Towards a K-theoretic characterization of graded isomorphisms between Leavitt path algebras.

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Classification of C*-algebras, flow equivalence of shift spaces, and graph and Leavitt path algebras
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Joint work with Pere Ara (Departament de Matemàtiques, Universitat Autònoma de Barcelona, Spain),

P. ARA, E. PARDO, Towards a K-theoretic characterization of graded isomorphisms between Leavitt path algebras, *J. K-Theory* **14** (2014), 203–245.

Main goal: Classify Leavitt path algebras up to isomorphism/Morita equivalence.

Key role: K-theoretic invariants & classification techniques of (essential) subshifts of finite type.

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Use K-theoretic invariants to classify Leavitt path algebras up to **graded** isomorphism/Morita equivalence.

Hazrat's Conjecture

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Outline

1. Preliminares
   - Some definitions on graphs
   - K-Theory for Leavitt path algebras
   - Fractional skew monoid rings
2. Forms of Hazrat's Invariant
   - Graded modules & K_0^{gr}
   - Shift equivalence
   - Strong shift equivalence
3. The main result

Outline

Graph: $E = (E^0, E^1, r, s)$, with E^0 vertices, E^1 edges, $r, s : E^1 \to E^0$ maps.

Sink: $v \in E^0$ with $s^{-1}(v) = \emptyset$.

Source: $v \in E^0$ with $r^{-1}(v) = \emptyset$.

Essential graph: E has neither sources no sinks.

Adjacency matrix:

$E^0 \times E^0$ matrix with $(A_E)_{v,w} = |s^{-1}(v) \cap r^{-1}(w)|$.

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Definition (Leavitt path algebra)

For a finite graph E and a field K, $L_K(E)$ is the universal algebra with:

Generators: $\{v \in E^0\}$ *and* $\{e, e^* \mid e \in E^1\}$,

Relations:

1. $vv' = \delta_{v,v'}v$ *for all* $v, v' \in E^0$.
2. $s(e)e = er(e) = e$ *for all* $e \in E^1$.
3. $e^*e' = \delta_{e,e'}r(e)$ *for all* $e, e' \in E^1$.
4. $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ *for every* $v \in E^0$ *that it is not a sink.*

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A ring, Γ group.

Definition (Γ-graded ring)

A is Γ-graded if $A = \bigoplus_{g\in\Gamma} A_g$ with:

1. *A_g abelian group for all $g \in \Gamma$.*
2. *A_1 subring, and A_g is A_1-bimodule for all $g \in \Gamma$.*
3. *$A_g \cdot A_h \subseteq A_{gh}$ for all $g, h \in \Gamma$.*

If $A_g \cdot A_h = A_{gh}$ for all $g, h \in \Gamma$, we say that A is strongly Γ-graded.

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Canonical $\mathbb{Z}$-grading on $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$:

The **degree** $n \in \mathbb{Z}$ component

$$L_K(E)_n = \text{span}\{\alpha\beta^* \mid \mathrm{length}(\alpha) - \mathrm{length}(\beta) = n\}.$$

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$K_0(A)$:= universal enveloping group of the abelian monoid of isomorphism classes in Proj-A.

E finite essential graph, $A = A_E^t$:

$$K_0(L_K(E)) \cong \operatorname{coker}(A - I) : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}.$$

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$L(E)_0$ is ultramatricial:

$$L(E)_0 \cong \varinjlim (L(E)_{0,n}, \phi_{n,n+1}).$$

$K_0(L(E)_{0,n}) \cong \mathbb{Z}^{E^0}$ and $K_0(\phi_{n,n+1}) \cong A_E^t$, whence

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Definition (Fractional skew monoid ring)

$R := A[t_+, t_-; \alpha]$ *for generators* t_+, t_- *satisfying:*

1. $t_- t_+ = 1$ *and* $t_+ t_- = p$;
2. $t_+ a = \alpha(a) t_+$ *for all* $a \in A$;
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This is the algebraic analog of Paschke's $A \rtimes_\alpha \mathbb{N}$.

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$R = \bigoplus_{i\in\mathbb{Z}} R_i$, with $R_i = At_+^i$ for $i > 0$ and $R_i = t_-^{-i}A$ for $i < 0$, while $A_0 = A$.

[Ara-Brustenga]: E finite graph with no sources, then $L(E) = L(E)_0[t_+, t_-; \alpha]$ for suitable $t_+, t_- \in L(E)$, α defined by $\alpha(a) = t_+ a t_-$ for all $a \in L(E)_0$.

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$$L^g(E) := L(E)_0[s_+, s_-, g \circ \alpha].$$

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For $g \in \Gamma$, $\mathcal{T}_g$: Gr-$A \to$ Gr-A ($\mathcal{T}_g(M) := M(g)$) induces an action $\mathcal{T}$ of Γ on Gr-A.

This action restricts to Gr-Proj-A. Thus, $K_0^{gr}(A)$ is $\mathbb{Z}[\Gamma]$-module by $(\mathcal{T}_g)_*[P] := [\mathcal{T}_g(P)]$.

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LPAs: $L_K(E)$ is $\mathbb{Z}$-graded. $K_0^{gr}(L_K(E))$ is $\mathbb{Z}[x, x^{-1}]$-module, where $x\cdot$ is $\mathcal{T} := \mathcal{T}_1$.

Hazrat's Conjecture

If E, F finite graphs, TFAE:

1. $L_K(E) \cong_{gr} L_K(F)$.
2. $(K_0^{gr}(L(E)), [1_{L(E)}]) \cong_{\mathbb{Z}[x,x^{-1}]} (K_0^{gr}(L(F)), [1_{L(F)}])$.

Theorem (Hazrat)

The Conjecture holds for finite polycephaly graphs.

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[Dade]: If A strongly Γ-graded, then $\mathcal{T}$ induces an action of Γ on mod-A_0.

These actions commute with the natural equivalence $(-)_0$: Gr-$A \to$ mod-A_0 ($M \mapsto M_0$).

$K_0(A_0)$ is also a $\mathbb{Z}[\Gamma]$-module, and $K_0^{gr}(A) \cong K_0(A_0)$ as $\mathbb{Z}[\Gamma]$-modules.

[Hazrat]: $L_K(E)$ strongly $\mathbb{Z}$-graded iff E has no sinks.

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E finite essential graph, then:

1. $L(E) = L(E)_0[t_+, t_-; \alpha]$.
2. $K_0^{gr}(L(E)) \cong_{\mathbb{Z}[x,x^{-1}]} K_0(L(E)_0)$.
3. $\alpha_* : K_0(L(E)_0) \to K_0(L(E)_0)$ is an ordered group isomorphism.

Lemma (Ara-P.)

If E is a finite essential graph, then the maps α_* and $\mathcal{T}_*$ are mutually inverse isomorphisms.

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Lemma (Ara-P.)

If E is a finite essential graph, then the maps α_* and $\mathcal{T}_*$ are mutually inverse isomorphisms.

Proposition (Ara-P.)

Let E, F be finite essential graphs, and α, β the respective corner isomorphisms. TFAE:

1. *There exists an isomorphism* $\Psi : (K_0(L(E)_0), K_0^+(L(E)_0)) \to (K_0(L(F)_0), K_0^+(L(F)_0))$ *such that* $\Psi \circ (\mathcal{T}_E)_* = (\mathcal{T}_F)_* \circ \Psi$
2. *There exists an isomorphism* $\Phi : (K_0(L(E)_0), K_0^+(L(E)_0)) \to (K_0(L(F)_0), K_0^+(L(F)_0))$ *such that* $\Phi \circ \alpha_* = \beta_* \circ \Phi$
3. *There is an ordered* $K_0^{gr}(L(E)) \cong_{\mathbb{Z}[x,x^{-1}]} K_0^{gr}(L(F))$.

Moreover, the result holds also in the order-unit preserving category.

Notation

We will denote the condition (1) in Proposition as

$$K_0(L(E)_0) \cong_{\mathbb{Z}[x,x^{-1}]} K_0(L(F)_0).$$

and condition (2) in Proposition as

$$(K_0(L(E)_0), \alpha_*) \cong (K_0(L(F)_0), \beta_*).$$

Shift equivalence (with lag $\ell \geq 1$)

$A \sim_{SE} B$: there exist matrices R, S such that $AR = RB$, $SA = BS$, $A^\ell = RS$ and $B^\ell = SR$.

Invariant associated to A**:** the dimension triple $(\Delta_A, \Delta_A^+, \delta_A)$.

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Invariant associated to A: the dimension triple $(\Delta_A, \Delta_A^+, \delta_A)$.

$\Delta_A \cong \varinjlim(\mathbb{Z}^n, A), \Delta_A^+ \cong \varinjlim((\mathbb{Z}^n)^+, A)$. The diagram

induces an ordered group isomorphism $\delta_A : \Delta_A \to \Delta_A$ by multiplication by A on the direct limit.

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$$\begin{array}{ccccccccc}
\mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n & \cdots\cdots\cdots\rightarrow & \Delta_A \\
\downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \delta_A} \\
\mathbb{Z}^n & \xrightarrow[A]{} & \mathbb{Z}^n & \xrightarrow[A]{} & \mathbb{Z}^n & \xrightarrow[A]{} & \mathbb{Z}^n & \cdots\cdots\cdots\rightarrow & \Delta_A
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induces an ordered group isomorphism $\delta_A : \Delta_A \to \Delta_A$ by multiplication by A on the direct limit.

Theorem (Krieger)

$A \sim_{SE} B$ iff there exists an ordered-group isomorphism $f : \Delta_A \to \Delta_B$ such that $\delta_B \circ f = f \circ \delta_A$.

Lemma (Ara-P.)

E be a finite essential graph, $A := A_E^t$, δ_A and α_* be the automorphism of $K_0(L(E)_0)$ defined above. Then, α_* is the inverse of δ_A.

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Theorem (Ara-P.)

E, F finite essential graphs, α, β the respective corner isomorphisms. Set $A := A_E^t$ and $B := A_F^t$, δ_A, δ_B the automorphisms of $K_0(L(E)_0)$ and $K_0(L(F)_0)$ defined above. TFAE:

1. $A \sim_{SE} B$.
2. $K_0(L(E)_0) \cong_{\mathbb{Z}[x,x^{-1}]} K_0(L(F)_0)$.
3. $(K_0(L(E)_0), \alpha_*) \cong (K_0(L(F)_0), \beta_*)$.
4. $(K_0(L(E)_0), \delta_A) \cong (K_0(L(F)_0), \delta_B)$.
5. *There is an ordered* $K_0^{gr}(L(E)) \cong_{\mathbb{Z}[x,x^{-1}]} K_0^{gr}(L(F))$.

Moreover, the equivalences also hold in the order-unit preserving category.

Given A, B matrices

Definition (Elementary strong shift equivalence)

There exist matrices R, S such that $A = RS$ and $B = SR$. We denote that as $(S, R) : A \approx B$

Definition (Strong shift equivalence of lag l)

$A \approx_l B$ if $(S_1, R_1) : A = A_0 \approx A_1, (S_2, R_2) : A_1 \approx A_2, \ldots, (S_l, R_l) : A_{l-1} \approx A_l = B$. If $S := S_l \cdots S_1$, $R := R_1 \cdots R_l$, we denote it by $(S, R) : A \approx_l B$.

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[Williams] E, F essential graphs. $A_E \approx_n A_F$ iff there exist graphs $E = D_0, D_1, \ldots, D_n, F = D_{n+1}$ such that either $D_i \to D_{i+1}$ or $D_{i+1} \to D_i$ is obtained via out/in-split/amalgamation graph moves.

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$A \approx_l B \Rightarrow A \sim_{SE} B$ of lag l (The converse does not hold [Kim-Roush]).

$(S,R) : A \approx_l B \Rightarrow$ there exists an isomorphism $\Phi : (\Delta_A, \Delta_A^+, \delta_A) \to (\Delta_B, \Delta_B^+, \delta_B)$ such that

$$\Phi(\iota_{n,\infty}(x)) = \iota_{m+n,\infty}(Sx)$$

for $x \in \mathbb{Z}^N$ ($m \in \mathbb{Z}^+$ fixed).

Φ induces an isomorphism $\Phi : K_0^{gr}(L(E)) \to K_0^{gr}(L(F))$, determined by $((S,R),m)$.

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Theorem (Ara-P.)

E and F finite essential graphs, $A := A_E^t$, $B := A_F^t$. Then:

1. *If $(S,R) : A \approx B$ and $m \in \mathbb{Z}^+$, then there exists graded Morita equivalence $\Psi : \mathrm{Gr}\text{-}L(E) \longrightarrow \mathrm{Gr}\text{-}L(F)$ such that $K_0^{gr}(\Psi)$ is induced by $((S,R),m)$.*
2. *If $\Psi(L(E)_{L(E)}) \cong L(F)_{L(F)}$, then there exists $\phi : L(E) \cong_{gr} L(F)$ such that $K_0^{gr}(\phi) = K_0^{gr}(\Psi)$.*

Outline

The equivalence between

$$(1) \quad K_0^{gr}(L(E)) \cong_{\mathbb{Z}[x,x^{-1}]} K_0^{gr}(L(F))$$

and

$$(2) \quad (K_0(L(E)_0), \alpha_*) \cong (K_0(L(F)_0), \beta_*)$$

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suggest a strategy to prove Hazrat's Conjecture for finite essential graphs.

Suppose that the commutative diagram of groups in (2)

$$\begin{array}{ccc} K_0(L(E)_0) & \xrightarrow{\alpha_*} & K_0(L(E)_0) \\ \Big\downarrow \Phi & & \Big\downarrow \Phi \\ K_0(L(F)_0) & \xrightarrow[\beta_*]{} & K_0(L(F)_0) \end{array}$$

with Φ isomorphism, lifts to a commutative diagram of algebras

$$\begin{array}{ccc} L(E)_0 & \xrightarrow{\alpha} & L(E)_0 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ L(F)_0 & \xrightarrow[\beta]{} & L(F)_0 \end{array},$$

with φ isomorphism of algebras such that $\varphi\alpha = \beta\varphi$.

We have $L(E) = L(E)_0[t_+, t_-; \alpha]$ and $L(F) = L(F)_0[s_+, s_-; \beta]$.

We can define map

$$\widehat{\varphi} : L(E)_0[t_+, t_-; \alpha] \to L(F)_0[s_+, s_-; \beta]$$

by $\widehat{\varphi}_{|L(E)_0} = \varphi$, $\widehat{\varphi}(t_+) = s_+$, $\widehat{\varphi}(t_-) = s_-$.

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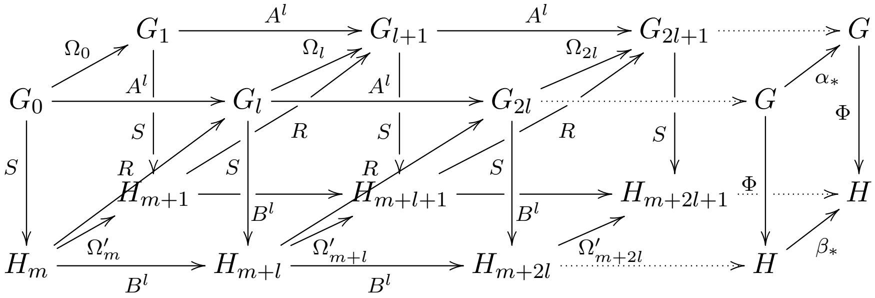
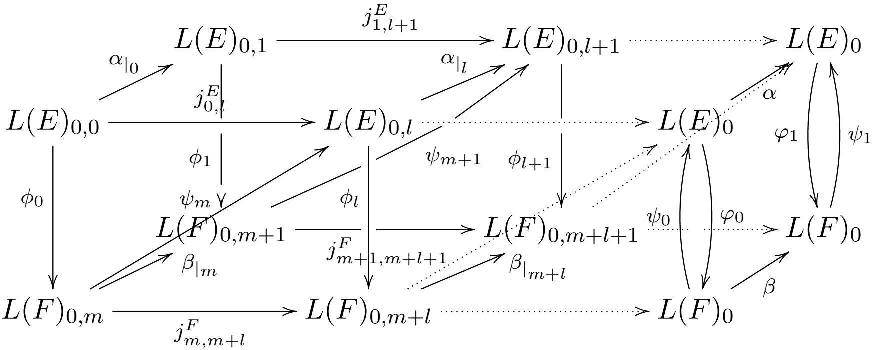
To do this we have, for a suitable $m \geq 1$, the following commutative diagram (tagged $(\dagger)$)



We lift, in an inductive way, diagram (†) to a commutative diagram of algebras



Thus, we have well-defined algebra isomorphisms

$$\varphi_0, \varphi_1 \colon L(E)_0 \to L(F)_0,$$

with inverses

$$\psi_0, \psi_1 \colon L(F)_0 \to L(E)_0$$

respectively.

Notice that $\beta \cdot \varphi_0 = \varphi_1 \cdot \alpha$.

Lemma

Let $g = \psi_0\varphi_1 \in \mathrm{Aut}(L(E)_0)$. Then g is a locally inner automorphism of $L(E)_0$.

So, using g we can define $L^g(E)$, and hence prove the main result.

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Theorem

E, F are finite essential graphs. TFAE:

1. $(K_0(L(E)_0), [1_{L(E)_0}]) \cong_{\mathbb{Z}[x,x^{-1}]} (K_0(L(F)_0), [1_{L(F)_0}])$.
2. *There exists a locally inner automorphism g of $L(E)_0$ such that $L^g(E) \cong_{gr} L(F)$.*

Proof.

$(2) \Rightarrow (1)$. If $L^g(E) \cong_{gr} L(F)$, there is an order-unit preserving isomorphism $K_0^{gr}(L^g(E)) \cong_{\mathbb{Z}[x,x^{-1}]} K_0^{gr}(L(F))$. So the result follows from main Theorem in previous section and the fact that $K_0^{gr}(L^g(E)) = K_0^{gr}(L(E))$.

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$(1) \Rightarrow (2)$ Set $g = \psi_0\varphi_1$. By Lemma, g is a locally inner automorphism of $L(E)_0$. Put $\alpha' = g\alpha$.

Observe that $\beta \cdot \varphi_0 = \varphi_0 \cdot \alpha'$. Indeed, we have

$$\varphi_0 \cdot \alpha' = \varphi_0 \cdot \varphi_0^{-1} \cdot \varphi_1 \cdot \alpha = \varphi_1 \cdot \alpha = \beta \cdot \varphi_0.$$

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Since

$$s_+\varphi_0(a)s_- = \beta(\varphi_0(a)) = \varphi_0(\alpha'(a))$$

$\forall a \in L(E)_0$, the universal property of $L(E)_0[t_+, t_-; \alpha']$ gives a unique algebra homomorphism

$$\varphi : L(E)_0[t_+, t_-; \alpha'] \to L(F)_0[s_+, s_-; \beta]$$

such that $\varphi|_{L(E)_0} = \varphi_0$, $\varphi(t_+) = s_+$ and $\varphi(t_-) = s_-$.

Clearly φ is a graded isomorphism, so we are done.

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Remark

1. If g inner automorphism of $L(E)_0$, $L(E) \cong_{gr} L^g(E)$.
2. There are locally inner automorphisms g of $L(E)_0$ such that $L(E) \ncong_{gr} L^g(E)$, and indeed such that $L^g(E) \ncong L(F)$ for any finite graph F.

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Note added in proof (April 2015):

[Ara-P.] By using a different method to lift diagrams, we are able to prove that

$$((S,R),m) : A \approx_1 B$$

induces a graded Morita equivalence from $L(E)$ onto $L(F)$. Moreover, if $S[1] = [1]$, then $L(E) \cong_{gr} L(F)$.

Towards a K-theoretic characterization of graded isomorphisms between Leavitt path algebras.

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COFEE?