Purely infinite étale groupoid C^*-algebras

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Classification of C^*-algebras, flow equivalence of shift spaces, and graph and Leavitt path algebras

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Outline

Separable C^*-algebras

Let X be a (countable) set.

- $\ell^2(X) = \{\xi : X \to \mathbb{C} : \sum_{x \in X} |\xi(x)|^2 < \infty\}$
  - If $X = \{1, 2\}$, $\ell^2(X) = \mathbb{C}^2$.
- There is an inner product on $\ell^2(X)$ given by

$$\langle \xi, \zeta \rangle = \sum_{x \in X} \overline{\xi(x)}\zeta(x) \quad \text{giving norm} \quad \|\xi\| = \sqrt{\langle \xi, \xi \rangle}.$$

- A linear map $T : \ell^2(X) \to \ell^2(X)$ (operator) is *bounded* if $\|T\| := \sup_{\|\xi\|=1} \|T\xi\| < \infty$.
  - Denote the set of bounded operators by $B(\ell^2(X))$
  - If $X = \{1, 2\}$, $B(\ell^2(X)) \cong M_2(\mathbb{C})$.
- Can add, multiply and scalar multiply bounded operators pointwise.
- Adjoint: $\forall T,\ \exists! T^*$ such that $\langle T\xi, \zeta \rangle = \langle \xi, T^*\zeta \rangle$.
- A C^*-algebra is a closed $*$-subalgebra of $B(\ell^2(X))$ for some X.
  - Projection: $p = p^2 = p^*$: think E_{11}
  - Partial isometry: s if s^*s, ss^* are projections: think E_{12}

Graph C^*-algebras

$E = (E^0, E^1, r, s)$ a directed graph
(row-finite, no sources: ie $0 < |r^{-1}(v)| < \infty$).

- $\alpha = \alpha_1\alpha_2\cdots$ a path if $s(\alpha_i) = r(\alpha_{i+1})$

$$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$$

- E^* is the set of finite paths, E^∞ is the set of infinite paths.
  - Denote by $Z(\alpha) = \alpha E^\infty = \{x \in E^\infty : x_1 \cdots x_{|\alpha|} = \alpha\}$
- $C^*(E)$ is generated by mutually orthoginal projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{s_e\}_{e\in E^1}$ which are universal for the Cuntz-Krieger relations:
  1. $s_e^* s_f = \delta_{e,f} p_{s(e)}$,
  2. $p_v = \sum_{r(e)=v} s_e s_e^*$.
- Cofinal: $\forall x \in E^\infty,\ v \in E^0\ \exists \alpha \in E^*, i \in \mathbb{N}$ such that $r(\alpha) = v,\ s(\alpha) = r(x_i)$.
- Condition L: If α a return path ($s(\alpha) = r(\alpha)$) then $\exists i \in \mathbb{N}$ such that $r(\alpha_i)E^1 - \{\alpha_i\} \neq \emptyset$.
- $C^*(E)$ simple $\Leftrightarrow$ E cofinal with Condition L. (Bates, et al 2000)

Kirchberg and Phillips classification

Let A be a C^*-algebra.

- For $a \in M_n(A)^+$, $b \in M_m(A)^+$, a is *Cuntz below* b, denoted $a \precsim b$, if there exists a sequence of elements x_k in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$ in norm.
- $a \in A^+$ is properly infinite if $a \oplus a \precsim a$.
  - if $a, b \in A^+$ $a \leq b$, then $a \precsim b$. (Kirchberg Rørdam '00)
- Heuristically, $a \in A^+$ is *properly infinite* if it has infinite range.
- A is *purely infinite* if every nonzero positive element is properly infinite.

Theorem (Kirchberg and Phillips '00)

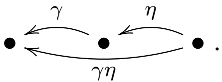
Two separable purely infinite, simple, nuclear C^*-algebras satisfying the UCT are isomorphic if and only if they are either both unital or nonunital and their ordered K-theory is isomorphic.

- Every simple graph C^*-algebra is either purely infinite or AF.
  - Contains return path $\Leftrightarrow$ $C^*(E)$ purely infinite. (Kumjian et al 1998)
- No characterization exists for generalizations of graph C^*-algebras.

Groupoid

A groupoid G is a small (arrows only) category in which every element is invertible.

- Mutliplication:



- Unit space: Identity arrows are identified with objects, we denote both by $G^{(0)}$.
- $r, s : G \to G^{(0)}$, taking γ to its range and source respectively.
  - γ, η composible iff $s(\gamma) = r(\eta)$.
- $xGy = \{\gamma : r(\gamma) = x, s(\gamma) = y\}$

Topological groupoids

- Topology
  - G second countable locally compact Hausdorff.
  - Composition and inversion are continuous.
- $B \subset G$ is a bisection if
  - B open,
  - $r(B), s(B)$ are open,
  - $r|_B, s|_B$ are homeomorphisms.
  - In particular $r|_B, s|_B$ are injective.
- G is étale if it has a basis of bisections.
- G étale implies
  - $r^{-1}(x)$ discrete.
    - ★ $\gamma \in r^{-1}(x)$ there exists open bisection B such that $\gamma \in B$.
    - ★ Since B a bisection $B \cap r^{-1}(x) = \{\gamma\}$.
  - $G^{(0)}$ clopen.

Graph Groupoid

Let E be a directed graph, $x, y \in E^\infty$.

- $x \sim_k y$ if there exists N such that for $i \geq N, x_{i+k} = y_i$.
  - $x \sim_k y$ if and only if there exists $\alpha, \beta \in E^*$, $z \in E^\infty$ with $|\alpha| - |\beta| = k$ and
$$x = \alpha z \qquad y = \beta z.$$
- Take $G_E = \{(x, k, y) : x \sim_k y\} \subset E^\infty \times \mathbb{Z} \times E^\infty$.
  - Multiplication: $(x, k, y)(y, \ell, z) = (x, k + \ell, z)$
  - Units: $(x, 0, x)(x, k, y) = (x, k, y) = (x, k, y)(y, 0, y)$.
    - ⋆ $E^\infty \leftrightarrow G_E^{(0)}$ by $x \leftrightarrow (x, 0, x)$.
  - Inverse: $(x, k, y)(y, -k, x) = (x, 0, x)$.

Basis: $Z(\alpha, \beta) = \{(\alpha z, |\alpha| - |\beta|, \beta z) : z \in s(\alpha)E^\infty\}$.

- Note:
$r(Z(\alpha, \beta)) = \alpha E^\infty \leftrightarrow Z(\alpha, \alpha), \quad s(Z(\alpha, \beta)) = \beta E^\infty \leftrightarrow Z(\beta, \beta)$
- $Z(\alpha, \beta)$ is compact.

Same construction works for higher rank graphs.

Étale groupoid C^*-algebras

Let G be an étale groupoid and $f, g \in C_c(G)$. Define

$$f * g(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma) \qquad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

$$1_{Z(\alpha,\beta)} * 1_{Z(\mu,\nu)}(x,k,y) = \sum_{(x,\ell,z)} 1_{Z(\alpha,\beta)}(x,\ell,z)1_{Z(\mu,\nu)}((x,\ell,z)^{-1}(x,k,y))$$

$$= \sum_{(x,\ell,z)} 1_{Z(\alpha,\beta)}(x,\ell,z)1_{Z(\mu,\nu)}(z,k-\ell,y)$$

$$= \begin{cases} 1 & \text{if } z = \mu z' = \beta x', y = \nu z', x = \alpha x', k = |\mu| - |\nu| + |\alpha| - |\beta| \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1_{Z(\alpha,\nu\mu')}(x,k,y) & \text{if} \quad \beta = \mu\mu' \\ 1_{Z(\alpha\beta',\nu)}(x,k,y) & \text{if} \quad \beta\beta' = \mu. \end{cases}$$

$$1^*_{Z(\alpha,\beta)}(x,k,y) = 1_{Z(\alpha,\beta)}(y,-k,x) = 1_{Z(\beta,\alpha)}(x,k,y)$$

Groupoid C^*-algebra

Regular Representation:

- For $x \in G^{(0)}$ consider $\ell^2(Gx)$, the left regular representation associated to x is charecterized by

$$L^x(f)\delta_\eta = f * \delta_\eta = \sum_{s(\gamma)=r(\eta)} f(\gamma)\delta_{\gamma\eta}$$

  where $f \in C_c(G)$ and δ_η is the Dirac delta function.

Define:

$$\|f\|_r = \sup_{x\in G^{(0)}} \|L^x(f)\|.$$

$C_r^*(G) \cong \overline{\bigoplus_{x\in G^{(0)}} L^x(C_c(G))}$.

- $C^*(E) \cong C_r^*(G_E)$ via $s_e \mapsto 1_{Z(e,s(e))}$.

Since $G^{(0)}$ is open in G,

- $C_c(G^{(0)}) \hookrightarrow C_c(G)$ via extension by 0;
- this inclusion extends to an inclusion $C_0(G^{(0)}) \hookrightarrow C_r^*(G)$.

Minimal groupoids

Definition

G is minimal *if $r(s^{-1}(x))$ is dense in $G^{(0)}$ for all $x \in G^{(0)}$.*

- Let E be a row-finite graph with no sources, then G_E is minimal if and only if E is cofinal. (Kumjian et al '97)
  - Suppose E is cofinal. Let $x \in E^\infty = G_E^{(0)}$ and $Z(\alpha)$ in $G^{(0)}$ open.
    - There exists a path μ and i such that $r(\mu) = s(\alpha)$ and $s(\mu) = r(x_{i+1})$.
    - Let $y = x_{i+1}x_{i+2}\cdots$ then $(\alpha\mu y, x_1 \cdots x_i y)$ has range in $Z(\alpha)$ and source x.
  - Suppose G_E is minimal, $x \in E^\infty$ $v \in E^0$
    - there exists $(\alpha y, k, \beta y)$ such that $\alpha y \in Z(v)$ and $\beta y = x$
    - Then $r(\alpha) = v$ and $s(\alpha) = r(y) = r(x_{|\beta|+1})$.

Topologically principal groupoids

Definition

G is topologically principal *if $X = \{x : xGx = \{x\}\}$ is dense in $G^{(0)}$. If $X = G^{(0)}$ then we say G is* principal*.*

- $xG_Ex \neq \{x\}$ if and only if $x = \alpha\mu\mu\cdots$ for some return path μ.
  - If $x = \alpha\mu\mu\cdots$ let $y = \mu\mu\cdots$ then $(\alpha\mu y, |\mu|, \alpha y) \in xG_Ex \setminus \{x\}$
  - If $(\alpha y, k, \beta y) \in xG_Ex \setminus \{x\}$, then $\alpha y = x = \beta y$.
  - WLOG assume $|\alpha| > |\beta|$ then for $\mu = \alpha_{|\beta|+1}\cdots\alpha_{|\alpha|}$, $x = \alpha\mu\mu\mu\cdots$.
- G_E is topologically principal if and only if E satisfies condition L. (Kumjian et al 1998)
  - If G_E topological principal and μ is a return path in E.
    - $y = \mu\mu\mu\cdots$. Then there exists $x \in Z(\mu)$ such that $xG_Ex = \{x\}$.
    - So $x = \mu z$ but $z \neq \mu\mu\cdots$.
    - Let i be the first index such that $x_i \neq \mu_j$ for some j. Then x_i is an entrance for μ: that is E satisfies L.
  - If E satisfies condition L and $Z(\alpha)$ open
    - Using condition L, construct a path $x = \alpha y$ with y not of the form $\beta\mu\mu\cdots$.
    - Then $xG_Ex = \{x\}$ and $x \in Z(\alpha)$: that is G_E topologically principal.

Locally Contracting

A étale groupoid G is *locally contracting* if for every $U \subset G^{(0)}$ open there exists a $V \subset U$ and a bisection B such that

1. $s(B) \subset \overline{V}$, and
2. $r(B) \subsetneq V$.

- If E is a graph and α is a return path with entrance e. Assume $r(e) = s(\alpha)$
- Consider $Z(\alpha\alpha, \alpha)$
  - $s(Z(\alpha\alpha, \alpha)) = Z(\alpha) \supset Z(\alpha\alpha) = r(Z(\alpha\alpha, \alpha))$
  - $Z(\alpha\alpha) \cap Z(\alpha e) = \emptyset$ so $Z(\alpha\alpha) \neq Z(\alpha)$.

Theorem (Anantharaman-Delaroche, '97)

If G is a locally contracting topologicially principal minimal étale groupoid then $C_r^*(G)$ is purely infinite.

Purely infinite

Theorem (B., Clark, Sierakowski '14)

For a second countable locally compact Hausdorff groupoid G that is topologically principal and minimal, $C_r^*(G)$ is purely infinite simple if and only if every nonzero positive element of $C_0(G^{(0)})$ is properly infinite.

Idea: Given $c \in C_r^*(G)$, use an argument of Anantharaman Delaroche (1997) and Lemma 2.2 of Kirchberg, Rørdam 2002 to construct a $b \in C_r^*(G)$ so that $b^*cb \in C_0(G^{(0)})$. Now b^*cb infinite implies c infinite.
Advantages of this theorem:

- Only have to check positive elements in a restricted abelian subalgebra of $C^*(G)$.

Disadvantage of the theorem:

- Theorem doesn't say anything about the groupoid structure.

Graphs

Let E be a row-finite cofinal graph that satisfies condition L.

- If $a \in C_0(E^\infty) = C_0(G_E^{(0)})$ is positive, then there exists $c \in \mathbb{C}$ and $\alpha \in E^*$ with $c1_{\alpha E^*} \leq a$.
- So a infinite if $1_{\alpha E^\infty}$ is infinite.
- Now $1_{\alpha E^*} = 1_{Z(\alpha,s(\alpha))}1^*_{Z(\alpha,s(\alpha))}$ is infinite if and only if $p_{s(\alpha)} \leftrightarrow 1_{s(\alpha)E^*} = 1^*_{Z(\alpha,s(\alpha))}1_{Z(\alpha,s(\alpha))}$ is.

Corollary (B., Clark, Sierakowski '14)

If E is a row-finite cofinal graph that satisfies condition L, then $C^*(E)$ is purely infinite simple if and only if p_v is infinite in $C^*(E)$ for all $v \in E^0$.

- Works for higher rank graphs too!

Constructing purely infinite groupoids

- Suppose G is a groupoid and $h : G \to G$ is an automorphism.
- Let $\mathcal{O}$ be the graph consisting of one vertex and countably many edges.
  - $C^*(\mathcal{O}) \cong \mathcal{O}_\infty$.
- Let $G_{\mathcal{O}}$ be the groupoid associated to $G_{\mathcal{O}}$ (slightly different construction from before).
- There is a map

$$c : G_{\mathcal{O}} \to \mathbb{Z} \quad \text{by} \quad (x, k, y) \mapsto k.$$

- Construct a semidirect product groupoid G_h^∞
  - $G_h^\alpha = \mathcal{O} \times G$ as a topological space.
  - $r(((x, k, y), \gamma)) = (x, r(\gamma)) \quad s(((x, k, y), \gamma)) = (y, h^k(s(\gamma)))$.
  - Operations:

$$((x, k, y), \gamma)) \cdot ((y, \ell, z), \eta) = ((x, k + \ell, z), \gamma h^{-k}(\eta)),$$
$$((x, k, y), \gamma))^{-1} = ((y, -k, x), h^k(\gamma^{-1})).$$

Properties of G_h^∞

Properties of G	Properties of h	Properties of G_h^∞
Principal	$r(s^{-1}(x)) = r(s^{-1}(h^k(x))) \Rightarrow k = 0$	Principal
	$\bigcup_{n\leq k} h^n(r(s^{-1}(x)))$ is dense in $G^{(0)}$	Minimal
Basis for $G^{(0)}$, $\mathcal{B}$, of compact open sets	$\forall V \in \mathcal{B} \ \ \exists \ell > 0$ such that $h^{-\ell}(V) \subset V$	locally contracting

By Anantharaman Delaroche '97 G_h^∞ is purely infinite simple if G and h satisfy the listed properties above.

Bratteli Diagrams

A Bratteli diagram E consists of vertices divided into levels V_n so that for each e in E^1 $s(e) \in V_i$ and $r(e) \in V_{i-1}$.

- Let $k_{vw} = |vE^1w|$ and enumerate $vE^1w = \{e^1_{vw}, e^2_{vw}, \cdots e^{k_{vw}}_{vw}\}$.
- Define
$$\tilde{h}(e^i_{vw}) = e^{(i+1 \mod (k_{vw}))}_{vw}$$
and $\tilde{h}$ fixes E^0.
- $\tilde{h}$ induces an automorphism on G_E via
$$h(x_1x_2\cdots, k, y_1y_2\cdots) = (\tilde{h}(x_1)\tilde{h}(x_2)\cdots, k, \tilde{h}(y_1)\tilde{h}(y_2)\cdots)$$
- Since $|r(\mu)Es(\mu)| < \infty$ and $h(Z(\alpha)) = Z(\tilde{h}(\alpha))$ there exists an ℓ with $h^\ell(Z(\alpha)) = Z(\alpha)$.
  - Thus G_E^∞ is locally contracting.

Bratteli Diagrams Continued

Assume for $v \in V_n$, $w \in V_{n+1}$ that $k_{vw} := |vE^1w| > n$. Then G_E^∞ is principal.

- Fix $x \in G^{(0)} = E^\infty$.
- Since E is acyclic, G_E is principal. Need to show $r(s^{-1}(x)) = r(s^{-1}(h^k(x))$ implies $k = 0$.
- If $r(s^{-1}(x)) = r(s^{-1}(h^k(x))$. This happens if and only if $\alpha x_m \cdots = \alpha \tilde{h}(x_n) \cdots$.Since $\tilde{h}$ fixes vertices we have $m = n$ and can thus assume

$$x = h^k(x).$$

- Thus $x_i = \tilde{h}^k(x_i)$ for all i.
- h^k permutes edges in $r(x_i)E^1s(x_i)$ mod $k_{r(x_i)s(x_i)}$ and $k_{r(x_i)s(x_i)} \to \infty$ we must have $k = 0$.

Thus if we assume E is cofinal. Then G_E is minimal and $\bigcup_{n \leq k} h^n(r(s^{-1}(x))) \subset r(s^{-1}(x))$ is dense for all x we get

$$C^*(G_E^\infty) \text{ is purely infinite simple.}$$

KK-equivalence

Theorem (B., Clark, Sierakowski, Sims)

Suppose G is a second countable locally compact amenable étale groupoid with a basis of compact open bisections and h and automorphism of G. Then the map

$$\iota_G : C_r^*(G) \to C_r^*(G_h^\infty) \quad \text{given by} \quad f \mapsto 1_{G_{\mathcal{O}}^{(0)}} \times f$$

induces an isomorphism of ordered K-theory.

Idea: Construct a $C_r^*(G)$ correspondence where $C_r^*(G_h^\infty)$ is the Toeplitz algebra and ι_G is the inclusion of $C_r^*(G)$ into the Toeplitz algebra. Then Theorem 4.4 of Pimsner 1997 gives the result.

Corollary (B., Clark, Sierakowski, Sims)

Suppose A is a Kirchberg algebra with $K_0(A)$ a simple dimension group and $K_1(A) = \{0\}$. Then there exists a topologically principal minimal étale groupoid G such that $C_r^*(G) \cong A$.

THANK YOU