Boundary Quotients of Semigroup C*-algebras

Charles Starling

uOttawa

May 14, 2015

Overview

- P - left cancellative semigroup
- Reduced $C_r^*(P)$ and Li's $C^*(P)$
- Li (2012, 2013) – studied $C^*(P)$ when $P \subset G$ (G group).
- What about when P does not embed in a group?
- $P \subset \mathcal{S}$, an *inverse semigroup* (always)
- $C^*(P)$ is an inverse semigroup algebra, with natural boundary quotient $\mathcal{Q}(P)$.
- Conditions on P which guarantee $\mathcal{Q}(P)$ simple, purely infinite.
- Self-similar groups

Semigroups

P countable semigroup (associative multiplication)

Left cancellative: $ps = pq \Rightarrow s = q$

Principal right ideal: $rP = \{rq \mid q \in P\}$

Elements of rP are right multiples of r

Assume $1 \in P$ (ie, P is a monoid)

Semigroups

Study P by representing on a Hilbert space, similar to groups.

$\ell^2(P)$ – square-summable complex functions on P.

δ_x – point mass at $x \in P$. Orthonormal basis of $\ell^2(P)$.

$v_p : \ell^2(P) \to \ell^2(P)$ bounded operator $v_p(\delta_x) = \delta_{px}$ (necessarily isometries)

$\{v_p\}_{p \in P}$ generate the reduced C*-algebra of P, $C_r^*(P)$

$v : P \to C_r^*(P)$ is called the left regular representation

Unlike the group case, considering all representations turns out to be a disaster

Li: we have to care for ideals.

Li's Solution

For $X \subset P$, then $e_X : \ell^2(P) \to \ell^2(P)$ is defined by

$$(e_X \xi)(p) = \begin{cases} \xi(p) & \text{if } p \in X \\ 0 & \text{otherwise.} \end{cases}$$

Note: $v_1 = e_P$

Note that in $\mathcal{B}(\ell^2(P))$,

$$v_p e_X v_p^* = e_{pX} \qquad v_p^* e_X v_p = e_{p^{-1}X}$$

If $p \in P$ and X is a right ideal, then

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

are right ideals too.

Li's Solution

$$pX = \{px \mid x \in X\} \quad p^{-1}X = \{y \mid py \in X\}$$

$\mathcal{J}(P)$ – smallest set of right ideals containing P, $\emptyset$, and closed under intersection and the above operations for all p – constructible ideals.

These are the ideals which are "constructible" inside $C_r^*(P)$.

1. $e_X e_Y = e_{X \cap Y}$
2. $e_P = 1$, $e_\emptyset = 0$
3. $v_p e_X v_p^* = e_{pX}$ and $v_p^* e_X v_p = e_{p^{-1}X}$

Li's Solution

$$pX = \{px \mid x \in X\} \qquad p^{-1}X = \{y \mid py \in X\}$$

$\mathcal{J}(P)$ – smallest set of right ideals containing P, $\emptyset$, and closed under intersection and the above operations for all p – constructible ideals.

These are the ideals which are "constructible" inside $C_r^*(P)$.

1. $e_X e_Y = e_{X \cap Y}$
2. $e_P = 1, \; e_\emptyset = 0$
3. $v_p e_X v_p^* = e_{pX}$ and $v_p^* e_X v_p = e_{p^{-1}X}$

Definition (Li)

$C^*(P)$ is the universal C-algebra generated by isometries $\{v_p \mid p \in P\}$ and projections $\{e_X \mid X \in \mathcal{J}(P)\}$ satisfying the above (and $v_p v_q = v_{pq}$).*

When does a semigroup embed in a group?

For $P \subset G$, we need cancellativity (left and right) $+$ *something*

Examples of *something*s which work:

- commutativity (Grothendieck group)
- $rP \cap qP \neq \emptyset$ for all r, q. (Ore condition)

When does a semigroup embed in a group?

For $P \subset G$, we need cancellativity (left and right) + *something*

Examples of *something*s which work:

- commutativity (Grothendieck group)
- $rP \cap qP \neq \emptyset$ for all r, q. (Ore condition)
- Rees conditions:
  - principal right ideals are comparable or disjoint, and
  - each principal right ideal is contained in only a finite number of other principal right ideals
- others...

Example: Free Semigroups

X finite set, $X^0 = \{\varnothing\}$, X^n words of length n in X.

$$X^* = \bigcup_{n \geq 0} X^n$$

This is the free semigroup on X, under concatenation.

$X^* \subset \mathbb{F}_X$, the free group.

Not Ore (unless $|X| = 1$): if $x, y \in X$ and $x \neq y$, we have $xX^* \cap yX^* = \emptyset$

$C^*(X^*) \cong C^*_r(X^*) \cong \mathcal{T}_{|X|}$

$\mathcal{T}_n$ Toeplitz algebra – generated by n isometries with orthogonal ranges.

Boundary Quotient

Simplification: suppose that for all $r, q \in P$, either $rP \cap qP = \emptyset$ or

$$rP \cap qP = sP \text{ some } s \in P$$

Then $\mathcal{J}(P) = \{sP \mid s \in P\} \cup \{\emptyset\}$.

Such semigroups are called Clifford semigroups, or right LCM semigroups.

Finite $F \subset P$ is a foundation set if for all $r \in P$, there is $f \in F$ with $fP \cap rP \neq \emptyset$.

Definition (Brownlowe, Ramagge, Robertson, Whittaker)

The boundary quotient $\mathcal{Q}(P)$ is the universal C-algebra generated by the same elements and relations as in Li's $C^*(P)$, and also satisfying*

$$\prod_{f \in F}(1 - e_{fP}) = 0 \text{ for all foundation sets } F.$$

Boundary Quotient

What the heck does "$\prod_{f \in F}(1 - e_{fP}) = 0$ for all foundation sets F." mean?

Boundary Quotient

What the heck does "$\prod_{f\in F}(1-e_{fP})=0$ for all foundation sets F." mean?

$D :=$ unital, commutative C*-algebra generated by $\{e_{rP}\}_{r\in P}$

Projections in D have a "greatest lower bound", "least upper bound", and "complement":

$$e\wedge f = ef \qquad e\vee f = e+f-ef \qquad \neg e = 1-e$$

ie, they form a Boolean algebra. Rearranging $\prod_{f\in F}(1-e_{fP})=0$ using de Morgan's laws gives

$$\bigvee_{f\in F} e_{fP} = 1.$$

Boundary Quotient

What the heck does "$\prod_{f\in F}(1-e_{fP})=0$ for all foundation sets F." mean?

$D :=$ unital, commutative C*-algebra generated by $\{e_{rP}\}_{r\in P}$

Projections in D have a "greatest lower bound", "least upper bound", and "complement":

$$e\wedge f = ef \qquad e\vee f = e+f-ef \qquad \neg e = 1-e$$

ie, they form a Boolean algebra. Rearranging $\prod_{f\in F}(1-e_{fP})=0$ using de Morgan's laws gives

$$\bigvee_{f\in F} e_{fP} = 1.$$

Free semigroup: $\mathcal{Q}(X^*)\cong\mathcal{O}_{|X|}$.

Even when P does not embed into a group, it embeds into an inverse semigroup.

A semigroup S is called an inverse semigroup if for every element $s \in S$ there is a unique element s^* such that

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*$$

Any set of partial isometries in a C*-algebra closed under multiplication and adjoint is an inverse semigroup.

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

C*-algebras that are $C^*_{\text{tight}}(S)$ for some S:

1. Graph C*-algebras & k-graph C*-algebras [Exel, 2008]

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

C*-algebras that are $C^*_{\text{tight}}(S)$ for some S:

1. Graph C*-algebras & k-graph C*-algebras [Exel, 2008]
2. Tiling C*-algebras [Exel, Gonçalves, S, 2012]

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

C*-algebras that are $C^*_{\text{tight}}(S)$ for some S:

1. Graph C*-algebras & k-graph C*-algebras [Exel, 2008]
2. Tiling C*-algebras [Exel, Gonçalves, S, 2012]
3. Katsura C*-algebras, Nekrashevych C*-algebras [Exel, Pardo, 2014]

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

C*-algebras that are $C^*_{\text{tight}}(S)$ for some S:

1. Graph C*-algebras & k-graph C*-algebras [Exel, 2008]
2. Tiling C*-algebras [Exel, Gonçalves, S, 2012]
3. Katsura C*-algebras, Nekrashevych C*-algebras [Exel, Pardo, 2014]
4. Carlsen-Matsumoto C*-algebras [S, 2015]

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

C*-algebras that are $C^*_{\text{tight}}(S)$ for some S:

1. Graph C*-algebras & k-graph C*-algebras [Exel, 2008]
2. Tiling C*-algebras [Exel, Gonçalves, S, 2012]
3. Katsura C*-algebras, Nekrashevych C*-algebras [Exel, Pardo, 2014]
4. Carlsen-Matsumoto C*-algebras [S, 2015]
5. Ample groupoid C*-algebras [Exel 2010] (S not countable)

Inverse Semigroups

Many C*-algebras of interest are generated by an inverse semigroup of partial isometries.

For a given S, we have

- $C^*(S)$ – universal C*-algebra of S (Toeplitz-type)
- $C^*_{\text{tight}}(S)$ – tight C*-algebra of S (Cuntz-type)

C*-algebras that are $C^*_{\text{tight}}(S)$ for some S:

1. Graph C*-algebras & k-graph C*-algebras [Exel, 2008]
2. Tiling C*-algebras [Exel, Gonçalves, S, 2012]
3. Katsura C*-algebras, Nekrashevych C*-algebras [Exel, Pardo, 2014]
4. Carlsen-Matsumoto C*-algebras [S, 2015]
5. Ample groupoid C*-algebras [Exel 2010] (S not countable)

$C^*(S)$ and $C^*_{\text{tight}}(S)$ come from étale groupoids, which can be analyzed

Inverse Semigroups

For our right LCM semigroup P,

$$\mathcal{S} := \{v_p v_q^* \mid p, q \in P\} \cup \{0\}$$

is closed under multiplication, and so is an inverse semigroup.

$$(v_p v_q^*)(v_r v_s^*) = \begin{cases} v_{pq'} v_{sr'}^* & \text{if } qP \cap rP = kP \text{ and } qq' = rr' = k \\ 0 & \text{if } qP \cap rP = \emptyset \end{cases}$$

Theorem

1. [Norling, 2014] $C^*(P) \cong C^*(\mathcal{S})$
2. [S, 2015] $\mathcal{Q}(P) \cong C^*_{tight}(\mathcal{S})$

Properties of $\mathcal{Q}(P)$

We know $\mathcal{Q}(P) \cong C^*_{\text{tight}}(\mathcal{S})$

Most of what we can say about $\mathcal{Q}(P)$ stems from knowing that $C^*_{\text{tight}}(\mathcal{S})$ comes from an étale groupoid $\mathcal{G}_{\text{tight}}$, a dynamical object.

One can formulate properties which guarantee that a groupoid algebra is simple, but they are topological and dynamical.

e.g. "$\mathcal{G}_{\text{tight}}$ is Hausdorff," "$\mathcal{G}_{\text{tight}}$ is minimal," "$\mathcal{G}_{\text{tight}}$ is essentially principal".

We translate these statements so that they are (mostly) algebraic properties.

Properties of $\mathcal{Q}(P)$

"$\mathcal{G}_{\text{tight}}$ is Hausdorff"

(H) For all $p, q \in P$, either

1. $pb \neq qb$ for all $b \in P$, or
2. There exists a finite $F \subset P$ with $pf = qf$ for all $f \in F$ and whenever $pb = qb$ there is an $f \in F$ such that $fP \cap bP \neq \emptyset$.

P satisfies condition (H) if the counterexamples to right cancellativity have a "finite cover".

P right cancellative $\Rightarrow$ P satisfies (H)

Properties of $\mathcal{Q}(P)$

"$\mathcal{G}_{\text{tight}}$ is Hausdorff"

(H) For all $p, q \in P$, either

1. $pb \neq qb$ for all $b \in P$, or
2. There exists a finite $F \subset P$ with $pf = qf$ for all $f \in F$ and whenever $pb = qb$ there is an $f \in F$ such that $fP \cap bP \neq \emptyset$.

P satisfies condition (H) if the counterexamples to right cancellativity have a "finite cover".

P right cancellative $\Rightarrow$ P satisfies (H)

"$\mathcal{G}_{\text{tight}}$ is minimal"

It turns out that it is always minimal.

Properties of $\mathcal{Q}(P)$

"$\mathcal{G}_{\text{tight}}$ is essentially principal"

$$P_0 = \{q \in P \mid qP \cap rP \neq 0 \text{ for all } r \in P\}$$

This is the core of P.

If P is Ore, $P_0 = P$

Properties of $\mathcal{Q}(P)$

"$\mathcal{G}_{\text{tight}}$ is essentially principal"

$$P_0 = \{q \in P \mid qP \cap rP \neq 0 \text{ for all } r \in P\}$$

This is the core of P.

If P is Ore, $P_0 = P \to$ cOre

Properties of $\mathcal{Q}(P)$

"$\mathcal{G}_{\text{tight}}$ is essentially principal"

$$P_0 = \{q \in P \mid qP \cap rP \neq 0 \text{ for all } r \in P\}$$

This is the core of P.

If P is Ore, $P_0 = P \to$ cOre

(EP) For all $p, q \in P_0$ and for every $k \in P$ such that

$$qkaP \cap pkaP \neq \emptyset$$

for all $a \in P$, there exists a foundation set F such that $qkf = pkf$ for all $f \in F$.

Properties of $\mathcal{Q}(P)$

Theorem (S)

Let P be a right LCM semigroup which satisfies (H). Then $\mathcal{Q}(P)$ is simple if and only if

1. *P satisfies (EP), and*
2. $\mathcal{Q}(P)(\cong C^*(\mathcal{G}_{tight})) \cong C_r^*(\mathcal{G}_{tight})$

So we see amenability plays a rôle here.

Properties of $\mathcal{Q}(P)$

Theorem (S)

Let P be a right LCM semigroup which satisfies (H). Then $\mathcal{Q}(P)$ is simple if and only if

1. *P satisfies (EP), and*
2. $\mathcal{Q}(P)(\cong C^*(\mathcal{G}_{tight})) \cong C^*_r(\mathcal{G}_{tight})$

So we see amenability plays a rôle here.

Theorem (S)

Let P be a right LCM semigroup which satisfies (H) and such that $\mathcal{Q}(P)$ is simple. Then $\mathcal{Q}(P)$ is purely infinite if and only if $\mathcal{Q}(P) \ncong \mathbb{C}$.

Example: Self-similar groups

Suppose we have an finite set X and

1. an action $G \times X^* \to X^*$ which preserves lengths, and
2. a restriction $G \times X \to G$

$$(g, x) \mapsto g|_x .$$

such that the action on X^* can be defined recursively

$$g(x\alpha) = (gx)(g|_x \alpha)$$

The pair (G, X) is called a self-similar action.

Example: Self-similar groups

Suppose we have an finite set X and

1. an action $G \times X^* \to X^*$ which preserves lengths, and
2. a restriction $G \times X \to G$

$$(g, x) \mapsto g|_x .$$

such that the action on X^* can be defined recursively

$$g(x\alpha) = (gx)(g|_x \alpha)$$

The pair (G, X) is called a self-similar action.
Restriction extends to words

$$g|_{\alpha_1\alpha_2\cdots\alpha_n} := g|_{\alpha_1}|_{\alpha_2}\cdots|_{\alpha_n}$$

$$g(\alpha\beta) = (g\alpha)(g|_\alpha \beta)$$

Example: The Odometer

$G = \mathbb{Z} = \langle z \rangle$
$X = \{0, 1\}$

Then the action of $\mathbb{Z}$ on X^* is determined by

$$z0 = 1 \qquad z|_0 = e$$

$$z1 = 0 \qquad z|_1 = z$$

A word α in X^* corresponds to an integer in binary (written backwards), and z adds 1 to α, ignoring carryover.

$$z(001) = 101 \qquad z|_{001} = e$$

$$z^2(011) = 000 \qquad z^2|_{011} = z$$

Example: Self-similar groups

Given (G, X), the product $X^* \times G$ with the operation

$$(\alpha, g)(\beta, h) = (\alpha(g\beta), g|_\beta h)$$

is a left cancellative semigroup – Zappa-Szép product $X^* \bowtie G$

Lawson-Wallis – all semigroups with Rees conditions arise like this, and so they embed in a group $\Leftrightarrow$ cancellative.

Many interesting examples are not cancellative!

Example: Self-similar groups

A word α is strongly fixed by $g \in G$ if $g\alpha = \alpha$ and $g|_\alpha = 1_G$. A strongly fixed word is minimal if no prefix is strongly fixed.

Proposition

$X^* \bowtie G$ satisfies (H) iff for all $g \neq 1_G$, there are only a finite number of minimal strongly fixed words for g.

$X^* \bowtie G$ is cancelative iff for all $g \neq 1_G$, g has no strongly fixed words.

Example: Self-similar groups

A word α is strongly fixed by $g \in G$ if $g\alpha = \alpha$ and $g|_\alpha = 1_G$. A strongly fixed word is minimal if no prefix is strongly fixed.

Proposition

$X^* \bowtie G$ satisfies (H) iff for all $g \neq 1_G$, there are only a finite number of minimal strongly fixed words for g.

$X^* \bowtie G$ is cancelative iff for all $g \neq 1_G$, g has no strongly fixed words.

The core of $X^* \bowtie G$ is $\{(\varnothing, g) \mid g \in G\} \cong G$.

Proposition

$X^* \bowtie G$ satisfies (EP) if the action of G on X^* is faithful.

If G is amenable, then the amenability condition is satisfied.

Example: The Odometer

The odometer has no strongly fixed words, so $X^* \bowtie \mathbb{Z}$ is cancelative (and satisfies (H)).

$X^* \bowtie \mathbb{Z}$ embeds into $BS(1,2)$.

$\mathcal{Q}(X^* \bowtie \mathbb{Z})$ is simple, purely infinite, and in fact isomorphic to $\mathcal{Q}_2$ (Brownlowe-Ramagge-Robertson-Whittaker)

If we modify the odometer by adding a strongly fixed letter B:

$$X_B = \{0, 1, B\} \qquad zB = B, \quad z|_B = e$$

$X_B^* \bowtie \mathbb{Z}$ satisfies (H), but is not cancelative.

$\mathcal{Q}(X_B^* \bowtie \mathbb{Z})$ is again simple and purely infinite.

- X. Li, Semigroup C*-algebras and amenability of semigroups, Journal of Functional Analysis 262 (2012) 4302 – 4340.
- X. Li, Nuclearity of semigroup C*-algebras and the connection to amenability, Advances in Mathematics 244 (2013) 626 – 662.
- C. Starling, Boundary quotients of C*-algebras of right LCM semigroups, to appear in Journal of Functional Analysis (2014) http://arxiv.org/abs/1409.1549
- N. Brownlowe, J. Ramagge, D. Robertson, M. F. Whittaker, Zappa–Szép products of semigroups and their C*-algebras, Journal of Functional Analysis 266 (6) (2014) 3937 – 3967.
- J. Brown, L. O. Clark, C. Farthing, A. Sims, Simplicity of algebras associated to étale groupoids, Semigroup Forum 88 (2) (2014) 433–452.