Algebras Associated to Ample Groupoids

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Group Actions

Group Actions

Suppose a group H acts on a set X.

Thus there is a map $\cdot : H \times X \to X$ such that for each $x \in X$ and $g, h \in H$ we have

- $h \cdot x \in X$,
- $g \cdot (h \cdot x) = gh \cdot x$ and
- $e \cdot x = x$.

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A **groupoid** is a set of morphisms between elements of a set X satisfying the above conditions.

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- For γ and α in G, $\gamma\alpha \in G$ if and only if $s(\gamma) = r(\alpha)$.

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$$s((h, x)) = x \quad \text{and} \quad r((h, x)) = h \cdot x.$$

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A groupoid that is an equivalence relation is called a **Principal Groupoid**.

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- A topological groupoid G is called **ample** if G has a base of compact open bisections.
- Assume G is a Hausdorff ample groupoid.

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- *The groupoid associated to an action of a group on a graph.
  (Exel-Pardo)

Groupoid of a directed Graph

Let $E = (E_0, E_1, r, s)$ be a directed graph consisting of

- a countable set of vertices E_0,
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- Shift equivalent with lag k: Let $k \in \mathbb{Z}$. Define a relation $\sim_k$ on the set of all infinite paths so that $x \sim_k y$ if and only if there exists $N \in \mathbb{Z}^+$ such that $x_i = y_{i+k}$ for all $i \geq N$.

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Assume E is row finite and no sources.

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MORPHISMS: Suppose x and y are infinite paths. There is a morphism from y to x if and only if $x \sim_k y$. In this case, we label the morphism (x, k, y).

The Cuntz groupoid

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Let E be the directed graph with one vertex and 2 edges.

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MORPHISMS: Elements of form (x, k, y) where x and y are sequences that are 'eventually' the same and differ only in index by a fixed integer k.

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$$x = (0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots) \text{ and}$$
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Note: The set of morphisms that begin and end at a particular unit u is called the isotropy group at u.

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The collection of all such $Z(\mu, \nu)$ give a base of compact open bisections. (Kumjian-Pask-Raeburn-Renault)

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Proposition

Suppose G is a Hausdorff ample groupoid. Then

$$A(G) = \{f \in C_c(G) : f \text{ is locally constant }\}.$$

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The complex **Steinberg algebra** of G is the convolution $*$-algebra

$$A(G) := \text{span}\{1_B \mid B \text{ is a compact open bisection}\} \subseteq C_c(G)$$

where addition and scalar multiplication are pointwise and convolution and * of characteristic functions is given by

$$1_B * 1_D = 1_{BD} \quad \text{and} \quad 1_B^* = 1_{B^{-1}}.$$

Proposition

Suppose G is a Hausdorff ample groupoid. Then

$$A(G) = \{f \in C_c(G) : f \textit{ is locally constant }\}.$$

Contributors: Steinberg, Exel, C-Farthing-Sims-Tomforde ... and many others.

Leavitt path algebras are Steinberg algebras

Proposition

Let E be a directed graph. Then there is an isomorphism from $L(E) \to A(G_E)$ such that

$$(**) \quad p_v \mapsto 1_{Z(v)}, \quad s_e \mapsto 1_{Z(e,s(e))}, \text{ and } \quad s_\mu s_{\nu^*} \mapsto 1_{Z(\mu,\nu)}.$$

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- Surjectivity is a little grubby.

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- Let Λ be a rank-2 Bratteli diagram that is cofinal and aperiodic. Then $A(G_\Lambda)$ is simple but is neither locally matricial nor purely infinite. (C-Flynn-an Huef)

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Contributors: J.H. Brown, C, Edie-Michell, Farthing, Sims, Steinberg and Tomforde

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- We can do better....but first some examples.

Crisp and Gow's collapsible subgraph

Proposition (C-Sims)

Let $F^0 \subseteq E^0$ such that:

(T1) *each vertex in F^0 is the range of at most one $y \in E^\infty$ such that the source of $y_i \notin F^0$ for all i;*

for each $x \notin F^0$ we have

(T2) *a path from the range of x to a vertex in F^0; and*

(T3) $|s^{-1}(r(x_i))| = 1$ *for all i.*

Suppose $H \subseteq G_E$ is the restriction of G_E to unit space $\{Z(v) : v \in F^0\}$. Then $A_R(G_E)$ is Morita equivalent to $A_R(H)$.

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- Can we characterize the ideals of $A_R(G)$?
- Can we find necessary and sufficient conditions for simplicity of $A(G)$ when G is not Hausdorff?

The End

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Thank you!