The faithful subalgebra

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joint work with
Jonathan H. Brown, Gabriel Nagy, Aidan Sims, and Dana Williams
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Classification of C^*-Algebras, etc.
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and conclude that ϕ is injective iff it is *nondegenerate*, i.e., injective on the "diagonal subalgebra" $\mathcal{D}$.

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[NR1] Nagy and Reznikoff, *Abelian core of graph algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 889–908.

[NR2] Nagy and Reznikoff, *Pseudo-diagonals and uniqueness theorems*, Proc. AMS (2013).

[BNR] Brown, Nagy, Reznikoff *A generalized Cuntz-Krieger uniqueness theorem for higher-rank graphs*, JFA (2013).

[BNRSW] Brown, Nagy, Reznikoff, Sims, and Williams, *Cartan subalgebras in C^*-algebras of Hausdorff étale groupoids* (2015).

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Generalizations: Exel crossed product algebras, Leavitt path algebras (Abrams, Ruiz, Tomforde), topological graph algebras (Katsura), Ruelle algebras (Putnam, Spielberg), Exel-Laca algebras, ultragraphs (Tomforde), Steinberg algebras (Brown, Clark, Farthing, Sims, etal.) Cuntz-Pimsner algebras, higher-rank Cuntz-Krieger algebras (Robertson-Steger), etc.

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A ***k*-graph** is a countable category Λ along with a "degree" functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *unique factorization property*:

For all $\lambda \in \Lambda$, and $m, n \in \mathbb{N}^k$, if $d(\lambda) = m + n$ then there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

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Example The set of finite paths in a directed graph, with $d(\alpha) =$ the length of α, forms a 1-graph.

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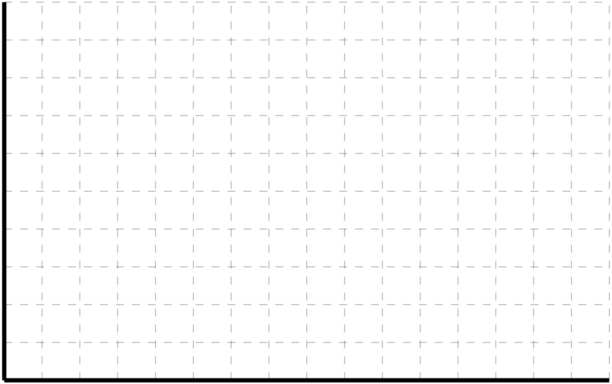

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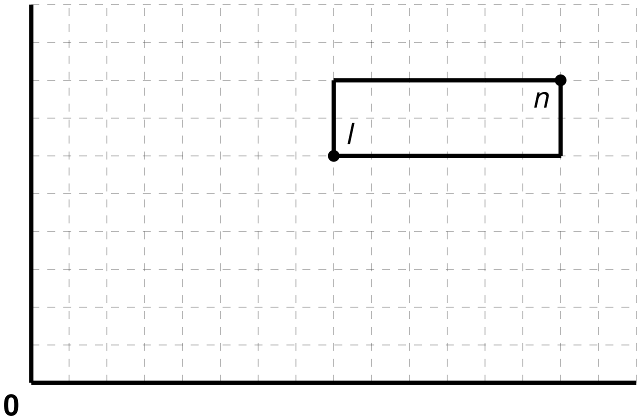

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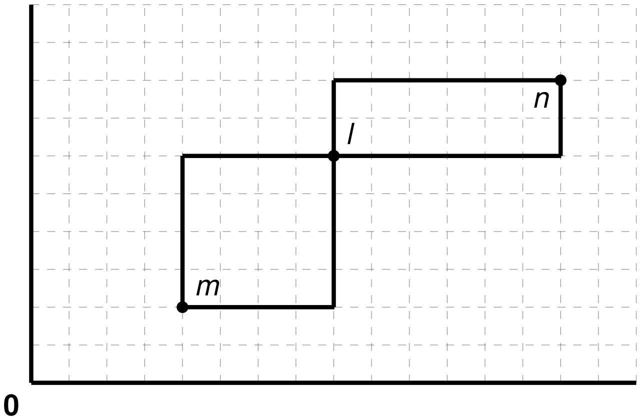

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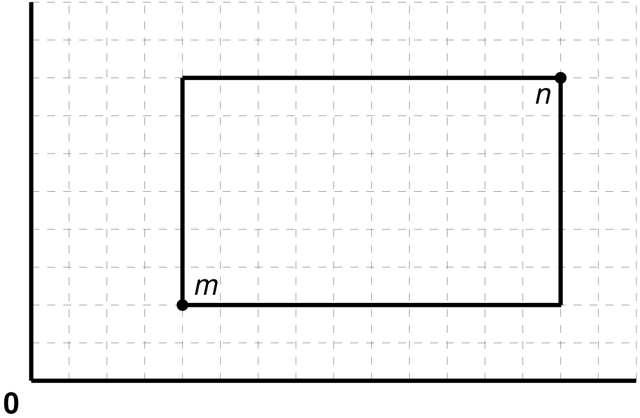

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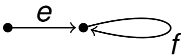

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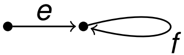

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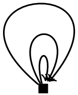

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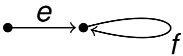

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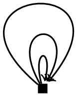

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Cuntz-Krieger ('80)
When the adjacency matrix A of G satisfies a "fullness" condition (I), $C^*(T_e \mid e \in G) \cong \mathcal{O}_A$, the Cuntz-Krieger algebra.

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No! Consider the cycle of length three, E. The map $\phi : C^*(E) \to M_3(\mathbb{C})$ given by

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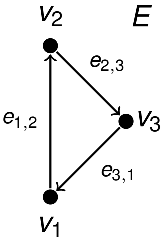
is a non-injective $*$-homomorphism.



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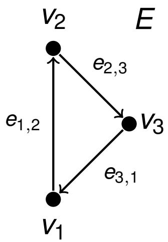

Cuntz-Krieger Uniqueness Theorem:

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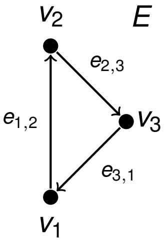

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When ϕ is nondegenerate and the graph satisfies

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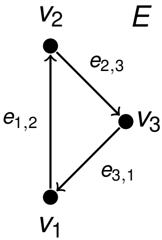

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Theorem Szymański (2001), Nagy-R (2010): Condition (L) can be replaced with a condition on the spectrum of $\phi(S_\lambda)$ for cycles λ without entry.

Aperiodicity – defined via the infinite path space Λ^{∞}

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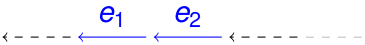

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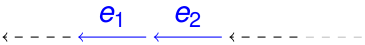

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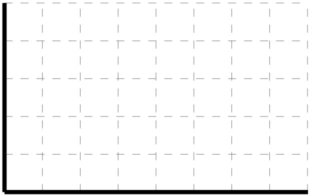

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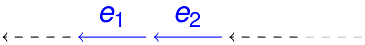

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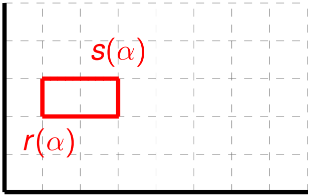

$x((1,2),(3,3)) = \alpha$
$d(\alpha) = (2,1)$

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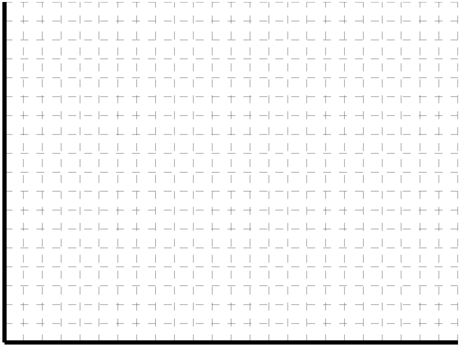

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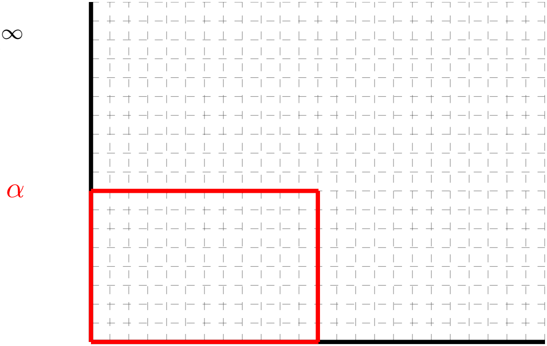

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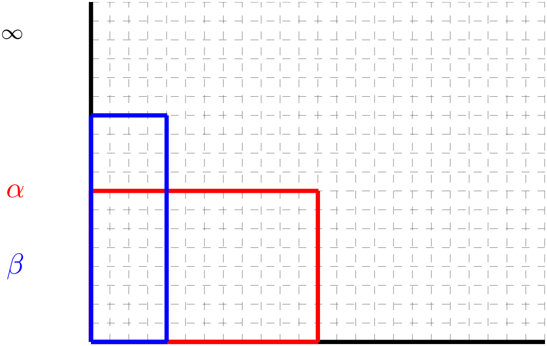

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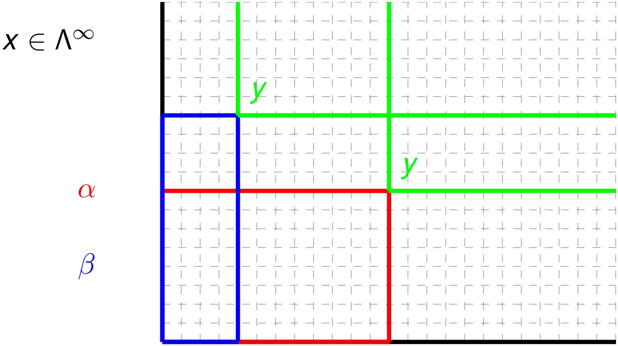

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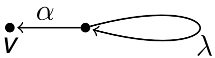

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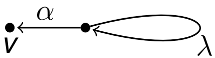

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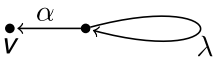

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Theorem Nagy-R (2010), Nagy-Brown-R (2013)
A $*$-homomorphism $\phi : C^*(\Lambda) \to \mathcal{A}$ is injective iff it is injective on the subalgebra $\mathscr{M} := C^*(S_\alpha S_\beta^* \mid \forall \gamma \in \Lambda^\infty \ \alpha\gamma = \beta\gamma\}$.

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Thm (Nagy-R, 2011) When G is a directed graph, $\mathscr{M} \subseteq C^*(G)$ is Cartan and satisfies AEP; i.e, it is a **pseudo-diagonal**.

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Theorem (Brown-Nagy-R-Sims-Williams, 2014)
Let $\mathcal{G}$ be a locally compact, amenable, Hausdorff, étale groupoid. If $\phi : C^*(\mathcal{G}) \to A$ is a C^*-homomorphism, then the following are equivalent.

(i) ϕ is injective.
(ii) ϕ is injective on $C^*((\mathrm{Iso}(\mathcal{G}))^\circ)$.

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$$s(x, d, y) = y = r(y, d', z) \qquad (x, d, y)^{-1} = (y, -d, x)$$
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- Recall: $C^*(\mathcal{G}_\Lambda)$ is a completion of $C_c(\mathcal{G}_\Lambda)$.
- The map $S_\alpha S_\beta^* \mapsto \chi_{Z(\alpha,\beta)}$ implements an isomorphism $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$ that restricts to an iso $\mathscr{M} \cong C^*(\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ)$.

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Thm (BNRSW, 2015) Let $\mathcal{G}$ be a Hausdorff, étale groupoid.

(a) If $(\mathrm{Iso}(\mathcal{G}))^\circ$ is closed, then the restriction map $f \mapsto f|_{\mathrm{Iso}(\mathcal{G})^\circ}$ extends to a faithful conditional expectation $E : C^*(\mathcal{G}) \to \mathcal{M} = C^*((\mathrm{Iso}(\mathcal{G}))^\circ)$.

(b) If $(\mathrm{Iso}(\mathcal{G}))^\circ$ is not closed, then there is no conditional expectation onto the subalgebra.

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Thm (NRBSW, 2014) All Cartan subalgebras satisfy the AEP.

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Then a $*$-homomorphism $\Phi : A \to B$ is injective iff $\Phi|_M$ is injective.

Thank you!

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