{\otimes}_{\min}\mathcal{S}/(I\hat{\otimes}_{\min}\mathcal{S}) \to (\mathcal{B}/I)\hat{\otimes}_{\min}\mathcal{S}),$$

is a complete order isomorphism.

Theorem (Kavruk, Paulsen, Todorov, Tomforde)

The following properties of an operator system are equivalent:

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Proposition

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For any operator system $\mathcal{T}$, we have that

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On the other hand,

$$\mathcal{S}_n \otimes_{\min} \mathcal{T} \subseteq_{\mathrm{c.o.i}} \mathcal{O}_n \otimes_{\min} \mathcal{T}.$$

□

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Let $\mathcal{S}$ be an operator system, $\mathcal{A}$ be a unital C^*-algebra, I be an ideal of $\mathcal{A}$, $q : \mathcal{A} \to \mathcal{A}/I$ be the quotient map and $\phi : \mathcal{S} \to \mathcal{A}/I$ be a unital completely positive map.

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1. $\mathcal{S}$ *has the OSLLP;*
2. $\mathcal{S} \otimes_{\min} B(\mathcal{H}) = \mathcal{S} \otimes_{\max} B(\mathcal{H})$.

So we need to show that $\mathcal{S}_n \otimes_{\min} B(\mathcal{H}) = \mathcal{S}_n \otimes_{\max} B(\mathcal{H})$.

We will use the isomorphism:

$$\mathcal{S}_n = \mathcal{E}_n / J.$$

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Lemma

The operator system $\mathcal{E}_n$ is C^*-nuclear, i.e. $\mathcal{E}_n \otimes_{\min} \mathcal{A} = \mathcal{E}_n \otimes_{\max} \mathcal{A}$ for each C^*-algebra $\mathcal{A}$.

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Proposition (Projectivity of the Max-Tensor Product)

Let $\mathcal{S}$, $\mathcal{T}$, $\mathcal{R}$ be operator systems and suppose $\psi : \mathcal{S} \to \mathcal{R}$ is a complete quotient map, then the map $\psi \otimes \mathrm{id}_{\mathcal{T}} : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{R} \otimes_{\max} \mathcal{T}$ is also a complete quotient map.

Hence, in order to show that the following diagram commute,

$$\begin{array}{ccc} \mathcal{E}_n \otimes_{\min} \mathcal{A} & \xrightarrow{\cong} & \mathcal{E}_n \otimes_{\max} \mathcal{A} \\ \Big\downarrow{\scriptstyle \phi\otimes \mathrm{id}_{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \phi\otimes \mathrm{id}_{\mathcal{A}}} \\ \mathcal{S}_n \otimes_{\min} \mathcal{A} & \xrightarrow{\mathrm{id}} & \mathcal{S}_n \otimes_{\max} \mathcal{A} \end{array},$$

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we just need to show that $\phi \otimes \mathrm{id}_{\mathcal{A}} : \mathcal{E}_n \otimes_{\min} \mathcal{A} \to \mathcal{S}_n \otimes_{\min} \mathcal{A}$ is a complete quotient map.

Proposition

Let $\mathcal{A} = B(\mathcal{K})$ *for an arbitrary Hilbert space* $\mathcal{K}$, *then the map* $\phi \otimes \mathrm{id}_{\mathcal{A}} : \mathcal{E}_n \otimes_{\min} \mathcal{A} \to \mathcal{S}_n \otimes_{\min} \mathcal{A}$ *is a complete quotient map.*

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The case when $n = \infty$ can be deduce by the inclusion $\mathcal{S}_n \subseteq_{\text{c.o.i}} \mathcal{S}_\infty$. So it follows that $\mathcal{S}_\infty$ has OSLLP by the following observation:

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If $A \in (\mathcal{S}_\infty \otimes_{\min} B(\mathcal{H}))^+$, then there exists $N \in \mathbb{N}$ such that $A \in (\mathcal{S}_N \otimes_{\min} B(\mathcal{H}))^+$.

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If $A \in (\mathcal{S}_\infty \otimes_{\min} B(\mathcal{H}))^+$, then there exists $N \in \mathbb{N}$ such that $A \in (\mathcal{S}_N \otimes_{\min} B(\mathcal{H}))^+$.

On the other hand, we have that

$$(\mathcal{S}_N \otimes_{\min} B(\mathcal{H}))^+ = (\mathcal{S}_N \otimes_{\max} B(\mathcal{H}))^+ \subseteq (\mathcal{S}_\infty \otimes_{\max} \mathcal{A})^+.$$

Definition

For an $n-$tuple of operators $(T_1, \ldots, T_n) \in B(\mathcal{H})$, their **joint numerical radius** is defined as:

$$w(T_1, \ldots, T_n) := \sup \left| \sum_{\alpha \in F_n^+} \sum_{j=1}^{n} \langle h_\alpha, T_j h_{g_j \alpha} \rangle \right|,$$

where F_n is the free group on n generators $g_1, \ldots, g_n$, and the supremum is taken over all families of vectors $\{h_\alpha\}_{\alpha \in F_n^+} \subseteq \mathcal{H}$ with $\sum_{\alpha \in F_n^+} \|h_\alpha\|^2 = 1$.

We can extend this notion of joint numerical radius for n-tuples in $B(\mathcal{H})$ to the category of C^*-algebras.

Definition

Let $\mathcal{A}$ be a C^*-algebra. The *joint numerical radius* of an n-tuple $(a_1, \ldots, a_n) \in \mathcal{A}$ is:

$$w(a_1, \ldots, a_n) := w(S_1 \otimes a_1^* + \cdots + S_n \otimes a_n^*),$$

where S_i's are Cuntz isometries.

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Theorem

A unital linear map $\phi : \mathcal{S}_n^d \to \mathcal{A}$ is completely positive if and only if

$$w(\phi(a_1)^*, \ldots, \phi(a_n)^*) \leq \frac{1}{2}.$$

Theorem (kavruk)

Let $\mathcal{S}$ be a finite dimensional. Then $\mathcal{S}$ is exact if and only if $\mathcal{S}^d$ has the lifting property, and vice versa.

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Theorem

Let $\mathcal{A}$ be a unital C^*-algebra and $J \lhd \mathcal{A}$ be an ideal. Suppose $T_1 + J, \dots, T_n + J \in \mathcal{A}/J$, then there exist $W_1, \dots, W_n \in \mathcal{A}$ with $W_i + J = T_i + J$ for each $1 \leq i \leq n$, such that $w(W_1, \dots, W_n) = w(T_1 + J, \dots, T_n + J)$.

Question

The operator space generated by Cuntz isometries is not of much interest (it is just the column Hilbert space). So what about the $*$-operator space,

$$\text{span}\{S_i, S_i^* : 1 \leq i \leq n\}?$$

Also, what about the operator system:

$$\text{span}\{I, S_i, S_i^*, S_iS_i^* : 1 \leq i \leq n\}, \quad \text{etc.}$$

Consider the Cuntz-Krieger algebra $\mathcal{O}_A$, which is universal the C^*-algebra generated by n partial isometries S_i satisfying

$$\sum_{i=1}^{n} S_i S_i^* = I, \quad S_i^* S_i = \sum_{i=1}^{n} A(i,j) S_j S_j^*, \quad A(i,j) = 0, 1.$$

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Question

Can we deal with the operator system $\mathcal{S}_A^n$ generated by the universal partial isometries from $\mathcal{O}_A$?