Graded Irreducible Representations of Leavitt Path Algebras

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6. Graded Self-injctive Leavitt path algebras

A **directed graph** $E = (E^0, E^1, r, s)$ consists of a set E^0 of **vertices**, a set E^1 of **edges** and maps r, s from E^1 to E^0. For each $e \in E^1$, say,

$\overset{u}{\bullet} \underset{e^*}{\overset{e}{\rightleftarrows}} \overset{v}{\bullet}$, $s(e) = u$ is called the **source** of e and $r(e) = v$ the **range** of e and e^* is called the ghost edge with $s(e^*) = v$ and $r(e^*) = u$. A **finite path** α of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n-1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$. A vertex u is called a **sink** if it emits no edges. If u is not a sink and emits finitely many edges, we say u is a **regular vertex**. If u emits infinitely many edges, we say u is an **infinite emitter.**

Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph and K be any field. The **Leavitt path algebra** $L_K(E)$ of the graph E with coefficients in K is the K-algerbra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

(3) (The "CK-1 relations") For all $e_i, e_j \in E^1$, $e_i^* e_i = r(e_i)$ and $e_i^* e_j = 0$ if $i \neq j$

(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$\sum_{e \in E^1, s(e) = v} ee^* = v$$

Notation: Here after, E will denote an arbitrary graph, L denotes $L_K(E)$ and all the modules we consider are left L-modules.

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- $a \in L \implies a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ where $k_i \in K, \alpha_i$ and β_i are paths.
- L is a $\mathbb{Z}$**-graded ring**, induced by $\deg(v) = 0$, $\deg(e) = 1$, $\deg(e^*) = -1$ for $v \in E^0, e \in E^1$ and $L = \bigoplus_{n \in \mathbb{Z}} L_n$ where each $L_n = \{a \in L : a = \sum k_i \alpha_i \beta_i^* \text{ with } |\alpha_i| - |\beta_i| = n\}$. The subgroups L_n satisfy $L_m L_n \subseteq L_{m+n}$ for all m, n. Elements of L_n are said to be **homogeneous**.

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- A left L-module M is a **graded L-module** if $M = \bigoplus_{n \in \mathbb{Z}} M_n$ where each M_n is an abelian group and for $s, t \in \mathbb{Z}$, $R_s M_t \subseteq M_{s+t}$.

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- For a $\mathbb{Z}$-graded module M we define, for any $k \in \mathbb{Z}$, the **k-shifted graded module** $M(k)$ as $M(k) = \bigoplus_{n \in \mathbb{Z}} (M(k))_n$, where $(M(k))_n = M_{k+n}$.

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- Upto isomorphism, $K[x, x^{-1}]$ has only one graded-simple $K[x, x^{-1}]$-module, namely itself, but has infinitely many non-graded simple modules.

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- **Definition**: A vertex v is called a **line point** if no vertex in $T_E(v)$ is either a bifuracating vertex or the base of a cycle. As a graph, $T_E(v)$ looks like $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$......, a finite or infinite straight line segment.

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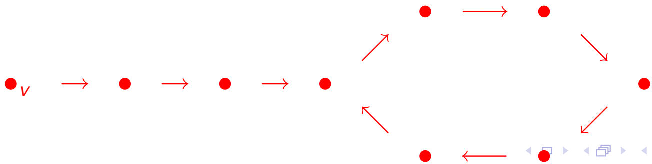

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- **Definition**: (Daniel Gonçalves and Danilo Royer 2011) Let E be an arbitrary graph. An E-**algebraic branching system** consists of a set X and a family of its subsets $\{X_v, X_e : v \in E^0, e \in E^1\}$ such that

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- (5) $\deg(\sigma_e(x)) = \deg(x) + 1$.

Let X be an algebraic branching system . Let $M(X)$ be the K-vector space having X as a basis. We make $M(X)$ a left L-module as follows: Define, for each vertex v and each edge e in E, linear transformations P_v, S_e and S_{e^*} on $M(X)$ as follows:

For all $x \in X$,

(I) $P_v(x) = \begin{cases} x, \text{ if } x \in X_v \\ 0, \text{ otherwise} \end{cases}$

(II) $S_e(x) = \begin{cases} \sigma_e(x), \text{ if } x \in X_{r(e)} \\ 0, \quad \text{otherwise} \end{cases}$

(III) $S_{e^*}(x) = \begin{cases} \sigma_e^{-1}(x), \text{ if } x \in X_e \\ 0, \text{ otherwise} \end{cases}$

The endomorphisms $\{P_u, S_e, S_{e^*} : u \in E^0, e \in E^1\}$ satisfy the defining relations (1) - (4) of the Leavitt path algebra L. This induces an algebra homomorphism ϕ from L to $End_K(M(X))$ mapping u to P_u, e to S_e and e^* to S_{e^*} . Then $M(X)$ can be made a left module over L via the homomorphism ϕ. We denote this L-module operation on $M(X)$ by $\cdot$.

Graded L-modules using branching systems

If X is a graded branching system, then define, for each $i \in \mathbb{Z}$, the homogeneous component

$$M(X)_i = \{\sum_{x \in X} k_x x \in M(X) : \deg(x) = i\}.$$

It is easy to see that

$$M(X) = \bigoplus_{i \in \mathbb{Z}} M(X)_i$$

and that $M(X)$ is a $\mathbb{Z}$-graded left L-module.
Next we illustrate constructions of graded irreducible representations using appropriate graded algebraic branching systems.

Graded simple but not simple

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- The corresponding L-module $M(X) = \bigoplus_{n \in \mathbb{Z}} (M(X))_n$ is $\mathbb{Z}$-graded where $(M(X))_n$ has the K-basis $\{pq^* \in X : |p| - |q| = n\}$. Denote it by N_{vc}

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- (i) The left L-module N_{vc} is a graded-simple L-module but is not a simple L-module.
- (ii) The annihilator P of N_{vc} is a graded prime ideal of L and is generated by $H(v = \{u \in E^0 : u \ngeq v\})$, but P is not a primitive ideal .

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- (iii) Conversely, every graded prime non-primitive ideal P is the annihilator of a graded simple L-module which is not simple , provided if E is row-finite or if E^0 is countable.

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- (i) The left L-module N_{vc} is a graded-simple L-module but is not a simple L-module.
- (ii) The annihilator P of N_{vc} is a graded prime ideal of L and is generated by $H(v = \{u \in E^0 : u \ngeq v\})$, but P is not a primitive ideal .
- (iii) Conversely, every graded prime non-primitive ideal P is the annihilator of a graded simple L-module which is not simple , provided if E is row-finite or if E^0 is countable.

- **Note**: In contrast, the annihilator of every simple L-module is always a primitive ideal of L.

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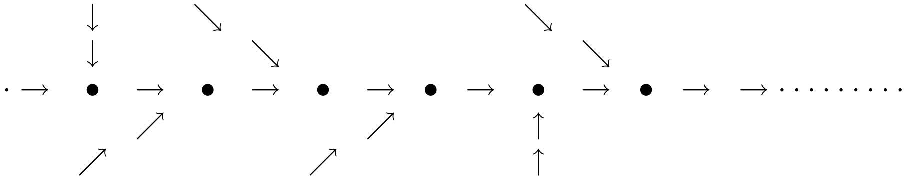
- This module is **a graded-simple module and also a simple module**

Let $p = e_1 e_2 \cdots e_n \cdots$ be an infinite path. Following X.W. Chen, define for each $n \geq 1$, $\tau^{>n}(p)$ to be the truncated infinite path $e_{n+1} e_{n+2} \cdots$. Let $[p] = \{\text{infinite paths } q : \tau^{>m}(q) = \tau^{>n}(p) \text{ for some } m, n\}$. We say q is **tail-equivalent** to p.



An infinite path p is said to be **rational** if it is tail equivalent to an finite path of the form $ccc \cdots$. where c is a closed path. An infinite path which is not rational is called an **Irrational path**.

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- If p is irrational, we can make $V_{[p]}$ a graded module by defining for any $q \in [p]$, $\deg(q) = m - n$ if $m > 0$ is the smallest integer such that $\tau^{>m}(q) = \tau^{>n}(p)$ for some n.

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- But if p is rational, $V_{[p]}$ is **simple but is not graded-simple:** (Suppose, on the contrary, $V_{[p]}$ is graded and $p = ccc\cdots$ with c a cycle. First show p is homogeneous. Then note $cp = p$. This implies $\deg(p) = |c| + \deg(p)$, a contradiction)

Graded-simple left ideals and the Graded Socle of Leavitt path algebras

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- **Theorem:** $Soc^{gr}(L)$ is the ideal generated by all the Laurent vertices and the line points in E and $Soc^{gr}(L) \cong_{gr} \bigoplus_{i \in I} M_{\Lambda_i}(K)(\bar{\alpha}_i) \oplus \bigoplus_{j \in J} M_{\Lambda_j}(K[x^{t_j}, x^{-t_j}])(\bar{\beta}_i)$ where Λ_i, Λ_j are arbitrary index sets , the t_j are positive integers and $\bar{\alpha}_i$,.$\bar{\beta}_i$ are grade shiftings.

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$$0 \leq I_1 \leq \cdots \leq I_\alpha \leq I_{\alpha+1} \leq \cdots \qquad (\alpha < \tau)$$

where τ is some ordinal, $I_1 = Soc(L)$ and, for each $\alpha \geq 1$, $I_{\alpha+1}/I_\alpha \cong M_{\Lambda_\alpha}(K[x, x^{-1}])$ where Λ_α is an arbitrary index set depending on α.

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- Let E be the graph with two vertices v, w and two edges e, f such that $v = s(e) = r(e) = s(f)$ and $r(f) = w$. Now distinct cycles are disjoint, so every simple module over $L = L(E)$ is finitely presented. If $I =< w >$,$L/I \cong_{gr} K[x, x^{-1}]$ is graded simple, but is not finitely presented as $I = Lw \oplus \bigoplus_{i=0}^{\infty} Lf^*(e^*)^i$ is an infinitely generated graded ideal.

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- On the other hand, as we shall see soon, if every graded-simple is fp , then every simple is also fp.

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- **Theorem**: (Abrams- K.M.R. 2010) If E is acyclic, then L is a directed union of graded subalgebras B_λ, each of which is a direct sum of finitely many matrix rings over K of finite order.

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Thank You !