Which k-Graphs Have AF C*-algebras?

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Given a k-graph, when is $C^*(\Lambda)$ AF?
When is $\mathcal{T}C^*(\Lambda)$ AF?
Is it ever the case that $C^*(\Lambda)$ is AF when $\mathcal{T}C^*(\Lambda)$ is not?

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When Λ is well behaved, (row finite, locally convex) an ideal $I(\Lambda)$ is generated by

$$\{t_v - \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^* : v \in \Lambda^0, i \in \{0, \ldots, k-1\}\}$$

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where e_i is the ith coordinate vector in $\mathbb{N}^k$.
Let q denote the quotient map. We write $q(t_\lambda) = s_\lambda$. Then $q(\mathcal{T}C^*(\Lambda)) = C^*(\Lambda)$.

Ordinary Graphs

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Theorem

TFAE

1. Λ *has no cycles;*
2. $C^*(\Lambda)$ *is AF;*
3. $\mathcal{T}C^*(\Lambda)$ *is AF;*
4. $\mathcal{T}C^*(\Lambda)$ *is finite.*

Cycles

Λ must not contain cycles.

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Since the AF property passes to quotients, it is enough to show that $C^*(\Lambda)$ is not AF.
However, we can easily observe that this is also an obstruction in $\mathcal{T}C^*(\Lambda)$. If λ is a cycle, $t_\lambda^* t_\lambda > t_\lambda t_\lambda^*$ so $\mathcal{T}C^*(\Lambda)$ is not finite.

Working with subsets of Λ

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Lemma

Λ has no cycles if and only if every finite set $E \subset \Lambda$ is contained in a finite self-invariant set.

Growth Lemma

Recall how multiplication works in $\mathcal{T}C^*(\Lambda)$. We will refer to an element $t_\mu t_\lambda^*$ as a monomial.

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Suppose $t_{\mu_1} t_{\lambda_1}^* t_{\mu_2} t_{\lambda_2}^* = t_{\mu'} t_{\lambda'}^*$. Then $d(\lambda') > d(\lambda_2)$ if and only if λ_1 is not an initial subpath of μ_2.

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Given a finite set of monomials S, a basis vector e_α can only be sent to finitely other basis vectors by elements of S.

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1. *Λ has no cycles;*
2. *$\mathcal{T}C^*(\Lambda)$ is finite;*
3. *$\mathcal{T}C^*(\Lambda)$ is quasidiagonal.*

It follows that there are many k-graphs for which $\mathcal{T}C^*(\Lambda)$ is finite but $C^*(\Lambda)$ is not, something which never happens with 1-graphs.

Locating finite dimensional subalgebras

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In a k-graph, even in the absence of cycles, a finite set of monomials can generate an infinite dimensional algebra.

Evans and Sims introduced *generalized cycles*, a certain class of infinite generating monomials and showed that they are an obstruction to $C^*(\Lambda)$ being AF.

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So it is sufficient to check sets of paths in $v\Lambda$.

Types of Finite Sets

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A set is *complete* if it is self-invariant and closed under minimal common extensions.

Observe that orthogonal sets are automatically complete.

Relating Finite Completions to Nilpotency

If $S \subset \{0, \ldots, k-1\}$, write $d(\lambda)_S$ to denote $(d(\lambda)_0 \cdot \chi_S(0), \ldots, d(\lambda)_{k-1} \cdot \chi_S(k-1))$.

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We will say that a pair of sets $E_0, E_1 \subset v\Lambda$ are *properly unbalanced* if there is a partition $S_0 \sqcup S_1$ of $\{0, \ldots, k-1\}$ such that for every $\lambda_0 \in E_0$ and $\lambda_1 \in E_1$, $d(\lambda_i)_{S_i} > d(\lambda_{1-i})_{S_i}$ for $i = 0, 1$.

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Lemma

Let Λ be a row-finite k-graph. When $E_0, E_1 \subset \Lambda$ are finite properly unbalanced orthogonal sets, then $E_0 \cup E_1$ has a finite completion if and only if $(\sum_{\mu \in E_0} t_\mu \sum_{\lambda \in E_1} t_\lambda^*)^n$ is 0 for some n.

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Theorem

Let Λ be a row-finite k-graph with no sources. Suppose $E_0 = \{\mu_1, \ldots, \mu_m\}$ and $E_1 = \{\lambda_1, \ldots, \lambda_m\} \subset v\Lambda$ are properly unbalanced orthogonal sets, and $(\sum_{i=1}^m t_{\mu_i} t_{\lambda_i}^*)^n$ is never 0 for any n. If either of $C^*(\Lambda_{S_i})$ is AF then $C^*(\Lambda)$ is not AF.

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Set $V = \sum_{i=1}^m t_{\mu_i} t_{\lambda_i}^*$. Note that this is a partial isometry.

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This must hold in an AF algebra, however it is unclear how to construct such a partial isometry or how to prove it cannot exist.

Sketching the Proof

Let Γ_k be the k-graph with $\Gamma_k^0 = \{v_n : n \in \mathbb{N}^k\}$ and a unique path of length $n_1 - n_2$ from v_{n_1} to v_{n_2} when $n_1 > n_2$.

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Consider the k-graph formed by $\Lambda_{S_i} \times \Gamma_{k-|S_i|}$. This has a copy of Λ_S on the graph induced by $\Lambda^0 \times v_n$ for each n.

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$H = \{\Lambda^0 \times v_n : n > 0\}$ is a hereditary and saturated set of vertices and the ideal it generates is Morita equivalent to $C^*(\Lambda_{S_i})$.
Finally, we assume that V completes to a unitary in $C^*(\Lambda)$ We then construct a proper isometry from V in $C^*(\Lambda_{A(S)})$.

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Consider the k-graph formed by $\Lambda_{S_i} \times \Gamma_{k-|S_i|}$. This has a copy of Λ_S on the graph induced by $\Lambda^0 \times v_n$ for each n. Modify this to put a copy of Λ at $\Lambda^0 \times v_0$ and call this $\Lambda_{A(S)}$.
$H = \{\Lambda^0 \times v_n : n > 0\}$ is a hereditary and saturated set of vertices and the ideal it generates is Morita equivalent to $C^*(\Lambda_{S_i})$.
Finally, we assume that V completes to a unitary in $C^*(\Lambda)$ We then construct a proper isometry from V in $C^*(\Lambda_{A(S)})$. Hence $C^*(\Lambda)$ is the quotient of a non-AF algebra by an AF algebra and is thus not AF.

The Canonical Ideal

Assume Λ is row-finite. Observe that $\mathcal{T}C^*(\Lambda_S)$ is a hereditary subalgebra of $I(\Lambda)$ for any proper subset $S \subset \{0, \ldots, k-1\}$.

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It follows that if $\mathcal{T}C^*(\Lambda_S)$ is not AF then neither is $\mathcal{T}C^*(\Lambda)$. Since the ideal it also not AF it does not necessarily follow that $C^*(\Lambda)$ is not AF, even if we know that $C^*(\Lambda_S)$ is not AF.
This problem does not occur in 2-graphs.

The Ideal In The 2-Graph Case

Proposition

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Corollary

Let Λ be a row-finite 2-graph. Then $\mathcal{T}C^*(\Lambda)$ is AF if and only if $C^*(\Lambda)$ is AF.

2-Graphs

We can strengthen our necessary condition in the case of 2-graphs.

Theorem

Let Λ be a row-finite 2-graph with no sources. Suppose $E_0 = \{\mu_1, \ldots, \mu_m\}$ and $E_1 = \{\lambda_1, \ldots, \lambda_m\} \subset v\Lambda$ are finite properly unbalanced orthogonal sets, and $(\sum_{i=1}^{m} t_{\mu_i} t_{\lambda_i}^*)^n$ is never 0 for any n. Then $C^*(\Lambda)$ is not AF.

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Recall our nilpotency-based sufficient condition for finite completions.

Lemma

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How To Construct Partial Isometries

Thus, if we can take an arbitrary finite subset E and find a pair E_0 and E_1 of properly unbalanced orthogonal sets which have finite completion if and only if E does, we can test finite completions in terms of nilpotency.

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Thus, if we can take an arbitrary finite subset E and find a pair E_0 and E_1 of properly unbalanced orthogonal sets which have finite completion if and only if E does, we can test finite completions in terms of nilpotency. If we can also put E_0 and E_1 into bijection then we can characterize the AF property in terms of nilpotency.

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Proposition

Let Λ be a fair row-finite 2-graph. If $(\sum_{\mu \in E_0} t_\mu \sum_{\lambda \in E_1} t_\lambda^*)^n$ is 0 for some n for every pair $E_0, E_1 \subset \Lambda$ of finite properly unbalanced sets then $C^*(\Lambda)$ is AF.

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Suppose E_0, E_1 form a perfect matching tree and write $S = s(E_0) \cap s(E_1)$. Choose a bijection $\Phi : E_0 \cap s^{-1}(S) \to E_1 \cap s^{-1}(S)$ which preserves sources.

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Nilpotency of V_Φ is independent of the choice of Φ. Thus we say a perfect matching tree is nilpotent when V_Φ is nilpotent for some Φ. If we assume that Λ contains lots of perfect matching trees then we can characterize when $C^*(\Lambda)$ is AF.

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Thank you.