K-theory for the tame C*-algebra of a separated graph

Pere Ara

Universitat Autònoma de Barcelona

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Joint work with Ruy Exel (UFSC)

Separated graphs

Definition

A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $r^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v:

$$r^{-1}(v) = \bigsqcup_{X \in C_v} X.$$

(In case v is a source, we take C_v to be the empty family of subsets of $r^{-1}(v)$.)
The constructions we introduce revert to existing ones in case $C_v = \{r^{-1}(v)\}$ for each $v \in E^0$. We refer to a *non-separated graph* in that situation.

The Leavitt path algebra of a separated graph

Definition

The *Leavitt path algebra of the separated graph* (E, C) is the $*$-algebra $L_{\mathbb{C}}(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the following relations:

(V) $vv' = \delta_{v,v'}v$ and $v = v^*$ for all $v, v' \in E^0$,

(E) $r(e)e = es(e) = e$ for all $e \in E^1$,

(SCK1) $e^*e' = \delta_{e,e'}s(e)$ for all $e, e' \in X$, $X \in C$, and

(SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \in C_v$, $v \in E^0$.

The C*-algebra of a separated graph

Definition (AG, Definition 1.5)

The *graph C*-algebra* of a separated graph (E, C) is the C*-algebra $C^*(E, C)$ with generators $\{v, e \mid v \in E^0,\ e \in E^1\}$, subject to the relations (V), (E), (SCK1), (SCK2). In other words, $C^*(E, C)$ is the enveloping C*-algebra of $L_{\mathbb{C}}(E, C)$.

The C*-algebra of a separated graph

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In case (E, C) is trivially separated, $C^*(E, C)$ is just the classical graph C*-algebra $C^*(E)$. There is a unique *-homomorphism $L_{\mathbb{C}}(E, C) \to C^*(E, C)$ sending the generators of $L_{\mathbb{C}}(E, C)$ to their canonical images in $C^*(E, C)$. This map is injective by [AG, Theorem 3.8(1)].

Example

Let $1 \leq m \leq n$. The C*-algebra $U^{\mathrm{nc}}_{m,n}$ was introduced and studied by McClanahan. It is the universal C*-algebra generated by the entries of an $m \times n$ unitary matrix U, i.e., $UU^* = I_m$ and $U^*U = I_n$. This algebra can be seen as a full corner of a separated graph C*-algebra: Consider the separated graph $(E(m,n), C(m,n))$, where $E(m,n)$ is the graph consisting of two vertices v, w and with

$$E(m,n)^1 = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\},$$

with $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j, and $C(m,n)$ consists of two elements $X = \{\alpha_1, \dots, \alpha_n\}$ and $Y = \{\beta_1, \dots, \beta_m\}$. Then

$$wC^*(E(m,n), C(m,n))w \cong U^{\mathrm{nc}}_{m,n}, \qquad \text{with } \beta_i^*\alpha_j \leftrightarrow u_{ij}.$$

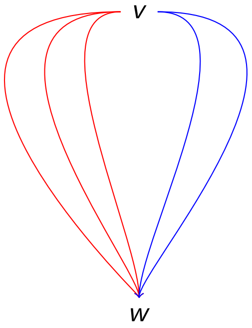

Figure: The separated graph $(E(2,3), C(2,3))$

The tame graph C*-algebra of a separated graph

The C*-algebra $C^*(E, C)$ for separated graphs behaves in quite a different way compared to the usual graph C*-algebras associated to non-separated graphs, the reason being that the final projections of the partial isometries corresponding to edges coming from different sets in C_v, for $v \in E^0$, need not commute. In order to resolve this problem, we make the following:

Definition (AE)

Let (E, C) be any separated graph. Let U be the multiplicative subsemigroup of $C^*(E, C)$ generated by $(E^1) \cup (E^1)^*$ and write $e(u) = uu^*$ for $u \in U$. Then the *tame graph C*-algebra* of (E, C) is the C*-algebra

$$\mathcal{O}(E, C) = C^*(E, C)/J\,,$$

where J is the closed ideal of $C^*(E, C)$ generated by all the commutators $[e(u), e(u')]$, for $u, u' \in U$.

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Observe that $J = 0$ in the non-separated case, so we get that $\mathcal{O}(E) = C^*(E)$ is the usual graph C*-algebra in this case.

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The K-theory of the C*-algebras $C^*(E, C)$ was computed in [AG] by Goodearl and Ara. Let (E, C) be a finitely separated graph. For $v, w \in E^0$ and $X \in C_v$, denote by $a_X(w, v)$ the number of arrows in X from w to v.

We denote by $1_C \colon \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}$ and $A_{(E,C)} \colon \mathbb{Z}^{(C)} \to \mathbb{Z}^{(E^0)}$ the homomorphisms defined by

$$1_C(\delta_X) = \delta_v \qquad \text{and} \qquad A_{(E,C)}(\delta_X) = \sum_{w \in E^0} a_X(w, v)\delta_w,$$

for $v \in E^0$, $X \in C_v$. (Here $(\delta_X)_{X\in C}$ and $(\delta_v)_{v \in E^0}$ denote the canonical basis of $\mathbb{Z}^{(C)}$ and $\mathbb{Z}^{(E^0)}$ respectively.)

With this notation, the K-theory of $C^*(E, C)$ has formulas which look very similar to the ones for the non-separated case (cf. [RaeSzy]):

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Theorem (AG, Theorem 5.2)

Let (E, C) be a finitely separated graph, and adopt the notation above. Then the K-theory of $C^*(E, C)$ is given as follows:

$$K_0(C^*(E,C)) \cong \operatorname{coker}\big(1_C - A_{(E,C)} : \mathbb{Z}^{(C)} \longrightarrow \mathbb{Z}^{(E^0)}\big), \tag{1}$$

$$K_1(C^*(E,C)) \cong \ker\big(1_C - A_{(E,C)} : \mathbb{Z}^{(C)} \longrightarrow \mathbb{Z}^{(E^0)}\big). \tag{2}$$

The K-theory of the tame graph C*-algebra

Theorem

Let (E, C) be a finitely separated graph. Then

1. *The canonical projection map $\pi\colon C^*(E,C) \to \mathcal{O}(E,C)$ induces a split monomorphism*

$$K_0(\pi)\colon K_0(C^*(E,C)) \to K_0(\mathcal{O}(E,C))$$

whose cokernel H is a torsion-free abelian group. Moreover, the group H is a free abelian group when E is a finite graph.

2. *The canonical projection map $\pi\colon C^*(E,C) \to \mathcal{O}(E,C)$ induces an isomorphism*

$$K_1(\mathcal{O}(E,C)) \cong K_1(C^*(E,C)) \cong \ker(1_C - A_{(E,C)}).$$

The proof is naturally divided into two main steps:

1. The case of finite bipartite separated graphs.
2. The general case.

The general case is obtained from Step (1) by using some direct limit arguments, and a fact from [AE]: Every (tame) graph C*-algebra is Morita-equivalent to a (tame) graph C*-algebra of a bipartite separated graph.

Bipartite separated graphs

Definition (AE)

Let E be a directed graph. We say that E is a *bipartite directed graph* if $E^0 = E^{0,0} \sqcup E^{0,1}$, with all arrows in E^1 going from a vertex in $E^{0,1}$ to a vertex in $E^{0,0}$.
A *bipartite separated graph* is a separated graph (E, C) such that the underlying directed graph E is a bipartite directed graph.

For a finite bipartite separated graph (E, C), the main technical tool we use is the construction (done in [AE]) of a sequence of finite bipartite separated graphs $\{(E_n, C^n)\}$ such that the graph C*-algebras $C^*(E_n, C^n)$ approximate the tame graph C*-algebra $\mathcal{O}(E, C)$, in the sense that

$$\mathcal{O}(E, C) \cong \varinjlim_{n} C^*(E_n, C^n).$$

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The generators of the free group H (the cokernel of $K_0(\pi)$) are given in terms of certain vertices of this sequence of graphs E_n.

The sequence (E_n, C^n)

We define $(E_0, C^0) = (E, C)$. Each finite bipartite separated graph (E_{n+1}, C^{n+1}) is obtained from the previous one (E_n, C^n) by a simple combinatorial algorithm.

So to show the result for K_1, we only have to prove:

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So to show the result for K_1, we only have to prove:

Theorem

Let (E, C) be a finite bipartite separated graph. Then the canonical map $\phi_0 \colon C^*(E, C) \to C^*(E_1, C^1)$ induces an isomorphism

$$K_1(\phi_0) \colon K_1(C^*(E, C)) \to K_1(C^*(E_1, C^1)).$$

The proof of the above theorem involves a computation of the index map for certain amalgamated free products.

Using this, we develop a concrete description of the isomorphism between $\ker(1_C - A_{(E,C)})$ and $K_1(C^*(E,C))$, which is then used to derive our result.

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Such a description was obtained by Carlsen, Eilers and Tomforde in [CET] for relative graph algebras of non-separated graphs, by using different techniques.

Example

The algebra U_n^{nc} is the C*-algebra generated by the entries of a universal $n \times n$ unitary matrix $U = [u_{ij}]$, see [McCla1]. The K-theory of U_n^{nc} was computed by McClanahan.

$K_1(U_n^{\mathrm{nc}})$ is a free abelian group generated by $[U]_1$.

With our approach, and writting $(E,C);=(E(n,n),C(n,n))$, we have

$$C^*(E,C)\cong M_{n+1}(U_n^{\mathrm{nc}}).$$

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Setting $\mathcal{O}(n, n) = \mathcal{O}(E, C)$, we obtain

$$K_1(\mathcal{O}_{n,n}) \cong K_1(U_n^{\mathsf{nc}}) \cong \ker\left(\begin{pmatrix} 1 & 1 \\ -n & -n \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2\right) \cong \mathbb{Z}.$$

We recover the fact that $K_1(U_n^{\mathrm{nc}})$ is generated by the class of $U = (u_{ij})$. Indeed, we obtain an explicit isomorphism

$$\lambda\colon \ker(1_C - A_{(E,C)}) \to K_1(C^*(E,C)),$$

and thus $K_1(C^*(E,C))$ is generated by $\lambda(x)$, where $x = \delta_X - \delta_Y$.

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$$Z = \begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix}, \qquad T = \begin{pmatrix} \beta_1 & \cdots & \beta_n \end{pmatrix}.$$

Thus $K_1(C^*(E,C))$ is generated by the class of the unitary $\sum_{i=1}^n \alpha_i\beta_i^*$ of $vC^*(E,C)v$.

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Thus $K_1(C^*(E,C))$ is generated by the class of the unitary $\sum_{i=1}^n \alpha_i\beta_i^*$ of $vC^*(E,C)v$.
The unitary $T^*Z = (\beta_i^*\alpha_j)$ in $M_n(wC^*(E,C)w)$ represents the same element and corresponds to (u_{ij}) under the canonical isomorphism $wC^*(E,C)w \cong U_n^{\mathrm{nc}}$.

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